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On Sufficient Conditions for a Graph to be Hamiltonian

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Summary.

A graph $G=(V, E)$ is called a complete semi-bigraph and denoted by $K'(l, m)$ if the vertex set can be partitioned into two subsets V_1 $(|V_1|=l)$ and V_2 $(|V_2|=m)$ such that $[u, v]\notin E$ for every $u, v\in V_1$ $(u\neq v)$, and $[v_1, v_2]\in E$ for every $v_1\in V_1$ and $v_2\in V_2$.

THEOREM. *Let $G=(V, E)$ be an undirected 2-connected graph with $n\geqq 3$ vertices and satisfying the following*:

$$[u, v]\notin E \Rightarrow d(u)+d(v)\geqq n-1 .$$

Then G is either hamiltonian or a complete semi-bigraph $K'\left(\frac{n+1}{2}, \frac{n-1}{2}\right)$. In particular, if n is even, then G must be hamiltonian.

1. Introduction.

We use the terminology of [1, 2], unless otherwise stated. Let G be a simple graph of order $n\geqq 3$. We denote by V, E and $d(v)$ respectively the set of all vertices, the set of all edges and the degree of a vertex v in G. G is called hamiltonian if it contains a cycle of length $|V|=n$ where $|V|$ means the order. Various sufficient conditions for a graph to be hamiltonian have been given in terms of the vertex degrees of the graph. We will need the following known results.

ORE'S THEOREM [4]. *If $G=(V, E)$ satisfies the conditions*:

$$[u, v]\notin E \Rightarrow d(u)+d(v)\geqq |n|, \tag{1}$$

then G is hamiltonian.

POSA'S THEOREM [1, 2]. *Let $G=(V, E)$ has $n\geqq 3$ vertices and satisfies the following conditions*:

(i) *for any k $\left(1\leqq k<\frac{n-1}{2}\right)$ it holds*

$$|\{v\in V:\ d(v)\leqq k\}|<k$$

(ii) *for odd n it holds*

$$\left|\left\{v\in V:\ d(v)\leqq\frac{n-1}{2}\right\}\right|\leqq\frac{n-1}{2}.$$

Then G is hamiltonian.

We recall that $G=(V, E)$ is called 2-connected if $G-\{v\}$ is connected for every $v\in V$.

DIRAC'S THEOREM [3]. *If G is a 2-connected graph with n vertices in which the degree of every vertex is at least d $\left(1<d\leqq\frac{n}{2}\right)$, then G contains a cycle with the length not less than $2d$.*

We can not weaken the condition (1) to the following:

$$(*)\qquad [u, v]\notin E\ \Rightarrow\ d(u)+d(v)\geqq|n|-1,$$

because there exists a counter example, that is, a graph constructed by a complete graph K_{n-1} and a vertex v_0 adjacent to only one vertex in K_{n-1}.

It is known that any hamiltonian graph is 2-connected. Hence it seems natural to restrict our consideration to 2-connected graphs only. The purpose of this paper is to investigate 2-connected graphs satisfying (*).

We shall call $G=(V, E)$ a complete semi-bigraph if there is a partition of V into two subsets $V_1=\{u_1, \cdots, u_l\}$ and $V_2=\{v_1, \cdots, v_m\}$ such that $[u_i, u_j]\notin E$ and $[u_i, v_j]\in E$ for any i, j. Such a graph G will be denoted by $K'(l, m)$ or $K'_{l,m}$. Clearly every complete bigraph is a complete semi-bigraph, and $K'(l, m)$ is not hamiltonian if $l>m$.

THEOREM. *Let $G=(V, E)$ be an undirected 2-connected graph with $n\geqq3$ vertices and satisfying the following:*

$$(*)\qquad [u, v]\notin E\ \Rightarrow\ d(u)+d(v)\geqq n-1.$$

Then G is either hamiltonian or a complete semi-bigraph $K'\left(\frac{n+1}{2}, \frac{n-1}{2}\right)$. Hence if n is even, then G must be hamiltonian.

2. Notations and proofs.

Let $G=(V, E)$ be a graph. We denote by $d_G(v)$ the degree of a vertex $v\in V$ if it is necessary to emphasize that v is considered as a vertex in G and by $\Gamma_G(v)$ the set of vertices in G adjacent to v. For a subset $A\subset V$, $\langle A\rangle$ denotes the induced subgraph by A, i.e. the graph with vertex set A whose edges are all those edges in E which connect two vertices in A. For an integer k $\left(1\leqq k<\frac{n-1}{2}\right)$ we decompose V into two subsets A_k and B_k defined by

$$A_k=\{v\in V:\ d(v)\leqq k\},\qquad B_k=\{v\in V:\ d(v)>k\}.$$

LEMMA 1. *Let* $G=(V, E)$ *be a 2-connected graph with* $n \geqq 3$ *vertices and satisfying the condition* (*). *If there exists an integer* k $\left(1 \leqq k < \frac{n-1}{2}\right)$ *such that*

$$|A_k| = k\,, \tag{2}$$

then $\langle A_k \rangle$ *is a complete graph,* $\langle B_k \rangle$ *is hamiltonian and moreover* G *is hamiltonian.*

PROOF. We write simply A, B instead of A_k, B_k respectively. Then $|A|=k$ and $|B|=n-k$. Let $A=\{v_1, v_2, \cdots, v_k\}$.

Step 1. For any two vertices v_i, v_j in A, we have $[v_i, v_j] \in E$ because of $d_G(v_i)+d_G(v_j) \leqq 2k < n-1$ and (*). Hence $\langle A \rangle$ is a complete graph K_k.

Step 2. As $d(v_i) \leqq k$ for $v_i \in A$, we know that

$$d_G(v_i)=k-1 \quad \text{or} \quad d_G(v_i)=k\,. \tag{3}$$

Since $\langle A \rangle$ is K_k, by (3) each vertex $v_i \in A$ is adjacent to at most one vertex in $\langle B \rangle$. Since G is 2-connected, there exist at least two edges joining $\langle A \rangle$ to $\langle B \rangle$ such that all the end vertices are different. We can write them, without loss of generality, by $[v_1, u_1]$, $[v_2, u_2]$ where u_1. $u_2 \in B$. Since any $u \in B$ $(\neq u_1)$ can not be adjacent to v_1 and u_1 can not be adjacent to v_2, we get

$$d_G(u) \geqq n-1-k \quad \text{for } u \in B \tag{4}$$

by our hypothesis (*).

(I) The case of $n \geqq 9$. When $u \in B$ is regarded as a vertex in G, we have by (4),

$$\sum_{u \in B} d_G(u) \geqq (n-k-1)(n-k)\,,$$

and when u is regarded as a vertex in $\langle B \rangle$,

$$\sum_{u \in B} d_{\langle B \rangle}(u) \geqq (n-k-1)(n-k)-k \tag{5}$$

is satisfied, because there are at most k edges joining $\langle A \rangle$ to $\langle B \rangle$.

Suppose that u, v in B $(u \neq v)$ are not adjacent to each other and put

$$d_{\langle B \rangle}(u)=(n-k-1)-\alpha, \quad d_{\langle B \rangle}(v)=(n-k-1)-\beta\,. \tag{6}$$

Now for these u and v let us consider a graph $G'=(B, E')$ satisfying the following conditions:

(i) it is a subgraph of the complete graph $K_{n-k}(B, \tilde{E})$,

(ii) $[u, v] \notin E'$,

(iii) E' consists of all edges of $\tilde{E}$ except α edges at u and β edges at v.

If we compare $|E'|$ with the number of edges in $\langle B \rangle$ and take account of (5) and (6), it follows that

$$2\{(\alpha-1)+(\beta-1)+1\} \leqq k\,. \tag{7}$$

On the other hand because of $n \geqq 9$ and $k < \dfrac{n-1}{2}$, we get

$$(n-k-2)-\frac{k+2}{2}=n-\frac{3}{2}k-3>n-\frac{3}{2}\cdot\frac{n-1}{2}-3=\frac{n-9}{4}\geqq 0$$

which implies

$$n-k-2\geqq\frac{k+2}{2}. \tag{8}$$

Thus using (6), (7) and (8) we obtain

$$d_{\langle B\rangle}(u)+d_{\langle B\rangle}(v)=(n-k)+(n-k-2)-(\alpha+\beta)\geqq n-k.$$

Moreover, since $n\geqq 9$ and $k<\frac{n-1}{2}$, we have $|B|\geqq 5>3$. Therefore we can apply Ore's theorem to the graph $\langle B\rangle$ and know that $\langle B\rangle$ is hamiltonian.

(II) The case of $n\leqq 8$. Decompose the set B into

$$B_1=\{v\in B: \Gamma_G(v)\cap A\neq\emptyset\} \quad\text{and}\quad B_2=\{v\in B: v\notin B_1\}.$$

Then $|B_1|\geqq 2$ follows from that G is 2-connected. Every $v\in B_2$ is adjacent to all vertices in $B-\{v\}$, because $d_G(v)\geqq n-k-1$ by (4) and moreover v is not adjacent to all vertices in A by the definition. If $|B_1|\leqq|B_2|$, $\langle B\rangle$ is hamiltonian.

We shall show that $n\geqq 6$. Assume $n<6$. Then we get $k=2$ and $|B|=3$, because $k\geqq 2$ and $k=|A|<|B|=n-k<4$. So by $|B_2|\geqq|B|-|A|=1$, there is a vertex $v\in B_2$ which must satisfy $2=k<d_G(v)=d_{\langle B\rangle}(v)=|B|-1=2$, that is a contradiction. Therefore $n\geqq 6$, and then we know $6\leqq n\leqq 8$.

Hence $k=2$ or 3, which follows from $2\leqq|A|<\frac{n-1}{2}$. If $k=2$, we have $|B_1|=|A|=2\leqq|B_2|$, and so $\langle B\rangle$ is hamiltonian. If $k=3$, $|B|$ is 4 or 5 since $3=|A|<|B|=n-3\leqq 5$. Assume $|B|=4$, then $B_2\neq\emptyset$ and for a vertex $v\in B_2$ we have $d_G(v)=n-k-1\geqq k+1=4$ which contradicts $d_G(v)\leqq|B|-1=3$. Assume $|B|=5$. There are two possibilities: $|B_1|=2$, $|B_2|=3$ and $|B_1|=3$, $|B_2|=2$. In the first case, $\langle B\rangle$ is hamiltonian because $|B_1|\leqq|B_2|$. The second implies that

$$\sum_{v\in B_1}d_G(v)-\sum_{v\in B_2}d_G(v)-|A|\geqq 1$$

because of $d_G(v)\geqq 4$ for every $v\in B_1$ and $d_G(v)=4$ for every $v\in B_2$. Hence there exists at least one edge joining two vertices in B_1, which implies that $\langle B\rangle$ is hamiltonian.

Step 3. We shall show that G is hamiltonian. Let us recall that a path which contains all vertices by strictly one time is called a hamiltonian path and, specially, if the end vertices coincide, it is called a hamiltonian cycle. For any 2 pathes $P_1=[v_1, \cdots, v_k]$ and $P_2=[w_1, \cdots, w_m]$ in G, denote by P_1+P_2 the path $[v_1, \cdots, v_{k-1}, w_1, \cdots w_m]$ provided that $v_k=w_1$.

Similarly to the proof of step 2, let us decompose $B=\{v\in X: d(v)>k\}$ into B_1 and B_2, and write $B_1=\{u_1, u_2, \cdots, u_p\}$, $B_2=\{w_1, w_2, \cdots, w_q\}$, respectively. Then $p+q=n-k$ and G consists of the complete graph $\langle A\rangle$, the hamiltonian graph $\langle B\rangle$ and the set C of edges joining $\langle A\rangle$ and $\langle B\rangle$ where C contains two edges such that $[u_1, v_1]$, $[u_2, v_2]$ ($v_1, \neq v_2\in A$ and $u_1, \neq u_2\in B_1$). Moreover B

consists of B_1 in which every vertex is adjacent to A and B_2 in which every vertex is not adjacent to A but adjacent to all vertices in B.

(I) The case of $p \leqq q$. Then there is a hamiltonian path $P_{\langle B\rangle}$ of B such that u_1 is the starting vertex and u_2 is the end one: $[u_1, w_1, u_3, w_2, u_4, w_3, \cdots u_p, w_{p-1}, w_p, w_{p+1}, \cdots w_q, u_2]$. Since $\langle A\rangle$ is a complete graph, there is a hamiltonian path $P_{\langle A\rangle}$ such that v_2 is the starting vertex and v_1 is the end one. Hence we get a hamiltonian cycle of G: $[v_1, u_1]+P_{\langle B\rangle}+[u_2, v_2]+P_{\langle A\rangle}$.

(II) The case of $p>q$. Let $C_{\langle B\rangle}$ be a hamiltonian cycle in $\langle B\rangle$ whose existence is assured by step 2. Then since a complete semi-bigraph $K'_{p,q}$ $(p>q)$ is not hamiltonian, $C_{\langle B\rangle}$ must contain at least one edge joining two vertices in B_1 which are denoted by u_i, u_j for some $i, \neq j$ $(1\leqq i, j \leqq p)$. Moreover we write by v_α, v_β the adjacent vertices in A to u_i, u_j respectively. We may assume that $v_\alpha \neq v_\beta$, and that there is a hamiltonian path $P_{\langle A\rangle}$ in $\langle A\rangle$ having v_β as the starting vertex and v_α as the end one. Now if we put $C_{\langle B\rangle}-[u_i, u_j]=p_{\langle B\rangle}$, then $[v_\alpha, u_i]+P_{\langle B\rangle}+[u_j, v_\beta]+P_{\langle A\rangle}$ is a hamiltonian cycle in G. Thus, Lemma 1 is proved.

LEMMA 2. *Let $G=(V, E)$ be a 2-connected graph with n vertices. If G satisfies the condition* (*), *then for* k $\left(1\leqq k<\frac{n-1}{2}\right)$

$$|A_k| \leqq k .$$

PROOF. Let put $m=\min_{v\in A_k} d(v)$. Clearly $m \leqq k$. We will first prove in the case $m \leqq k-1$. Let pick up a vertex $v_0 \in A_k$ such that $d(v_0) \leqq k-1$. Then for each $v\in A_k$ $(v\neq v_0)$ we have $[v_0, v]\in E$ by (*), which implies $|A_k| \leqq |\Gamma_G(v_0)|+1 \leqq k$. In the other case $m=k$, similarly to the above we obtain $|A_k| \leqq k+1$. Suppose $|A_k|=k+1$. Then since $d(v)=k$ for every $v\in A_k$ and $\langle A_k\rangle$ is complete, any vertex in A_k can not be adjacent to the vertices in $V-A_k$. This is a contradiction to the 2-connectedness of G, so we get $|A_k| \leqq k$.

REMARK. Lemma 2 is true if G is connected.

PROOF OF THEOREM.

Case 1: n is odd. If k satisfies $1\leqq k<\frac{n-1}{2}$, we have already shown in Lemma 2 that $|A_k| \leqq k$. Moreover if there exists a k such that $|A_k|=k$, then G is hamiltonian by Lemma 1. Therefore it is sufficient to prove the theorem in the case of $|A_k|<k$ for every k $\left(1\leqq k<\frac{n-1}{2}\right)$.

Now put $m=\frac{n-1}{2}$ and $A_m=\{v\in X\colon d(v)\leqq m\}$. If $|A_m|\leqq m$, using Posa's theorem we can conclude that G is hamiltonian. This leaves only the case $|A_m|>m$ to consider. The inequality $|A_m|>m$ implies $A_{m-1}=\emptyset$, for if $A_{m-1}\neq\emptyset$, there is a vertex $v\in X$ satisfying $d(v)\leqq m-1$, which must be adjacent by (*) to all vertices in A_m, so $m-1\geqq d(v)\geqq |A_m|-1>m-1$, which is a contradiction.

Therefore we get

$$d(v) \geqq m = \frac{n-1}{2} \quad \text{for every vertex } v \in X. \tag{9}$$

By Dirac's theorem there is a cycle in G with length at least $2m = n-1$. Let C be one of cycles in G with the largest length. Then $|C| \geqq n-1$. If $|C| = n$, C is a hamiltonian cycle.

Now we shall prove that if $|C| = n-1$, G is $K'\left(\frac{n+1}{2}, \frac{n-1}{2}\right)$. Let denote C by $[v_1, v_2, \cdots, v_n]$ with $v_1 = v_n$ and v_0 the only one vertex not contained in C. Since the length of C is maximum, C does not contain two vertices with succeeding number (with respect to mod $n-1$) which are adjacent to v_0. Hence by (9), we may consider that v_0 is adjacent to all v_i with odd i and not adjacent to all v_i with even i. Let put

$$V_1 = \{v_i : i \text{ is even}\} \cup \{v_0\} \quad \text{and} \quad V_2 = \{v_i : i \text{ is odd}\}.$$

Then $|V_1| = m+1$ and $|V_2| = m$. Now for any pair of vertices v_i, v_j $(0 \neq i < j)$ in V_1, we have $[v_i, v_j] \notin E$ because if $[v_i, v_j] \in E$, G has a hamiltonian cycle: $[v_0, v_{j+1}, v_{j+2}, \cdots, v_m, v_1, v_2, \cdots, v_i, v_j, v_{j-1}, \cdots, v_{i+1}, v_0]$, which is a contradiction to the majority of the length of C. On the other hand, for any $v \in V_1$ $d(v) \geqq m$ implies that v must be adjacent to all vertices in V_2. Thus G is a complete semi-bigraph $K'\left(\frac{n+1}{2}, \frac{n-1}{2}\right)$.

Case 2: n is even. Similarly to the case of odd, using Posa's theorem, we can conclude that G is hamiltonian.

From the theorem for even n, we have the following.

COROLLARY. *Let $G = (V, E)$ be an undirected 2-connected graph with even $n \geqq 3$ vertices and satisfying the following:*

$$[u, v] \notin E \Rightarrow d(u) + d(v) \geqq n-1.$$

Then G is hamiltonian.

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