On the Summation of Linearly Ordered Series

Kanesiroo Iseki

Department of Mathematics, Faculty of Science,
Ochanomizu University, Tokyo
(Received December 8, 1980)

Introduction.

Let N and $\boldsymbol{R}$ denote, respectively, the set of the positive integers and that of the real numbers, throughout this paper. An infinite sequence of real numbers may be regarded as a map of N into $\boldsymbol{R}$, where the set N is considered to be ordered by its natural ordering. Generalizing this, we obtain the notion of a linearly ordered sequence with real-valued terms, as follows.

Let Ω be a nonvoid countable (i.e. finite or enumerable) set of indices and suppose Ω furnished with a linear ordering. We shall understand by a *linearly ordered sequence* (with real-valued terms) defined on Ω, any map of the set Ω into the real line $\boldsymbol{R}$.

We are interested in finding a process by which, under certain conditions, the terms of a linearly ordered sequence can be summed in the given order of the indices. When we deal with summation, we shall speak of linearly ordered *series* instead of *sequence*, in accordance with the usual wording. Our summation process for linearly ordered series will be essentially based on the integration theory, and it is convenient to begin by obtaining certain properties of functions of a real variable.

§ 1. Terminology and notation.

We shall understand by a *linear set* any subset of $\boldsymbol{R}$. When we speak of a *closed interval* $[a, b]$ or an *open interval* (a, b), we shall always suppose a and b to be finite real numbers such that $a<b$. A *function*, by itself, will mean a map of $\boldsymbol{R}$ into $\boldsymbol{R}$, unless explicitly stated otherwise. In other respects, we shall generally conform to Saks [1] in terminology and notation. Thus, a *sequence* (of numbers, of sets, etc.), by itself, will signify a nonvoid countable one and so may be finite as well as infinite. Again, if $F(x)$ is a function and I is a closed interval, the increment of $F(x)$ over I will be denoted by $F(I)$, while we shall write $F[X]$ for the image of a linear set X under the function F. We shall, however, deviate from Saks [1] in using the notation $F(J)$ in the case of an open interval J also, this denoting the increment of F over the closure of J.

We shall often consider a linear compact set containing at least two points. Such a set will be called CT *set* for short. We shall understand by *interval spanned* by a CT set, the minimal closed interval containing this set. Throughout the paper, the letters Q and K will represent, respectively, a CT set and the interval spanned by it. Plainly, both the extremities of K belong to Q.

§ 2. Preliminaries.

Given a function $F(x)$ and a CT set Q, denote by J a generic open interval contiguous to Q, and by D the union of all intervals J. (If there is no J, then D is the void set.) We construct a function $L(x)$ subject to the following conditions:

(i) *$L(x)=F(x)$ whenever $x\in\boldsymbol{R}-D$,*
(ii) *$L(x)$ is linear in x on the closure of each J.*

This function L, which is plainly uniquely determined, will be called *linear modification* of the function F *with respect to the set Q.* The following properties of $L(x)$ are obvious:

THEOREM 1. *Let $L(x)$ be the linear modification of a function $F(x)$ with respect to a* CT *set Q.*

(i) *If $F(x)$ is continuous on the set Q, then $L(x)$ is continuous on the closed interval K spanned by Q;*

(ii) *$L(x)$ fulfils the condition* (N) *of Luzin on the open set $K-Q$. Consequently, $L(x)$ fulfils the same condition on the whole interval K, provided that $|F[Q]|=0$;*

(iii) *$L(x)$ is derivable at all points of $K-Q$ and we have, for every open interval J contiguous to Q, the relations*

$$F(J)=L(J)=\int_J L'(x)dx\,, \qquad |F(J)|=|L(J)|=\int_J |L'(x)|\,dx\,.$$

For later use, we quote now the following two theorems from Saks [1], p. 227 and p. 228.

THEOREM 2. *In order that a function $F(x)$ which is both continuous and* BV (*i.e. of bounded variation*) *on a compact set E, be* AC *on E, it is necessary and sufficient that $F(x)$ fulfil on this set the condition* (N) *of Luzin.*

THEOREM 3. *In order that a function F which is continuous on a closed interval I be* AC *on this interval, it is necessary and sufficient that F fulfil on I the condition* (N) *of Luzin and that its derivative exist almost everywhere on I and be summable on I.*

§ 3. Functions which are AC on a CT set of measure zero.

The results of this and the next section will constitute the kernel of our whole theory.

THEOREM 4. *Given a* CT *set* Q *of measure zero, denote by* J *a generic open interval contiguous to* Q. *In order that a function* F *which is continuous on* Q, *be* AC *on* Q, *it is necessary and sufficient that*

$$|F[Q]|=0 \quad \text{and} \quad \sum_J |F(J)|<+\infty$$

simultaneously. When this is the case, we have also $\sum_J F(J)=F(K)$, *where* K *means the closed interval spanned by* Q.

PROOF. Let $L(x)$ be the linear modification of $F(x)$ with respect to the set Q.

(a) Necessity. Suppose $F(x)$ to be AC on Q. Then $F(x)$ fulfils the condition (N) on Q, where $|Q|=0$ by hypothesis. Hence we have $|F[Q]|=0$, and it follows by articles (i) and (ii) of Theorem 1 that, on the interval K, the function $L(x)$ is both continuous and subject to the condition (N).

We shall show next that $L(x)$ is BV on K. For this purpose, consider an arbitrary sequence $x_0<x_1<\cdots\cdots<x_n$ $(n\in\boldsymbol{N})$, finite and increasing, of points of K. It suffices to ascertain that the sum $\sum_{i=1}^{n}|L(x_i)-L(x_{i-1})|$ is bounded. Since $L(x)$ is linear on every J, we may suppose, without loss of generality, that all the points $x_0, x_1, \cdots\cdots, x_n$ belong to the set Q. It follows that

$$\sum_{i=1}^{n}|L(x_i)-L(x_{i-1})|=\sum_{i=1}^{n}|F(x_i)-F(x_{i-1})|\leqq \mathrm{V}(F;Q).$$

But $\mathrm{V}(F;Q)<+\infty$, inasmuch as every function which is AC on a bounded set is necessarily BV on the same set.

The above results, combined with Theorem 2, show that $L(x)$ is AC on the interval K. The derivative $L'(x)$ is therefore summable on K. Noting that $|Q|=0$, we thus find, by article (iii) of Theorem 1, that

$$\sum_J|F(J)|=\sum_J\int_J|L'(x)|dx=\int_K|L'(x)|dx<+\infty.$$

which completes the necessity proof.

It also follows, from the above arguments, that

$$\sum_J F(J)=\sum_J\int_J L'(x)dx=\int_K L'(x)dx=L(K)=F(K).$$

(b) Sufficiency. Suppose that a function F is continuous on Q and fulfils both $|F[Q]|=0$ and $\sum_J|F(J)|<+\infty$. By articles (i) and (ii) of Theorem 1, the linear modification $L(x)$ of $F(x)$ is, on the interval K, both continuous and subject to the condition (N). Moreover, article (iii) of the same theorem, together with the hypothesis $|Q|=0$, shows that $L(x)$ is derivable almost everywhere on K and that

$$\int_K|L'(x)|dx=\sum_J\int_J|L'(x)|dx=\sum_J|F(J)|<+\infty.$$

The derivative $L'(x)$ is thus summable on K. It follows, on account of Theorem 3, that $L(x)$ is AC on K. Hence $F(x)$ is AC on Q, and this completes the sufficiency proof.

THEOREM 5. *Let Q be a* CT *set of measure zero, and let J denote a generic open interval contiguous to Q. Suppose that F and G are two functions subject to the following conditions:*

(a) *$F(x)$ is absolutely continuous on Q,*
(b) *$G(x)$ is continuous on Q and we have $|G[Q]|=0$,*
(c) *$F(J)=G(J)$ for every interval J introduced above.*

We then have $F(I)=G(I)$ for every closed interval I with extremities belonging to Q; or, what amounts to the same thing, the difference $F(x)-G(x)$ is a constant throughout the set Q.

PROOF. Let K be the closed interval spanned by Q. Condition (a) and Theorem 4 together imply that

$$\sum_J |F(J)| < +\infty \quad \text{and} \quad F(K)=\sum_J F(J).$$

It follows by condition (c) that $\sum_J |G(J)| < +\infty$. This, combined with condition (b), enables us to apply the same Theorem 4 to the function $G(x)$, and we obtain

$$F(K)=\sum_J F(J)=\sum_J G(J)=G(K).$$

This being so, consider any closed interval I with extremities belonging to Q. We find at once that, in the hypothesis of the present theorem, the set Q may be replaced by the intersection $Q\cap I$. The interval K considered above is then replaced by I. It follows, from what has already been proved, that $F(I)$ equals $G(I)$, which completes the proof.

§4. Functions which are GAC on a CT set of measure zero.

We are concerned, in this section, with generalizing the foregoing theorem to the case of GAC (i. e. generalized absolutely continuous) functions. For this purpose, we quote firstly the following theorem from p. 233 of Saks [1].

THEOREM 6. *In order that a function which is continuous on a nonvoid closed set E, be* GAC *on E, it is necessary and sufficient that every nonvoid closed subset of E contain a portion on which the function is* AC.

THEOREM 7. *We may weaken the condition* (a) *of Theorem* 5 *to the following form:* (a*) *$F(x)$ is* GAC *on the set Q.*

Thus, let Q be a CT *set of measure zero, and let J denote a generic open interval contiguous to Q. Suppose that F and G are two functions such that the above condition* (a*) *as well as the following two conditions are fulfilled:*

(b) *$G(x)$ is continuous on Q and we have $|G[Q]|=0$,*
(c) *$F(J)=G(J)$ for every interval J introduced above.*

We then have $F(I)=G(I)$ for every closed interval I with extremities belonging to Q, so that $G(x)$, too, must be GAC *on Q.*

PROOF. Let us denote by K the closed interval spanned by the set Q, and by $L(x)$ and $M(x)$ the linear modifications of $F(x)$ and $G(x)$, respectively, with respect to Q. These two modifications are continuous on K on account of article (i) of Theorem 1. An interval (closed or open) contained in K will be temporarily called *interval of constancy*, if the difference $L(x)-M(x)$ is a constant on the whole of it. For example, every interval J of our theorem has this property, in virtue of condition (c).

This being premised, let S be the union of all open intervals of constancy. S is then a nonvoid open set contained in the interval K and, as such, decomposes into its component open intervals, which we shall denote generically by H. We are going to show that every H is an interval of constancy. For this purpose, it is clearly sufficient to verify that every closed interval A situated in a fixed component H is an interval of constancy.

There exists, for such an interval A, a positive number δ such that any two points p and q of A both belong to one of the open intervals of constancy, provided only that $|p-q|<\delta$. Indeed, if the contrary were true, we could extract from A two infinite sequences of points, say $\langle p_n\rangle$ and $\langle q_n\rangle$, such that $\lim_n |p_n-q_n|=0$ and that, for each $n\in N$, no open interval of constancy would contain both p_n and q_n. We may, without loss of generality, suppose that both these sequences converge to a common limit, say ξ, belonging to A. Since $\xi\in H$, there would exist an open interval of constancy containing ξ. Both p_n and q_n would belong to this interval for n sufficiently large, and we should thus arrive at a contradiction.

Writing $A=[\alpha, \beta]$, suppose that $\alpha=x_0<x_1<\cdots\cdots<x_n=\beta$. It follows immediately from the above that, if $\max(x_i-x_{i-1})$ is sufficiently small, each of the n intervals $[x_{i-1}, x_i]$ is contained in an open interval of constancy. The difference $L(x)-M(x)$ must therefore be a constant on the whole interval A.

We have thus confirmed that every component interval H of the union S of all open intervals of constancy is itself one of these intervals. As we may observe, all the intervals J of our theorem are contained in S.

Let us write $E=K-S$, so that E is a CT set of measure zero contained in Q and spanning the interval K. Moreover, the open intervals contiguous to this set E are no other than the components H of S.

Now, every H being an interval of constancy, so must be the closure of H, too, by continuity of $L(x)-M(x)$ on K. If, therefore, E consists merely of the extremities of K, then $L(x)-M(x)$ will be a constant on the whole interval K, and the conclusion of our theorem will follow directly. Consequently we may, in what follows, assume E to contain at least one interior point of K, and it suffices to derive a contradiction from this assumption.

All points of E that are interior to K must then be accumulation points of E; in fact, the contrary would lead at once to absurdity. On the other hand, the function $L(x)$ is GAC on the set E, by condition (a*) of our theorem. As is plainly possible, let us take now, in the interior of K, a closed interval K_1 whose extremities belong to E and whose intersection with E is a perfect set. It follows by Theorem 6 that this intersection $E \cap K_1$ contains a portion on which $L(x)$ is AC. We then can choose in K_1 a closed interval K_2 whose extremities belong to E and whose intersection with E is a perfect set. Writing $P = E \cap K_2$ for short, we find easily that

(1) *P is an infinite* CT *set contained in E and, as such, is of measure zero;*
(2) *the function $L(x)$ is* AC *on this set P;*
(3) *the function $M(x)$ is continuous on P and we have $M[P]=0$;*
(4) *every open interval B contiguous to P is also contiguous to E, so that $L(B)=M(B)$.*

These properties of the set P, combined with Theorem 5, ensure that the difference $L(x)-M(x)$ is a constant, say k, on the whole set P. On the other hand, this difference is continuous on the interval K and further, on account of the above property (4), all open intervals contiguous to P are intervals of constancy. It follows immediately that $L(x)-M(x)=k$ all over the interval K_2 defined above. But property (1) requires that K_2, being the interval spanned by P, should contain infinitely many points of P, and hence, of E. This evidently contradicts the definition of the set E, and the proof is complete.

§ 5. A theorem connected with the Denjoy integration.

The following theorem will not be used later on; but we shall prove it here, because it is closely related to the interesting problem of generalizing the Denjoy integral.

THEOREM 8. *Suppose that $F(x)$ is a function which is* GAC *on a closed interval K, and that $G(x)$ is a function which is, on this interval,* (i) *continuous,* (ii) *subject to the condition* (N), *and* (iii) *approximately derivable almost everywhere.*

If $F'_{ap}(x)=G'_{ap}(x)$ almost everywhere on K, then the difference $F(x)-G(x)$ is a constant on the whole interval K, so that $G(x)$ also turns out to be GAC *on this interval.*

PROOF. Let A be the set of all interior points of K at each of which both $F(x)$ and $G(x)$ are approximately derivable, so that $|K-A|=0$ by hypothesis. We fix a positive number ε and we consider an arbitrary point, say ξ, of this set A. Writing

$$l=F'_{ap}(\xi)=G'_{ap}(\xi)$$

for brevity, we find at once that, in the interior of K, there are arbitrarily short closed intervals, say C, containing the point ξ and satisfying the inequalities

$$\left|\frac{F(C)}{|C|}-l\right|<\varepsilon \quad \text{and} \quad \left|\frac{G(C)}{|C|}-l\right|<\varepsilon .$$

These conditions together imply that $|F(C)-G(C)|<2|C|\varepsilon$.

Since ξ is an arbitrary point of A, the aggregate of the above intervals C covers A in the Vitali sense. By Vitali's covering theorem, therefore, we can extract from this aggregate a countable disjoint sequence of intervals C_1, C_2, …… whose union S fulfils $|A-S|=0$. We now write J_n for the interior of the closed interval C_n for $n=1, 2, \cdots\cdots$, so that

$$|F(J_n)-G(J_n)|<2|J_n|\varepsilon .$$

We then have $|K-D|=0$ for the union D of these intervals J_n; in fact, clearly $K-D\subset(K-A)\cup(A-S)\cup(S-D)$, where each summand set is of measure zero.

The set D just defined is open and its components are precisely the intervals $J_1, J_2, \cdots\cdots$. Again, the set $K-D$ is a CT set of measure zero and spans the interval K. We shall write $Q=K-D$.

Let us associate with each J_n the number δ_n determined by

$$|J_n|\delta_n=F(J_n)-G(J_n) ;$$

this condition, together with the inequality mentioned above, shows that $|\delta_n|<2\varepsilon$. This being so, let us define, on the real line $\boldsymbol{R}$, a function $\varphi(x)$ by the following condition:

$$\varphi(x)=\delta_n \text{ for } x\in J_n \quad \text{and} \quad \varphi(x)=0 \text{ for } x\in\boldsymbol{R}-D .$$

Thus defined, $\varphi(x)$ is a bounded function which is measurable (and hence summable) on every finite interval. Let $\Phi(x)$ be an indefinite integral of $\varphi(x)$; then $\Phi(x)$ is absolutely continuous on K and we have, for every J_n,

$$\Phi(J_n)=|J_n|\delta_n=F(J_n)-G(J_n) .$$

We also have $|\Phi(K)|\leqq 2|K|\varepsilon$ on account of the inequality $|\delta_n|<2\varepsilon$.

Thus, if we write $H(x)=F(x)-\Phi(x)$, we see directly that this function $H(x)$ is GAC on K and fulfils the condition $H(J_n)=G(J_n)$ for every J_n. By hypothesis, furthermore, $G(x)$ is continuous on K and satisfies the condition (N) on K, so that $|G[Q]|=0$ in particular, where $Q=K-D$ (see above). We therefore have $H(K)=G(K)$, in virtue of Theorem 7. It then follows that $F(K)-G(K)=\Phi(K)$, where $|\Phi(K)|\leqq 2|K|\varepsilon$ as already mentioned. We thus obtain the inequality $|F(K)-G(K)|\leqq 2|K|\varepsilon$, and this implies the equality $F(K)=G(K)$, since ε is an arbitrary positive number.

If I is an arbitrary closed interval lying in K, then, in the above proof for $F(K)=G(K)$, we may plainly replace the interval K by I throughout. We thus derive $F(I)=G(I)$, which completes the proof.

§ 6. Order isomorphism and equivalent functions.

Let A and B denote nonvoid linear sets in this section. If $x<y$ whenever $x\in A$ and $y\in B$, we write $A<B$ and we say that A *precedes* B. This relation may simply be expressed by $a<B$, in case A is a singletonic set $\{a\}$, and by $A<b$ when $B=\{b\}$.

We shall mean by $A\leqq B$ that either $A<B$ or $A=B$. If $\mathfrak{M}$ is any nonvoid aggregate of nonvoid linear sets, this relation $\leqq$ is evidently a partial ordering for $\mathfrak{M}$, and will be called its *natural ordering*. In particular, $\mathfrak{M}$ is linearly ordered by its natural ordering, iff (i. e. if and only if), of two arbitrary distinct elements of $\mathfrak{M}$, one necessarily precedes the other. This certainly occurs when $\mathfrak{M}$ consists of the components of a nonvoid open set in $\boldsymbol{R}$.

THEOREM 9. *If E and E' are two arbitrary linear sets which are nonvoid and closed, then every increasing map of E onto E', supposed existing, is necessarily continuous.*

By an increasing map we always mean one which is strictly increasing. The proof of this theorem is immediate and may be omitted.

THEOREM 10. *Given two nondense* CT *sets Q and Q', denote by J a generic open interval contiguous to Q, and by $\mathfrak{M}$ the aggregate of all the intervals J. Further, let J' and $\mathfrak{M}'$ be defined similarly for Q'. We suppose that $\mathfrak{M}$ and $\mathfrak{M}'$ are order isomorphic with respect to their natural linear orderings.*

If φ is an order isomorphism of $\mathfrak{M}$ onto $\mathfrak{M}'$, there exists an increasing map $\theta(x)$ of Q onto Q', such that, for any two points u, v of Q and any interval $J\in\mathfrak{M}$, the relation $u<J<v$ is always equivalent to $u'<J'<v'$, where $u'=\theta(u)$, $v'=\theta(v)$, $J'=\varphi(J)$.

This map θ is uniquely determined and must, of itself, be continuous.

PROOF. Writing $[a, b]$ and $[a', b']$ for the closed intervals spanned, respectively, by Q and Q', we set firstly $\theta(a)=a'$ and $\theta(b)=b'$. Let us consider next an arbitrary point $\xi\in Q$ other than a and b. We shall denote generically by A any of the intervals $J\in\mathfrak{M}$ fulfilling $J<\xi$, and by B any of the intervals $J\in\mathfrak{M}$ fulfilling $\xi<J$, so that we always have $A'<B'$, where we write $A'=\varphi(A)$ and $B'=\varphi(B)$. The set Q' being nondense, it is readily seen that there is a unique point $\xi'\in Q'$ satisfying the relation $A'<\xi'<B'$ for every pair A, B. We now define $\theta(\xi)=\xi'$.

We verify, without difficulty, that the map $\theta(x)$ thus defined fulfils the requirements of our theorem. Finally, the uniqueness of $\theta(x)$ is obvious, and its continuity is a direct consequence of Theorem 9.

REMARK. It is obvious but noteworthy that, with the notation of the theorem, the relation $J=(u, v)$ is always equivalent to $J'=(u', v')$.

THEOREM 11. *Given two* CT *sets Q and Q', and given an increasing map $\theta(x)$ of Q onto Q', suppose that F and Φ are two functions fulfilling the condition $F(x)=\Phi(x')$ for all $x \in Q$, where we write $x'=\theta(x)$ for short. Then the function F is continuous on Q, iff the function Φ is so on Q'.*

REMARK. The functions F and Φ of the theorem are, so to speak, *equivalent to each other* under the map $\theta(x)$.

PROOF. The map $\theta(x)$, together with its inverse map, is continuous on account of Theorem 9, whence the assertion follows at once.

THEOREM 12. *Suppose that, in the foregoing theorem, both Q and Q' are especially* CT *sets af measure zero. The function F is then* AC *on Q, iff the function Φ is so on Q'. Furthermore, we may replace here the property* AC *by* GAC.

PROOF. Both $\theta(x)$ and its inverse map are increasing as well as continuous, and we have $|Q|=|Q'|=0$ by hypothesis. Hence, by Theorem 2, these maps are both absolutely continuous. In view of their biuniqueness, we find immediately that absolute continuity of F on Q is equivalent to that of Φ on Q'.

To deduce the latter half of the assertion, it suffices to assume F to be GAC on Q and to show the same property of Φ on Q'. We can express the set Q as the union of a sequence of sets $Q_1, Q_2, \cdots\cdots$ on each of which the function F is AC and, by continuity of F, each set Q_n may be supposed to be CT. Then the set $\theta[Q_n]$ is also CT, and it follows, from what was already proved, that the function Φ is AC on $\theta[Q_n]$. Since Q' is the union of the sets $\theta[Q_n]$ and since Φ is continuous on Q' in virtue of the preceding theorem, we conclude that Φ is GAC on Q', which completes the proof.

Throughout the rest of this paper, we shall mean by a *linearly ordered set*, one which is nonvoid and countable. The following theorem might perhaps be a known result; we shall, however, give an outline of its proof, because we cannot, at present, ascertain the locality of the possible relevant literature.

THEOREM 13. *Given any linearly ordered set Ω, there always exists a linear* CT *set Q of measure zero, such that Ω is order isomorphic with the aggregate $\mathfrak{M}$ of the open intervals contiguous to Q, where we regard $\mathfrak{M}$ as linearly ordered by its natural ordering.*

PROOF. Supposing, as we plainly may, the set Ω to be infinite, let us enumerate all the elements of Ω in a distinct infinite sequence, say $\omega_1, \omega_2, \cdots\cdots$ Two distinct elements ω_k, ω_l of Ω will be termed to be *adjacent*, if there is no ω_n such that $\omega_k<\omega_n<\omega_l$ or $\omega_l<\omega_n<\omega_k$.

Let us write $U=[0, 1]$. It is sufficient to construct within U a disjoint infinite sequence of open intervals $J_1, J_2, \cdots\cdots$ whose union occupies almost all points of U and for which the relation $J_p<J_q$ is always equivalent to $\omega_p<\omega_q$, the indices p, q being arbitrary.

Suppose $H_1, \cdots\cdots, H_n$ to be a finite disjoint sequence of open intervals contained in U, and let k, p, q denote arbitrary positive integers $\leqq n$. The sequence $H_1, \cdots\cdots, H_n$ will, for the nonce, be termed to be *n-admissible*, if it satisfies the following conditions:

(i) *the relation* $H_p < H_q$ *is always equivalent to* $\omega_p < \omega_q$;
(ii) H_p *and* H_q *abut each other (i.e. have one extremity in common), iff* ω_p *and* ω_q *are adjacent*;
(iii) *the left-hand [or right-hand] extremity of* H_k *coincides with that of* U, *iff* ω_k *is the leftmost [or rightmost] element of the linearly ordered set* Ω.

In order to construct the aforesaid infinite sequence of open intervals J_1, $J_2, \cdots\cdots$, we fix a positive number $\lambda < 1$ and we proceed by induction. Let us choose firstly a 1-admissible interval J_1 with length $|J_1| > \lambda |U| = \lambda$, as is clearly possible. When we have determined the n intervals $J_1, J_2, \cdots\cdots, J_n$, we choose the next interval J_{n+1} in the following manner. There certainly exist open intervals (or, perhaps, an open interval) J such that the sequence $J_1, \cdots\cdots, J_n$, J is $(n+1)$-admissible, as we may see without difficulty. Every such interval J is clearly contained in one and the same closed interval $I \subset U$ disjoint with $J_1, \cdots\cdots, J_n$. Of course, this interval I is uniquely determined. We now take for J_{n+1} an arbitrary J of the above kind with length $|J| > \lambda |I|$.

The sequence $J_1, J_2, \cdots\cdots$ thus obtained fulfils all our requirements. The detailed verification of this may be left to the reader.

§7. Summation of linearly ordered series.

Suppose $f(\omega)$ to be a linearly ordered sequence defined on an indexing set Ω; in other words, let Ω be a linearly ordered set (nonvoid and countable, as already remarked) and let f be a map of Ω into the real line. We are concerned with the summation of the linearly ordered series $\sum_\omega f(\omega)$, where ω ranges over the set Ω.

DEFINITION. Given Ω and f as above, let us associate with Ω an arbitrary linear set Q satisfying the conditions of the foregoing theorem. We shall say that the linearly ordered series $\sum_\omega f(\omega)$ is *Luzin convergent*, if there exists at least one function $F(x)$ subject to the following three conditions and if, further, two such functions F necessarily differ, over the set Q, only by an additive constant.

(i) *The function* $F(x)$ *is continuous on the set* Q;
(ii) *the image* $F[Q]$ *is of measure zero*: $|F[Q]| = 0$;
(iii) *we have* $F(J) = f(\omega)$ *for every* $\omega \in \Omega$, *where* J *is that open interval (contiguous to* Q*) which corresponds to the index* ω *under the order isomorphism of the foregoing theorem.*

When the above is the case, we define the *Luzin sum* of our series by the formula:

$$\sum_{\omega} f(\omega)=F(K),$$

where K means the interval spanned by the set Q. Clearly, the increment $F(K)$ does not depend on the choice of the function F.

The epithet "Luzin" in the above terminology indicates the importance of the condition (N) of Luzin in our theory.

The following theorem is a direct consequence of Theorems 10 and 11.

THEOREM 14. *The above notion of Luzin convergence, as well as the value of the Luzin sum, depends neither on the choice of the set Q nor on that of the relevant order isomorphism.*

Suppose given a linearly ordered set Ω of indices and a map $f(\omega)$ of Ω into the real line. We are going to consider a few particular cases of Luzin convergence of the linearly ordered series $\sum_{\omega} f(\omega)$.

This series will be called *absolutely convergent*, if $\sum_{\omega} |f(\omega)| < +\infty$. The following theorem results immediately from Theorems 4 and 5.

THEOREM 15. *Every linearly ordered series which is absolutely convergent, is Luzin convergent, and its Luzin sum coincides with its ordinary sum.*

Let Q be any linear CT set of measure zero, such that Ω is isomorphic with the aggregate of the open intervals contiguous to Q (see Theorem 13). We shall call the linearly ordered series $\sum_{\omega} f(\omega)$ to be *Denjoy convergent*, if there exists a function $F(x)$ which is GAC on the set Q and for which the relation $F(J)=f(\omega)$ holds for every $\omega \in \Omega$, where J means the open interval corresponding to ω under the above order isomorphism. By Theorems 10 and 12, we see at once that this definition does not does not depend on the choice of the set Q or of the relevant order isomorphism. Evidently, *a linearly ordered series is Denjoy convergent, whenever it is absolutely convergent.*

The function $F(x)$ appearing above must be continuous on Q, since it is GAC on Q; we have further $|F[Q]|=0$, since Q is of measure zero and since every function which is GAC on a set must fulfil the condition (N) of Luzin on the same set. Thus $F(x)$ satisfies the conditions (i)$\sim$(iii) in the definition of Luzin convergence. This, in conjunction with Theorem 7, leads at once to the following result:

THEOREM 16. *Every linearly ordered series which is Denjoy convergent, is necessarily Luzin convergent.*

With the same notation as above, let $L(x)$ be the linear modification (see §2) of the function $F(x)$ with respect to the set Q. Evidently, $L(x)$ is GAC on the closed interval K spanned by Q and derivable almost everywhere on K. The derivative $L'(x)$ is then Denjoy integrable on K and we have

$$\sum_{\omega} f(\omega) = F(K) = \int_K L'(x)dx\,.$$

This explains our terminology "Denjoy convergent".

THEOREM 17. *Every infinite series* $\sum_{n=1}^{\infty} a_n$ *whose terms* a_n *are real numbers and which converges in the ordinary sense, is always Denjoy convergent and its Luzin sum coincides with its ordinary sum.*

PROOF. Let S be the ordinary sum of the series under consideration and let us write

$$D = \bigcup_{n=1}^{\infty} J_n\,, \qquad \text{where } J_n = \left(\frac{n-1}{n}, \frac{n}{n+1}\right),$$

$$\varphi(x) = \begin{cases} n(n+1)a_n & \text{if } x \in J_n\,, \\ 0 & \text{if } x \in \boldsymbol{R} - D\,, \end{cases} \qquad F(x) = \begin{cases} \int_0^x (t)dt & \text{if } x < 1\,, \\ S & \text{if } x \geqq 1\,. \end{cases}$$

We readily see that the function $F(x)$, thus defined, is GAC on the unit interval $U = [0, 1]$ and fulfils the relation $F(J_n) = a_n$ for all $n \in \boldsymbol{N}$. Moreover, the set $Q = U - D$ is closed and countable, and spans the interval U. The series $\sum_{n=1}^{\infty} a_n$ is thus Denjoy convergent, and its Luzin sum equals its ordinary sum, since $F(U) = F(1) - F(0) = S$. This completes the proof.

Let it be remarked finally that we are, at present, unable to decide whether or not the following assertions are true:

(i) *There exist linearly ordered series which are Luzin convergent without being Denjoy convergent.*

(ii) *Let* $f(\omega)$ *and* $g(\omega)$ *be two maps of a linearly ordered set* Ω *into the real line, and let us write* $h(\omega) = f(\omega) + g(\omega)$. *If the series* $\sum_{\omega} f(\omega)$ *and* $\sum_{\omega} g(\omega)$ *are both Luzin convergent, then so is the series* $\sum_{\omega} h(\omega)$ *also and we have the equality*

$$\sum_{\omega} h(\omega) = \sum_{\omega} f(\omega) + \sum_{\omega} g(\omega)\,.$$

It is easy to see that the above assertion (ii) holds good if we replace there the Luzin convergence by the Denjoy convergence.

Reference

[1] S. Saks: *Theory of the Integral*, Warszawa-Lwów (1937).