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# On Absolute Class Fields of Certain Algebraic Number Fields

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## § 0. Introduction.

In [9], H. M. Stark showed that there exist only finitely many totally imaginary fields of fixed degree with class number one, which are quadratic extensions of totally real fields. In [11], K. Uchida proved the finiteness of imaginary abelian fields with class number one (see also [6]). Moreover it is known that absolute values of discriminants of such imaginary fields with class number one of degree $\geqq 6$ have effectively computable upper bounds. For a special case, in [12], K. Uchida determined all the imaginary abelian fields of type $(2, 2, \cdots, 2)$ with class number one. On the other hand, it is well known that there exist only finitely many imaginary quadratic fields whose absolute class fields are equal to their genus fields ([7, Satz 25.5]). Recently, M. Bhaskaran [1] constructed the narrow genus field of any algebraic number field. Our problem is: how many fields are there whose absolute class fields are equal to their genus fiels? In the present paper, we extend the above finiteness theorems to the fields whose absolute class fields are equal to their genus fields (§ 1, Theorem 1 and 2) and determine all such imaginary abelian fields of type $(2, 2, \cdots, 2)$ of degree $\geqq 8$ (§ 2, Theorem 4).

It is my pleasant duty to express hearty gratitude to Professor M. Fujiwara for his continual guidance.

## Notations and definitions.

| | |
|---|---|
| $\boldsymbol{Z}$, $\boldsymbol{Q}$ and $\boldsymbol{R}$ | the rational integer ring, the rational field and the real number field respectively. |
| $k$ and $K$ | finite number fields contained in the complex number field. |
| $\tilde{K}$ | the absolute class field of $K$. |
| $h(K)=[\tilde{K}:K]$ | the class number of $K$. |
| $E_K$ and $W_K$ | the groups of units of $K$ and of roots of unity in $K$ respectively. |
| $E_K{}^*$ | the group generated by the fundamental units of all real quadratic fields contained in $K$. |
| $w_K=\sharp W_K$ | the number of the elements of $W_K$. |

$R(K)$ the regulator of $K$.
$D_K$ the absolute value of the discriminant of $K$.
$D$, $D_i$ ($i=1, 2, \cdots, m$) in §§2 and 3
discriminants of quadratic fields.
$h_D=h(\boldsymbol{Q}(\sqrt{D}))$.
$w_D=w_{\boldsymbol{Q}(\sqrt{D})}$.
$K_0=K\cap\boldsymbol{R}$ the maximal real subfield of an imaginary field $K$.

The following notations are used only for totally imaginary quadratic extensions $K$ of totally real fields $K_0$.

$q=(E_K : W_K E_{K_0})$ the unit index of $K/K_0$.
$D_{K/K_0}=D_K/D_{K_0}^2$ the absolute norm of the relative different of $K/K_0$.
$h_1(K)=h(K)/h(K_0)$ the first factor of the class number $h(K)$.

Genus fields and genus numbers are defined as follows.

DEFINITION. *Let $k$ be a subfield of an algeblaic number field $K$. The genus field $K^*(k)$ of $K$ over $k$ is the maximal extension of $K$ satisfying the conditions:*
(i) *no prime divisor of $K$ ramifies in $K^*(k)/K$,*
(ii) *there is an abelian extension $k_1/k$ such that $K^*(k)=Kk_1$.*
*The genus number $g(K/k)$ of $K$ over $k$ is $[K^*(k): K]$. Especially, we call $K^*=K^*(\boldsymbol{Q})$ the genus field of $K$ and $g(K)=g(K/\boldsymbol{Q})$ the genus number of $K$.*

## §1. The finiteness theorems.

In this section, we assume that $K$ is a totally imaginary quadratic extension of a totally real field $K_0$.

The genus number $g(K/k)$ of a normal extension $K/k$ is given by Y. Furuta [4]. Lemma 1 is an immediate consequence of his formula.

LEMMA 1. (i) *Let $K$ be a totally imaginary quadratic extension of a totally real field $K_0$, then*

$$g(K/K_0)=h(K_0)2^{r-1+g^*},$$

*where $r$ is the number of distinct prime ideals in $K_0$ ramified in $K/K_0$ and $2^{g^*}=(N_{K/K_0}(K) : E_{K_0}^2)$, $0\leqq g^*\leqq[K_0 : \boldsymbol{Q}]$.*

(ii) *Let $k$ be a real quadratic field, then*

$$g(k)=2^{r-2}\delta_{D_k},$$

*where $r$ is the number of distinct primes ramified in $k/\boldsymbol{Q}$ and $\delta_{D_k}$ is defined as*

$$\delta_{D_k}=\begin{cases} 2 & \text{if } N_{k/Q}(k)\ni -1 \\ 1 & \text{otherwise} \end{cases}.$$

LEMMA 2. *Let $k$ be an algebraic number field of degree $m$, $\mathfrak{a}$ an integral ideal in $k$ and $r$ the number of distinct prime ideals dividing $\mathfrak{a}$. Then there exists an* *effectively computable constant $r(m)$ depending only on $m$ such that if $r>r(m)$,*

$$\log N_{k/Q}(\mathfrak{a})>(3m)^{-1}r(\log r-\log m).$$

PROOF. Let us put all primes in increasing order $p_1=2$, $p_2=3$, $\cdots$. Let $r'$ denote the number of distinct primes dividing $N_{k/Q}(\mathfrak{a})$. Then $r\leqq mr'$ and

$$\log N_{k/Q}(\mathfrak{a})\geqq \sum_{i=1}^{r'}\log p_i \geqq \sum_{\substack{p \\ \sqrt{p_{r'}}\leqq p<p_{r'}}} \log\sqrt{p_{r'}},$$

so that

$$\log N_{k/Q}(\mathfrak{a})\geqq \frac{1}{2}\log p_{r'}(r'-\sqrt{p_{r'}}). \tag{1}$$

On the other hand, the prime number theorem shows effectively that

$$\log p_{r'}(r'-\sqrt{p_{r'}}) \sim p_{r'} \sim r'\log r' \qquad (r'\to\infty). \tag{2}$$

(1) and (2) prove the lemma. Q. E. D.

Though the following Brauer-Siegel theorem is essential, it is ineffective.

THEOREM A (Brauer-Siegel theorem [2, Theorem 2]). *For the fields $k$ of fixed degree,*

$$\log(h(k)R(k)) \sim \log\sqrt{D_k} \qquad (D_k\to\infty).$$

H. M. Stark gave an effective lower bound for $h(K)$ but what he obtained is actually an estimate of the first factor $h_1(K)$ of $h(K)$.

THEOREM B (H. M. Stark [9]). *Let $K$ be a totally imaginary field of degree $2n$ containning a totally real subfield $K_0$ of degree $n$. For $\varepsilon$ in the range $0<\varepsilon<1/2$, there is an effectively computable function $c(\varepsilon)>0$ such that*

$$h_1(K)>(ng_K)^{-1}c(\varepsilon)^n D_{K_0}{}^{1/2-1/n-\varepsilon} D_{K/K_0}{}^{1/2-1/2n},$$

*where $g_K=1$ if there exists a sequence of fields $\boldsymbol{Q}=k_0\subset k_1\subset\cdots\subset k_m=K_0$ each normal over the preceding field and $g_K=n!$ otherwise.*

Using the above theorems, we can prove the following theorem.

THEOREM 1. *There are only finitely many totally imaginary fields $K$ of a fixed degree $2n$ whose absolute class fields $\tilde{K}$ are equal to their genus fields $K^*(K_0)$. Moreover, if $2n\geqq 6$, $D_K$ of such fields $K$ have an effectively computable upper bound depending only on $n$.*

PROOF. In case $n>2$, Theorem B and Lemma 2 show the inequality

$$h_1(K)>2^{r-1+g^*} \tag{3}$$

for almost all $K$, where $r$ and $g^*$ are those in Lemma 1 (i). When $n\leqq 2$, the inequality $R(K)\leqq 2R(K_0)$ and Theorem A show that

(4) $$\frac{\log(h(K)R(K))}{\log\sqrt{D_K}}\leqq\frac{\log(2h_1(K))}{\log\sqrt{D_K}}+\frac{1}{2}+\varepsilon$$

for almost all $K$ and for any $0<\varepsilon<1/4$. (4) and Theorem A show that

(5) $$\frac{1}{3}\log\sqrt{D_K}\leqq\log h_1(K)$$

for almost all $K$ of degree 2 or 4. Thus, in case $n\leqq 2$,

(6) $$h_1(K)>2^{r-1+g^*}$$

for almost all $K$ by virtue of (5) and Lemma 2. (3), (6) and Lemma 1 (i) prove the theorem. Q.E.D.

The case $n=1$ of Theorem 1 is Satz 25.5 of [7]. The following corollary can be easily obtained from Theorem 1.

COROLLARY. *There are only finitely many totally imaginary fields $K$ of a fixed degree whose absolute class fields $\tilde{K}$ are equal to their genus fields $K^*$. Moreover, $D_K$ have an effectively computable upper bound depending only on the degree $[K:\boldsymbol{Q}]$ if $[K:\boldsymbol{Q}]>4$.*

For abelian fields $K$, the conductor-discriminant formula gives a better lower bound of $D_K$. Therefore we can show that we don't have to fix the degree $[K:\boldsymbol{Q}]$ in the above theorem and corollary (cf. Theorem 2 and its proof). The following lemma was proved by K. Uchida in [11].

LEMMA 3 (K. Uchida). *Let $f_K$ denote the conductor of an abelian field $K$ of degree $2n$. Then*

$$D_K\geqq f_K{}^n .$$

THEOREM 2. *There exist only finitely many fields $K$ whose absolute class fields $\tilde{K}$ are imaginary abelian over $\boldsymbol{Q}$. And the conductors $f_K$ of such fields $K$ have an effectively computable upper bound if $[K:\boldsymbol{Q}]\geqq 6$.*

PROOF. We first show that

(7) $$h_1(K)>2^{r-1+g^*}$$

for any large $n$, where $r$ and $g^*$ are those in Lemma 1 (i). Note that $f_K\to\infty$ when $n\to\infty$, and that Theorem B and Lemma 3 imply the inequality

(8) $$h_1(K)>\frac{c(\varepsilon)^n}{n}f_K{}^{n/4-1/2-\varepsilon n/2} .$$

Let $r'$ be the number of distinct primes dividing $f_K$. Then clearly,

(9) $$2^{r-1+g^*}<2^{n(r'+1)} .$$

Lemma 2, (8) and (9) show the inequality (7) for any large $n$. Thus, we proved that there exist only finitely many imaginary abelian fields $K$ whose absolute class fields $\tilde{K}$ are equal to $K^*(K_0)$. On the other hand, by the definition of genus fields $K^*$, $\tilde{K}=K^*$ if and only if $\tilde{K}$ is abelian over $\boldsymbol{Q}$. Those facts prove the theorem. Q. E. D.

For imaginary cyclic biquadratic fields $K$, the effectiveness of Theorem 2 is also true (cf. [11, Proof of Theorem 1′]).

## § 2. Imaginary abelian fields of type (2, 2, ⋯ , 2).

Our purpose now is to determine all the imaginary abelian fields of type (2, 2, ⋯ , 2) of degree $\geqq 8$ whose absolute class fields are abelian over $\boldsymbol{Q}$. Dirichlet's class number formula and prime discriminants of quadratic fields will fully be used. Henceforth we will mainly be concerned with abelian fields of type (2, 2, ⋯ , 2) of degree $2^m$.

LEMMA 4. *Let $K$ be an abelian field and $k$ a subfield of $K$. Then $h(K)=g(K)$ implies $h(k)=g(k)$.*

PROOF. Obvious.

LEMMA 5. *Let $G$ be an abelian group of type (2, 2, ⋯ , 2) of order $2^m$ and $\hat{G}$ its character group. Then the rank of the $(2^m-1)\times 2^m$ matrix*

$$S(m)=(\chi(\tau))_{\substack{\tau\in G,\, \tau\neq 1\\ \chi\in\hat{G}}}$$

*is $2^m-1$ and all the minors of order $(2^m-1)$ of $S(m)$ have the same absolute value.*

PROOF. Easily verified.

Let $s(m)$ denote the absolute value of any minor of order $2^m-1$ of $S(m)$. Then Dirichlet's class number formula shows the following:

LEMMA 6. *Let $K$ be an imaginary abelian field of type (2, 2, ⋯ , 2) of degree $2^m$. Assume that $h(K)=g(K)$, then*

$$h(K)=Q\prod_{D>0}\delta_D\frac{1}{s(m-1)}\frac{1}{w}2^{\sum_D r_D-2^m-2^{m-1}+2},$$

*where the product $\prod_{D>0}$ runs through the discriminants $D>0$ of all the real quadratic subfields of $K$, the sum $\sum_D$ runs through the discriminants of all the quadratic subfields of $K$, $r_D$ is the number of primes dividing $D$ and $Q$, $\delta_D$ and $w$ are as follows.*

$$Q=(E_K:W_KE_K{}^*),$$

$$\delta_D=\begin{cases}2 & \textit{if } N(\boldsymbol{Q}(\sqrt{D}))\ni -1\\ 1 & \textit{otherwise}.\end{cases}$$

$$w=\begin{cases} 2^{2^{m-1}-2} & if\ K\supset \boldsymbol{Q}(\sqrt{-1}, \sqrt{-2}) \\ 2^{2^{m-1}-1} & otherwise. \end{cases}$$

PROOF. From the formula $\zeta_K(s)=\zeta(s)\prod_D L(s, (D/\circ))$ and Lemma 5, it follows that

$$h(K)=Q\frac{1}{s(m-1)}\frac{1}{w}\prod_D h_D,$$

where the product $\prod_D$ runs through the discriminants $D$ of all the quadratic subfields of $K$. Lemmas 5, 4 and 1 (ii) and equality $g(\boldsymbol{Q}(\sqrt{D}))=2^{r_D-1}$ for $D<0$ prove the lemma. Q.E.D.

A prime discriminant $P$ is either $-4$, $\pm 8$ or $(-1)^{(p-1)/2}$ for an odd prime number $p$. It is well known that discriminants of quadratic fields can uniquely be factorized into prime discriminants. Let $D=P_1P_2\cdots P_t$ be such a factorization of discriminant $D$. We call $P_i$ $(i=1, 2, \cdots, t)$ be the prime discriminant factors of $D$.

LEMMA 7. *Let $K$ be an abelian field of type $(2, 2, \cdots, 2)$ of degree $2^m$. Then there exists a set of discriminants $D_i$ and prime discriminants $P_i$ $(i=1, 2, \cdots, m)$ such that*

(i) $K=\boldsymbol{Q}(\sqrt{D_1}, \cdots, \sqrt{D_m})$,

(ii) *$P_i$ is a prime discriminant factor of $D_i$ but of $D_j$ $(i\neq j)$,*

(iii) *$D_j$ $(j=3, 4, \cdots, m)$ are all odd,*

(iv) *if $D_2$ is even, then $D_1$ is also even and $P_1=-4$, $P_2=-8$.*

To prove Lemma 7, we introduce the symbol $( , )_d$ for convenience.

DEFINITION. *Let $D$ and $D'$ denote discriminants of quadratic fields and $P$ and $P'$ the even prime discriminant factors of $D$ and $D'$ respectively. We define the symbol $( , )_d$ as follows.*

$$(-4, \pm 8)_d=(\pm 8, -4)_d=\mp 2, \qquad (-8, 8)_d=(8, -8)_d=-4,$$

$$(D, 1)_d=(1, D)_d=1, \qquad (1, 1)_d=1, \qquad (D, D)_d=D,$$

$$(D, D')_d=(P, P')_d\{\pm g.c.d.(D/P, D'/P')\},$$

*where the signature in $\{\}$ must be taken so that the term in $\{\}$ is the discriminant of a quadratic field.*

It is easily verified that $(P, D)_d=(D, P)_d=P$ if and only if $P$ is a prime discriminant factor of $D$. We shall list the pricipal properties of the symbol $( , )_d$ in Lemma 8. The proof is easy and left to the readers.

LEMMA 8. *Let $D_1$, $D_2$, $D$ and $D'$ be discriminants of quadratic fields and $P$ a prime discriminant. Then*

(i) *if $D_1=DD'$, then $(D_1, D_2)_d=(D, D_2)_d(D', D_2)_d$,*

(ii) *$D_1D_2/(D_1, D_2)_d^2 \in \boldsymbol{Z}$ and if it is not a discriminant, then it is equal to* 1.

(iii) *if $P$ is a prime discriminant factor of $D_1$ and $(P, D_2)_d$ is a discriminant, then $(D_1D_2/(D_1, D_2)_d^2, P)_d$ is not a discriminant.*

(iv) *if $P$ is a prime discriminant factor of $D_1$ and $(P, D_2)_d=1$, then $(D_1D_2/(D_1, D_2)_d^2, P)_d=P$.*

PROOF OF LEMMA 7. We may set $K=\boldsymbol{Q}(\sqrt{D_1}, \cdots, \sqrt{D_m})$. In case both of 8 and $-8$ are prime discriminant factors of $D_j$ $(j=1, 2, \cdots, m)$, fix one of the discriminants $D_i$ for which $-8$ is a prime discriminant factor. Let us call it $D$ and set

$$D_j'=\begin{cases} DD_j/(D, D_j)_d^2 & \text{if } (8, D_j)_d=8 \\ D_j & \text{otherwise .} \end{cases}$$

Then $K=\boldsymbol{Q}(\sqrt{D_1'}, \cdots, \sqrt{D_m'})$ and $(8, D_j')_d$ $(j=1, 2, \cdots, m)$ are $-4$'s or not discriminants by Lemma 8 (iii). And so 8 is not a prime discriminant factor of $D_j'$ $(j=1, 2, \cdots, m)$. In case both of $-4$ and 8 are prime discriminant factors of $D_j$ $(j=1, 2, \cdots, m)$, similarly, we can replace $D_j$ by $D_j'$ $(j=1, 2, \cdots, m)$ whose prime discriminant factors are not 8's. Hence we assume that if there are more than two even prime discriminant factors of $D_j$, they are not 8's and that

$$\begin{cases} (D_1, -4)_d=-4,\ P_1=-4 & \text{if } (-4, D_j)_d=-4 \text{ for some } j=1, 2, 3, \cdots, m \\ (D_1, 2)=2,\ (P_1, 2)=2 & \text{if there is one and only one even prime} \\ & \text{discriminant factor of } D_j\ (j=1, 2, \cdots, m) \\ P_1 \text{ is any one of } P \text{ such that } (P, D_1)_d=P & \\ & \text{otherwise.} \end{cases}$$

For $j\neq 1$, set

$$D_j'=\begin{cases} D_1D_j/(D_1, D_j)_d^2 & \text{if } (P_1, D_j)_d=\textit{discriminant} \\ D_j & \text{otherwise .} \end{cases}$$

Then by Lemma 8 (iii), $(P_1, D_j')_d$ $(j=2, 3, \cdots, m)$ are not discriminants. Obviously, $K=\boldsymbol{Q}(\sqrt{D_1}, \sqrt{D_2'}, \cdots, \sqrt{D_m'})$ and $(P_1, D_1)_d=P_1$. Moreover if there are even prime discriminant factors of $D_2', \cdots, D_m'$, then they have to be $-8$'s. In such a case we may assume $(D_2', -8)_d=-8$. In other case, $D_2'$ is odd and $D_1\neq D_2'$, and we set

$$\begin{cases} P_2=-8 & \text{if } (D_2', -8)_d=-8 \\ P_2 \text{ is any one of } P \text{ such that } (P, D_2')_d=P & \text{otherwise.} \end{cases}$$

We use $D_i$ again in state of $D_i'$ and set

$$D_j'=\begin{cases} D_2D_j/(D_2, D_j)_d^2 & \text{if } (P_2, D_j)_d=\textit{discriminant}\ (j\neq 2) \\ D_j & \text{otherwise .} \end{cases}$$

Then we have

(i) $K=\boldsymbol{Q}(\sqrt{D_1'}, \cdots, \sqrt{D_m'})$,

(ii) $P_i$ is a prime discriminant factor of $D_i$ but of $D_j$ ($j\neq 1$, $i=1, 2$, $j=1, 2, \cdots, m$),

(iii) $D_j'$ ($j=3, 4, \cdots, m$) are all odd,

(iv) if $D_2'$ is even, then $D_1'$ is also even and $P_1=-4$, $P_2=-8$.

(iii) can easily be checked. (ii) is shown by Lemma 8 (iii), (iv). Induction on $t$ shows that there is a set of discriminants $D_j$ ($j=1, 2, \cdots, m$) and prime discriminants $P_i$ ($i=1, 2, \cdots, t$, $3\leqq t\leqq m$) such that

(i) $K=Q(\sqrt{D_1}, \cdots, \sqrt{D_m})$,

(ii) $P_i$ is a prime discriminant factor of $D_i$ but of $D_j$ ($j\neq i$),

(iii) $D_j$ ($j=3, 4, \cdots, m$) are all odd,

(iv) if $D_2$ is even, then $D_1$ is also even and $P_1=-4$, $P_2=-8$.

The rest of the proof is left to the readers.

Let $K=Q(\sqrt{D_1}, \cdots, \sqrt{D_m})$, where $D_i$ ($i=1, 2, \cdots, m$) is chosen as in Lemma 7, and let $r$ be the number of distinct prime discriminant factors of $D_i$ ($i=1, 2, \cdots, m$) and $e_p$ the ramificasion index of a prime $p$ in $K/\boldsymbol{Q}$. Furthermore, note that $D_2$ being even if and only if $e_2=4$.

LEMMA 9. *Let $K$, $r$ and $e_p$ be as in the above remark and $\sum_D r_D$ be as in Lemma* 6. *Then*

$$\sum_D r_D=\begin{cases} 2^{m-1}r & \text{if } e_2\leqq 2 \\ 2^{m-1}r-2^{m-2} & \text{if } e_2=4. \end{cases}$$

PROOF. Let us choose $D_i$ ($i=1, 2, \cdots, m$) as in Lemma 7 and let $r_0$, $r_1$, $r_2$ and $r_3$ be the numbers of distinct prime discriminant factors of $(D_3, D_4, \cdots, D_m)$, $(D_1, D_3, D_4, \cdots, D_m)$, $(D_2, D_3, D_4, \cdots, D_m)$ and $(D_1D_2/(D_1, D_2)_d{}^2, D_3, D_4, \cdots, D_m)$ respectively. Then

$$r_1+r_2+r_3-r_0=\begin{cases} 2r & \text{if } D_2 \text{ is odd} \\ 2r-1 & \text{if } D_2 \text{ is even}. \end{cases} \tag{10}$$

On the other hand, let $k=\boldsymbol{Q}(\sqrt{D_3}, \sqrt{D_4}, \cdots, \sqrt{D_m})$, and let $D(0)$, $D(1)$, $D(2)$ and $D(3)$ denote symbolically the discriminants of quadratic subfields of $k$, $k(\sqrt{D_1})$, $k(\sqrt{D_2})$ and $k(\sqrt{D_1D_2})$ respectively, then

$$\sum_D r_D=\sum_{D(1)} r_{D(1)}+\sum_{D(2)} r_{D(2)}+\sum_{D(3)} r_{D(3)}-2\sum_{D(0)} r_{D(0)}, \tag{11}$$

where $r_{D(j)}$ ($j=0, 1, 2, 3$) is the number of primes dividing $D(j)$. (11), (10) and induction on $m$ prove the lemma. Q. E. D.

From Lemmas 6 and 9, we have

LEMMA 10. *Let $K$ be an imaginary abelian field of type* (2, 2, $\cdots$, 2) *of degree $2^m$. If $h(K)=g(K)$, then*

$$h(K)=\frac{Q(\prod_{D>0}\delta_D)}{s(m-1)}\begin{cases} 2^{2^{m-1}r-2^{m+1}+3} & \text{if } e_2\leqq 2 \\ 2^{2^{m-1}r-2^{m-2}-2^{m+1}+3} & \text{if } e_2=4,\ K\not\supset \boldsymbol{Q}(\sqrt{-1}\sqrt{-2}). \\ 2^{2^{m-1}r-2^{m-2}-2^{m+1}+4} & \text{if } K\supset \boldsymbol{Q}(\sqrt{-1}, \sqrt{-2}) \end{cases}$$

Taking into account the inequality $g(K)\geqq h'(K)=h(K)/Q(\prod_{D>0}\delta_D)$, we obtain an upper bound of $r$ for given $m$. For example, in case $m=2$,

$$h'(K)=\begin{cases} 2^{2r-5} & \text{if } e_2\leqq 2 \\ 2^{2r-6} & \text{if } e_2=4,\ K\neq \boldsymbol{Q}(\sqrt{-1},\sqrt{-2}). \end{cases}$$

and $g(K)=2^{r-2}$, so

$$r\leqq\begin{cases} 3 & \text{if } e_2\leqq 2 \\ 4 & \text{if } e_2=4 \text{ and } K\neq \boldsymbol{Q}(\sqrt{-1},\sqrt{-2}). \end{cases}$$

As K. Uchida determined all the imaginary abelian fields of type $(2, 2, \cdots, 2)$ with class number one, we consider only the case $h\geqq 2$ i.e. the case $r\geqq m+1$.

First factor $h_1(K)$ is written as $h_1(K)=q(1/w)\prod_{D>0} h_D$, where $q=(E_K : W_K E_{K_0})$, and therefore

$$h(K)\geqq\frac{h_1(K)}{q}=\begin{cases} 2^{\sum_{D<0} r_D-2m+1} & \text{if } K\not\supset \boldsymbol{Q}(\sqrt{-1},\sqrt{-2}) \\ 2^{\sum_{D<0} r_D-2m+2} & \text{if } K\supset \boldsymbol{Q}(\sqrt{-1},\sqrt{-2}). \end{cases}$$

Now, let $P_j$ $(j=1, 2, \cdots, r)$ be all the prime discriminant factors of $D_i$ $(i=1, 2, \cdots, m)$, then

$$K^*=\begin{cases} \boldsymbol{Q}(\sqrt{P_1},\cdots,\sqrt{P_r}) & \text{if } K \text{ is imaginary} \\ \boldsymbol{Q}(\sqrt{P_1},\cdots,\sqrt{P_r})\cap \boldsymbol{R} & \text{if } K \text{ is real ,} \end{cases}$$

and $h(K)\geqq h(K_0)$.

Using above remarks and Lemma 7, we write down the tables of all imaginary abelian fields $K$ of type $(2, 2, \cdots, 2)$ of degree 4 whose absolute class fields $\tilde{K}$ are abelian and $\tilde{K}\neq K$.

**Table I**

| | $D_1$ | $D_2$ | $P_1$ | $P_2$ | $P_3$ | $Q=q$ |
|---|---|---|---|---|---|---|
| 1 | $P_1P_3$ | $P_2P_3$ | $-$ | $-$ | $+$ | 1 |
| 2 | $P_1P_3$ | $P_2$ | $-$ | $-$ | $+$ | 1 |
| 3 | | | $-$ | $-$ | $-$ | 1 |

**Table II**

| | $D_1$ | $D_2$ | $P$ | $Q=q$ |
|---|---|---|---|---|
| 1 | $-4P$ | $-8P$ | $+$ | 1 |
| 2 | $-4P$ | $-8$ | $-$ | 2 |
| 3 | | | $+$ | 1 |
| 4 | $-4$ | $-8P$ | $-$ | 2 |
| 5 | | | $+$ | 1 |

**Table III**

| | $D_1$ | $D_2$ | $P_1$ | $P_2$ | $Q=q$ | discriminants of imaginary quadratic subfields of $K$ | |
|---|---|---|---|---|---|---|---|
| 1 | $-4P_1$ | $-8P_2$ | $-$ | $+$ | 1 | $-8P_2$, | $8P_1P_2$ |
| 2 | | | $+$ | $-$ | 1 | $-4P_2$, | $8P_1P_2$ |
| 3 | $-4P_1P_2$ | $-8P_2$ | $-$ | $+$ | 1 | $-8P_2$, | $8P_1$ |
| 4 | | | $-$ | $-$ | 1 | $-4P_1P_2$, | $8P_1$ |
| 5 | $-4P_1$ | $-8P_1P_2$ | $+$ | $-$ | 1 | $-4P_1$, | $8P_2$ |
| 6 | | | $-$ | $-$ | 1 | $-8P_1P_2$, | $8P_2$ |
| 7 | $-4P_1P_2$ | $-8$ | $-$ | $-$ | 1 | $-4P_1P_2$, | $-8$ |
| 8 | | | $+$ | $-$ | 1 | $-8$ | $8P_1P_2$ |
| 9 | $-4$ | $-8P_1P_2$ | $-$ | $-$ | 1 | $-4$, | $-8P_1P_2$ |
| 10 | | | $+$ | $-$ | 1 | $-4$, | $8P_1P_2$ |

In the tables, $D_i$ $(i=1, 2)$ are the discriminants of quadratic fields such that $K=\boldsymbol{Q}(\sqrt{D_1}, \sqrt{D_2})$, where $K$ is an imaginary bicyclic biquadratic field whose absolute class field is equal to its genus field, and $P_j$ $(j=1, 2, 3)$ and $P$ are the prime discriminant factors of $D_i$. Furthermore, in Table I, at most only one of $P_i$ $(i=1, 2, 3)$ is even, in Table II, $P$ is odd and in Table III, $P_i$ $(i=1, 2)$ are both odd.

THEOREM 3. *Let $K$ be an imaginary abelian field of type* $(2, 2, \cdots, 2)$ *of degree* $2^m$. *Suppose that the absolute class field $\tilde{K}$ of $K$ is equal to the genus field $K^*$, then $m\leqq 3$.*

PROOF. If $\tilde{K}=K^*$, from $g(K)\geqq h'(K)$, it follows that

$$r\leqq\begin{cases} 4 & \text{if } e_2\leqq 2 \\ 5 & \text{if } e_2=4 \end{cases}$$

for $m=4$. There is no imaginary abelian field of type (2, 2, 2, 2) with class number one (see [12]), so if $\tilde{K}=K^*\neq K$ and $m=4$, then it must holds that $r=5$ and $e_2=4$. On the other hand, the fields such that $m=4$, $r=5$ and $e_2=4$ contain at least one imaginary bicyclic biquadratic field which is not in any of Tables I, II and III. It is a contradiction to Lemma 4. Thus, the theorem was proved.

Q. E. D.

Now, it suffices to consider the case $m=3$ for our purpose. The remarks above Table I, Lemma 7 and Satz 29 of [5] give Table IV of all imaginary abelian fields $K$ of type (2, 2, 2) whose absolute class fields $\tilde{K}$ are abelian and $\tilde{K}\neq K$.

**Table IV**

| $D_1$ $D_2$ $D_3$ | $P_1$ $P_2$ | $Q$ | $q$ | discriminants of imaginary quadratic subfields of $K$ |
|---|---|---|---|---|
| $-4P_2$ $-8P_2$ $P_1$ | $-$ $+$ | 2 | 1 | $-4P_2$, $-8P_2$, $8P_1$, $P_1$ |

In Table IV, $P_i$ $(i=1, 2)$ are both odd.

By using Table IV, Lemma 4 for all the quadratic subfields $k$ of $K$, the result of K. Uchida on imaginary abelian fields of type $(2, 2, \cdots, 2)$ with class number one and the list of all the imaginary quadratic fields with class number two, we obtain

THEOREM 4. *The imaginary abelian fields of type* $(2, 2, \cdots, 2)$ *of degree* $\geqq 8$ *whose absolute class fields are abelian over* $\boldsymbol{Q}$ *are as follows*:

$(-4, -8, 5)$, $(-4, -3, 5)$, $(-4, -7, 5)$, $(-4, -7, 13)$, $(-8, -3, 5)$,

$(-8, -7, 5)$, $(-3, -7, 5)$, $(-3, -11, 8)$, $(-3, -11, 17)$, $(-4, -8, -3)$,

$(-4, -8, -11)$, $(-4, -3, -7)$, $(-4, -3, -11)$, $(-4, -3, -19)$,

$(-4, -7, -19)$, $(-8, -3, -7)$, $(-3, -11, -19)$,

$(-4\cdot 5, -8\cdot 5, -3)$, $(-4\cdot 5, -8\cdot 5, -11)$,

*where* $(D_1, D_2, D_3)$ *means the field* $\boldsymbol{Q}(\sqrt{D_1}, \sqrt{D_2}, \sqrt{D_3})$.

## § 3. Some remarks.

The following theorem is an immediate consequence of the estimates of $r$ in the preceding section.

THEOREM 5. *If the absolute class field* $\tilde{K}$ *of an imaginary abelian field* $K$ *is of type* $(2, 2, \cdots, 2)$ *of degree* $\geqq 2^5$, *then* $K$ *is a quadratic field.*

We will give some remarks on imaginary bicyclic biquadratic fields and imaginary quadratic fields.

For bicyclic biquadratic fields, from the table in [3], it can be seen that all the fields in Table I or II are known except for in I 3. The number of such fields are

19 fields in I 1,

85 fields in I 2,

one field in II 1,

2 fields in II 2,

3 fields in II 3,

2 fields in II 4,

2 fields in II 5.

Moreover, it is known that at least 26 fields are in I 3. On the other hand, Theorem 3 of [8] shows that all the fields in Table I, II or III except for the fields in I 3, III 4 or III 7 can be determined.

For quadratic fields, T. Tatuzawa obtained the following estimate.

THEOREM C (T. Tatuzawa [10]). *Let D be the discriminant of an imaginary quadratic field. Then, for a given natural number N, if*

$$-D \geqq 2100 \cdot N^2 \cdot \log^2(13 \cdot N),$$

*then* $h_D > N$ *with at most one exception.*

Theorem C shows that if an imaginary quadratic field $K$ has the absolute class field $\tilde{K}$ being abelian over $\boldsymbol{Q}$, then $[\tilde{K} : \boldsymbol{Q}] \leqq 2^9$, and

$$D_K \leqq 4 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdots 29$$

with at most one exception.

From Theorem 5, above remark and Table III, we can easily obtain the following:

THEOREM 6. *If the absolute class field* $\tilde{K}$ *of an imaginary abelian field* $K$ *is of type* $(2, 2, \cdots, 2)$, *then* $[\tilde{K} : \boldsymbol{Q}] \leqq 2^9$ *with at most one exception and* $K$ *is* $\leqq 8 \cdot 3 \cdot 5 \cdot 7 \cdots 29$ *with at most four exceptions.*

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