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α-Submanifold in a Locally Conformal Kählerian Manifold

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§ 0. Introduction.

A $2n$-dimensional Hermitian manifold is said to be locally conformal kählerian (l.c.k. manifold) if its metric is locally conformal to kählerian metrics. In 1976, I. Vaisman [1] introduced an l.c.k. manifold and showed that the Hopf manifold is a typical example of l.c.k. manifolds. Since the Hopf manifold is an l.c.k. manifold which can not admit a kählerian metric, l.c.k. manifolds seem to constitute an interesting class among all complex manifolds.

On the other hand, let N^{2n+1} be a Sasakian manifold with contact form η, $\tilde{D}$ a distribution defined by $\eta(X)=0$, then we know the following results:

(1) The dimension of any integrable distribution $D\subset\tilde{D}$ is at most n. ([3])

(2) Let $D\subset\tilde{D}$ be an n-dimensional integrable distribution of a Sasakian space form and N_0 a maximal integral manifold of D. Then N_0 is totally geodesic provided that N_0 is compact minimal and the square length of the second fundamental form is bounded by a certain number. ([4])

Since an l.c.k. manifold resembles to a contact manifold in a sense, it is a problem to study in such manifolds the character of a distribution corresponding to $\tilde{D}$.

In § 1, we shall give some basic properties in an l.c.k. structure (J, g, α). In § 2, we shall derive an integrability condition for a distribution D defined by $\alpha(X)=0$, $\alpha(JX)=0$, and show that the dimension of any integrable distribution in D is at most $n-1$. If there exists such a distribution of dimension $n-1$, we shall call it α-distribution and its maximal integral manifold α-submanifold. In § 3, we shall show that if M^{2n} is a conformally flat l.c.k. manifold satisfying $\nabla\alpha=0$, then the similar theorems to (1) and (2) stated above hold on α-submanifold. In § 4, we shall show that there exists α-submanifold in the Hopf manifold with $|h|^2=\dfrac{(n-1)^2}{2n-3}$, where h is the second fundamental form of the immersion.

Throughout the paper, manifolds, vector fields and tensor fields are assumed to be C^∞.

§ 1. L.c.k. manifold.

Let M^{2n} be a $2n$-dimensional manifold. We denote by $\mathfrak{X}(M)$ the Lie algebra of all vector fields on M^{2n}. A complex structure J on M^{2n} is by definition a tensor field of type (1, 1) on M^{2n} satisfying the following two conditions:

$$J^2X=-X, \tag{1.1}$$

$$[X, Y]+J[JX, Y]+J[X, JY]-[JX, JY]=0, \tag{1.2}$$

where $X, Y \in \mathfrak{X}(M)$.

A $2n$-dimensional manifold with a complex structure is called a complex manifold. If a complex manifold M^{2n} admits a Riemannian metric g which satisfies the condition

$$g(JX, JY)=g(X, Y) \tag{1.3}$$

for $X, Y \in \mathfrak{X}(M)$, then we call $\{M^{2n}, J, g\}$ a Hermitian manifold.

Putting

$$J^b(X, Y)=g(X, JY)$$

for $X, Y \in \mathfrak{X}(M)$, we have

$$J^b(X, Y)=-J^b(Y, X).$$

This means that J^b is a differential 2-form which will be called the fundamental 2-form of $\{M^{2n}, J, g\}$. A Hermitian manifold $\{M^{2n}, J, g\}$ is called a kählerian manifold if $\nabla J=0$, where ∇ is the Riemannian connection. A Hermitian manifold $\{M^{2n}, J, g\}$ is called an l.c.k. manifold if the metric is locally conformal to kählerian metrics. More precisely, for each point of M there exists a neighbourhood U and a local function ρ on U such that $g^*=e^{-2\rho}g$ is a kählerian metric.

If $g^{*\prime}=e^{-2\rho'}g$ is a kählerian metric in another neighbourhood U', then $g^{*\prime}$ is homothetic to g^* on $U\cap U'$. As ρ-ρ' is constant on $U\cap U'$, the collection $\{d\rho\}$ defines a closed differential 1-form which will be denoted by α.

Now we shall investigate some relations between the Riemannian connection ∇ with respect to g and ∇^* with respect to $g^*=e^{-2\rho}g$.

Taking account of the definition of ∇_X:

$$\begin{aligned}2g(\nabla_X Y, Z)&=Xg(Y, Z)+Yg(X, Z)-Zg(X, Y)\\&\quad+g([X, Y], Z)+g([Z, X], Y)+g(X, [Z, Y])\end{aligned}$$

for $X, Y, Z \in \mathfrak{X}(M)$, and the similar expression of ∇^*_X, we have

$$g(\nabla^*_X Y, Z)=g(\nabla_X Y, Z)-\alpha(X)g(Y, Z)-\alpha(Y)g(X, Z)+\alpha(Z)g(X, Y). \tag{1.4}$$

On the other hand, $\nabla^* J^{*b}=0$ holds for g^*, where

$$J^{*b}(X, Y)=g^*(X, JY)=e^{-2\rho}J^b(X, Y).$$

From (1.4) and

$$(\nabla^*_X J^{*b})(Y, Z)=e^{-2\rho}\{(\nabla^*_X J^b)(Y, Z)-2\alpha(X)J^b(Y, Z)\},$$

$\nabla^* J^{*b}=0$ is equivalent to the following equations:

(1.5) $(\nabla_X J^b)(Y, Z)=\alpha(Z)J^b(X, Y)-\alpha(Y)J^b(X, Z)+\alpha(JY)g(X, Z)-\alpha(JZ)g(X, Y)$.
Hence we know that if M^{2n} is an l.c.k. manifold, then there exist a tensor field J of type (1, 1), Riemannian metric g and a closed differential 1-form α satisfying (1.1)$\sim$(1.3) and (1.5).

Conversely it is easy to prove that a manifold which admits (J, g, α) satisfying (1.1)$\sim$(1.3) and (1.5) and $d\alpha=0$ is an l.c.k. manifold. We call α the fundamental 1-form of the l.c.k. manifold.

We shall mean by $\{M^{2n}, J, g, \alpha\}$ a $2n$-dimensional l.c.k. manifold with structure $\{J, g, \alpha\}$.

Let us introduce a 1-form β by

$$\beta(X)=\alpha(JX) \qquad \text{for } X\in\mathfrak{X}(M),$$

then (1.5) can be written as

$$(1.6)\qquad (\nabla_X J^b)(Y, Z)=\alpha(Z)g(X, JY)-\alpha(Y)g(X, JZ)$$
$$+\beta(Z)g(X, Y)-\beta(Y)g(X, Z).$$

Defining $\alpha^{\#}, \beta^{\#}\in\mathfrak{X}(M)$ by

$$\alpha(X)=g(X, \alpha^{\#}), \qquad \beta(X)=g(X, \beta^{\#}),$$

we easily obtain the following five formulas:

$$(1.7)\qquad \beta^{\#}=-J\alpha^{\#}, \qquad \beta(JX)=-\alpha(X).$$

$$(1.8)\qquad \alpha(\beta^{\#})=\beta(\alpha^{\#})=0.$$

$$(1.9)\qquad (\nabla_X\beta)(Y)=|\alpha^{\#}|^2 J^b(X, Y)+\alpha(X)\beta(Y)-\alpha(Y)\beta(X)+(\nabla_X\alpha)(JY).$$

$$(1.10)\qquad \nabla_{\alpha^{\#}}J^b=0.$$

(1.11) If $\nabla_X\alpha=0$, then

$$(\nabla_X\beta)(Y)=|\alpha^{\#}|^2 J^b(X, Y)+\alpha(X)\beta(Y)-\alpha(Y)\beta(X).$$

§2. Submanifold in an l.c.k. manifold satisfying $\nabla\alpha=0$.

Let $\{M^{2n}, J, G, \alpha\}$ be an l.c.k. manifold satisfying $\bar{\nabla}\alpha=0$, where $\bar{\nabla}$ is the Riemannian connection with respect to G. We may assume that $|\alpha|=1$ without loss of generality.

Now let us consider the distribution given by

$$D=\{X\in\mathfrak{X}(M)|\quad \alpha(X)=0, \quad \beta(X)=0\}.$$

PROPOSITION 2.1. *If D_0 is an involutive distribution such that $D_0\subset D$, then*

D_0 is anti-invariant, i.e, $X\in D_0 \Rightarrow JX \notin D_0$.

PROOF. Since D_0 is involutive, we have

$$\beta([X, Y])=0 \qquad \text{for} \quad X, Y\in D_0 .$$

On the other hand,

$$\beta([X, Y])=2g(Y, JX)$$

follows from (1.11), and we obtain

$$g(Y, JX)=0 \qquad \text{for} \quad X, Y\in D_0 ,$$

which means that D_0 is anti-invariant. Q.E.D.

We are interested in the distribution D_0 stated in proposition 2.1 and investigate its integral manifold.

PROPOSITION 2.2. *The dimension of D_0 is at most $n-1$.*

PROOF. Suppose that the dimension of D_0 is r and M_0 is a maximal integral manifold of D_0. Let $E_1, \cdots, E_r$ be the local orthonormal basis of $\mathfrak{X}(M_0)$. Since D_0 is anti-invariant, we have

$$G(E_i, JE_j)=0 , \qquad i, j=1, \cdots, r .$$

Since $E_i\in D_0$, we have $\alpha(E_i)=0$, $\beta(E_i)=0$, i.e, $G(\alpha^\#, E_i)=0$, $G(\beta^\#, E_i)=0$. On the other hand, from (1.6) and (1.7), we have

$$G(\alpha^\#, \beta^\#)=0 , \qquad G(JE_i, \alpha^\#)=0 , \qquad G(JE_i, \beta^\#)=0 .$$

Thus, $E_1, \cdots, E_r, JE_1, \cdots, JE_r, \alpha^\#, \beta^\#$ constitute a local orthonormal basis of $\mathfrak{X}(M)$, and hence $r\leqq n-1$. Q.E.D.

From now on, we shall only consider the distribution D_0 with dimension $n-1$. Let $i: M_0 \to M$ be an integral manifold of D_0 and for the sake of simplicity we call M_0 *α-submanifold*.

Let $E_1, \cdots, E_{n-1}$ be a local orthonormal basis of $\mathfrak{X}(M_0)$ and $\xi_1, \cdots, \xi_{n+1}$ be a local orthonormal normal vector fields.

By the above argument, we can put

$$\xi_i=JE_i , \qquad i=1, \cdots, n-1 ,$$

$$\xi_n=\alpha^\# , \qquad \xi_{n+1}=\beta^\# .$$

We denote the induced metric on M_0 by g, that is,

$$g(X, Y)i=G(X, Y) \qquad \text{for} \quad X, Y\in D_0 .$$

Let ∇ and $\overline{\nabla}$ be Riemannian connections with respect to g and G, and D be the normal connection of M_0. Then the Gauss and Weingarten formulas are as follows:

$$\text{(2.1)} \qquad \overline{\nabla}_X Y=\nabla_X Y+h(X, Y) \qquad \text{for} \quad X, Y\in D_0 ,$$

(2.2) $$\bar{\nabla}_X\xi=-H_\xi(X)+D_X\xi \qquad \text{for } X\in D_0,\ \xi\in D_0^\perp ,$$

where h is the second fundamental form of M_0 and H_ξ is the second fundamental form with respect to a normal vector field ξ.

If we denote H_i instead of H_{ξ_i}, $(i=1, \cdots, n+1)$, it holds that

$$h(X, Y)=\sum_{i=1}^{n+1} g(H_i(X), Y)\xi_i .$$

Now we have following propositions.

PROPOSITION 2.3. (a) *The second fundamental form with respect to* $\alpha^\#$, $\beta^\#$ *are identically zero, that is*

(2.3) $$H_n=H_{n-1}=0 .$$

(b) *If* $X\in\mathfrak{X}(M_0)$, *then*

(2.4) $$D_X\alpha^\#=0 , \qquad D_X\beta^\#=JX .$$

PROOF. From (2.2) and $\bar{\nabla}\alpha=0$, we have

$$0=\bar{\nabla}_X\xi_n=-H_n(X)+D_X\xi_n , \qquad \text{for} \quad X\in D_0 ,$$

which implies that $H_n(X)=0$, $D_X\xi_n=0$ for any $X\in D_0$. Similarly we have from (2.2)

$$\bar{\nabla}_X\beta^\#=-H_{n+1}(X)+D_X\xi_{n+1} .$$

On the other hand, we can obtain from (1.10),

$$G(\bar{\nabla}_X\beta^\#, Y)=G(Y, JX) ,$$

which means $$\bar{\nabla}_X\beta^\#=JX .$$ Q.E.D.

PROPOSITION 2.4. *For any* X, $Y\in D_0$, *we have*

(2.5) $$H_{JX}(Y)=H_{JY}(X) .$$

PROOF. Since X, $Y\in D_0$, we have

$$(\bar{\nabla}_X J)(Y)=G(X, Y)\beta^\# .$$

Substituting the above equation into $\bar{\nabla}_X(JY)=(\bar{\nabla}_X J)(Y)+J(\bar{\nabla}_X Y)$, we obtain

$$\bar{\nabla}_X(JY)=G(X, Y)\beta^\#+J(\nabla_X Y)+J(h(X, Y)) .$$

On the other hand, we have

$$\bar{\nabla}_X(JY)=-H_{JY}(X)+D_X(JY) ,$$

because JX is a normal vector field for any $X\in D_0$.

Therefore by comparing the tangential parts of above two equations, we obtain (2.5). Q.E.D.

PROPOSITION 2.5 $$\operatorname{tr}\,(\sum_j H_j^2)^2=\sum_{i,j}(\operatorname{tr}\,H_iH_j)^2$$

PROOF. We put

$$H_{j,kl}=g(H_j(E_k),\ E_l)\,,$$

where $E_1, \cdots, E_{n-1}$ is a local orthonormal basis of $\mathfrak{X}(M_0)$.

By definition of H_j and from (2.3) and (2.5), we have

$$H_{j,kl}=H_{j,lk}\,, \qquad H_{n,kl}=H_{n+1,kl}=0\,, \qquad H_{i,jk}=H_{j,ik}\,.$$

Hence it holds that

$$\begin{aligned}\operatorname{tr}(\sum_j H_j^2)^2&=\operatorname{tr}(\sum H_{i,jk}H_{i,kl})^2\\ &=\sum H_{i,jk}H_{i,kl}H_{m,ln}H_{m,nj}\\ &=\sum_{i,j}(\operatorname{tr} H_iH_j)^2\,,\end{aligned}$$

where the sums are taken over all repeated indices. Q.E.D.

§ 3. α-submanifold in a conformally flat l.c.k. manifold.

Let M^{2n} be an l.c.k. manifold satisfying $\nabla\alpha=0$, then at each point of M^{2n} there exists a neighbourhood where $G^*=e^{-2\rho}G$ is a kählerian metric for a suitable local function ρ and $\alpha=d\rho$ is the fundamental 1-form. Denoting the curvature tensor with respect to G and G^* by $\bar{R}$ and R^* respectively, we have by straightforward calculation

$$\begin{aligned}(3.1)\quad G(R^*_{XY}Z,\ W)=&G(\bar{R}_{XY}Z,\ W)+\alpha(X)\alpha(W)G(Y,\ Z)+\alpha(Y)\alpha(Z)G(X,\ W)\\ &-\alpha(Y)\alpha(W)G(X,\ Z)-\alpha(X)\alpha(Z)G(Y,\ W)-G(X,\ W)G(Y,\ Z)\\ &+G(Y,\ W)G(X,\ Z)\,.\end{aligned}$$

From now on, we assume that M^{2n} is conformally flat, and look for the exact form of curvature tensor in terms of G and α.

As conformally flat kählerian metric is flat, G^* satisfies $R^*_{XY}Z=0$. Hence from (3.1), we have

$$\begin{aligned}(3.2)\quad G(\bar{R}_{XY}Z,\ W)=&\alpha(X)\alpha(Z)G(Y,\ W)-\alpha(Y)\alpha(Z)G(X,\ W)+\alpha(Y)\alpha(W)G(X,\ Z)\\ &-\alpha(X)\alpha(W)G(Y,\ Z)+G(X,\ W)G(Y,\ Z)-G(Y,\ W)G(X,\ Z)\,.\end{aligned}$$

The curvature tensor of M_0 with respect to g will be denoted by R, and the equation of Gauss is given by

$$(3.3)\quad g(R_{XY}Z,\ W)=G(\bar{R}_{XY}Z,\ W)+G(h(X,\ W),\ h(Y,\ Z))-G(h(X,\ Z),\ h(Y,\ W)).$$

From (3.2) and (3.3), we obtain

$$\begin{aligned}(3.4)\quad g(R_{XY}Z,\ W)=&g(X,\ W)g(Y,\ Z)-g(Y,\ W)g(X,\ Z)\\ &+\sum_i\{g(H_i(X),\ W)g(H_i(Y),\ Z)-g(H_i(Y),\ W)g(H_i(X),\ Z)\\ &\text{for}\quad X,\ Y,\ Z,\ W\in\mathfrak{X}(M_0)\,.\end{aligned}$$

The Ricci tensor $Ric(X, Y)$ and the scalar curvature k of M_0 are as follows:

(3.5) $$Ric(X, Y)=(n-2)g(X, Y)+\sum_i (\mathrm{tr}\, H_i)g(H_i(X), Y)-\sum_i g(H_i(X), H_i(Y)).$$

(3.6) $$k=(n-2)(n-1)+\sum_i (\mathrm{tr}\, H_i)^2-|h|^2.$$

The sectional curvature $\rho_{M_0}(X, Y)$ of M_0 determined by an orthonormal pair X and Y of D_0 is given by

(3.7) $$\rho_{M_0}(X, Y)=1-\frac{1}{|X|^2|Y|^2}\sum_i \{g(H_i(X), Y)^2-g(H_i(X), X)g(H_i(Y), Y)\}.$$

Thus, we obtain the following two propositions immediately.

PROPOSITION 3.1. *Let M^{2n} be conformally flat. If M_0 is minimal, then*

$$k\leqq(n-2)(n-1),$$

with equality if and only if M_0 is totally geodesic.

PROPOSITION 3.2. *Let M^{2n} be conformally flat. If M_0 is totally geodesic, then the sectional curvature of M_0 is identically* 1.

THEOREM 3.3. *Let M_0 be an α-submanifold in a conformally flat l. c. k. manifold $\{M^{2n}, J, G, \alpha\}$ satisfying $\overline{\nabla}\alpha=0$ and $|\alpha|=1$. If M_0 is minimal, then the followings are equivalent to one another.*

(a) *M_0 is totally geodesic.*

(b) $Ric=(n-2)g$.

(c) $k=(n-2)(n-1)$.

(d) $\rho_{M_0}(X, Y)=1$ *for any orthogonal pair X and Y in D_0.*

PROOF. It is immediate from (3.5), (3.6) and (3.7) that (a) $\Rightarrow$ (b), (c), (d). (b) $\Rightarrow$ (a), (c) $\Rightarrow$ (a) and (b) $\Rightarrow$ (c) are obvious.

Hence it is sufficient to show only (d) $\Rightarrow$ (b). Let X_1 be an arbitrary unit vector field and choose $X_2, \cdots, X_{n-1}$ such that $X_1, X_2, \cdots, X_{n-1}$ is an orthonormal basis of $\mathfrak{X}(M_0)$. Then from (d), we have

$$Ric(X_1, X_1)=n-2.$$

This completes the proof. Q.E.D.

Now for the second fundamental form h, we shall define the covariant derivative $'\nabla_X h$ by

(3.8) $$('\nabla_X h)(Y, Z)=D_X(h(Y, Z))-h(\nabla_X Y, Z)-h(Y, \nabla_X Z).$$

Then the Codazzi equation of M_0 is given by

$$(\bar{R}_{XY}Z)^N=('\nabla_X h)(Y, Z)-('\nabla_Y h)(X, Z),$$

and the left hand side is zero because of (3.2). Hence we have

(3.9) $$('\nabla_X h)(Y, Z)=('\nabla_Y h)(X, Z).$$

Next, denoting $R^{\perp}$ the curvature tensor of normal connection D, the Ricci equation of M_0 is as follows.

$$G(\bar{R}_{XY}\xi, \eta)=G(R^{\perp}_{XY}\xi, \eta)-g([H_{\xi}, H_{\eta}]X, Y),$$

where $X, Y\in D_0$, $\xi, \eta\in D_0^{\perp}$. Hence from (3.2), we obtain

$$G(R^{\perp}_{XY}\xi, \eta)=g([H_{\xi}, H_{\eta}]X, Y). \tag{3.10}$$

LEMMA 3.4. *Let M_0 be a minimal α-submanifold in a conformally flat l. c. k. manifold M. Then we have*

$$\frac{1}{2}\Delta|h|^2=(n-1)|h|^2-\sum_{i,j}\operatorname{tr}(H_iH_j-H_jH_i)^2-\sum_{i,j}(\operatorname{tr}H_iH_j)^2+|'\nabla h|^2$$
$$=(n-1)|h|^2-3\sum_{i,j}(\operatorname{tr}H_iH_j)^2+2\sum_{i,j}\operatorname{tr}(H_iH_j)^2+|'\nabla h|^2.$$

PROOF. Since M_0 is minimal, we have from (3.8) and (3.9)

$$\frac{1}{2}\Delta|h|^2=\sum\{R_{ij}H_{k,jl}H_{k,li}-R_{ijkl}H_{m,il}H_{m,jk}+R^{\perp}_{ijkl}H_{k,im}H_{l,jm}\}+|'\nabla h|^2,$$

where R_{ijkl}, R_{ij}, $R^{\perp}_{ijkl}$ are the components of R, Ric and $R^{\perp}$.

Hence from (3.4), (3.5) and (3.6), we have

$$\frac{1}{2}\Delta|h|^2=(n-1)|h|^2-2\sum H_{i,jk}H_{i,kl}H_{m,ln}H_{m,nj}-\sum H_{i,jk}H_{i,lm}H_{n,jk}H_{n,lm}$$
$$+2\sum H_{i,jk}H_{i,lm}H_{m,jl}H_{n,km}+|'\nabla h|^2$$
$$=(n-1)|h|^2+\sum_{i,j}\operatorname{tr}(H_iH_j-H_jH_i)^2-\sum_{i,j}(\operatorname{tr}H_iH_j)^2+|'\nabla h|^2.$$

The second equation follows from the above equation and proposition 2.5.

Q.E.D.

We have known the following lemma.

LEMMA 3.5. [5] *Let A and B be symmetric $n\times n$-matrices. Then we have $-\operatorname{tr}(AB-BA)^2\leqq 2\operatorname{tr}A^2\operatorname{tr}B^2$, and the equality holds for non zero matrices A and B if and only if A and B can be transformed simultaneously by an orthogonal matrix into scalar multiples of $\tilde{A}$ and $\tilde{B}$ respectively, where*

$$\tilde{A}=\left(\begin{array}{cc|c} 0 & 1 & \\ 1 & 0 & 0 \\ \hline \multicolumn{2}{c}{0} & 0 \end{array}\right) \qquad \tilde{B}=\left(\begin{array}{cc|c} 1 & 0 & \\ 0 & -1 & 0 \\ \hline \multicolumn{2}{c}{0} & 0 \end{array}\right)$$

Moreover, if A_1, A_2, A_3 are $n\times n$ symmetric matrices, and if

$$-\operatorname{tr}(A_iA_j-A_jA_i)^2\leqq 2\operatorname{tr}A_i^2A_j^2, \qquad 1\leqq i, j\leqq 3, \qquad i\neq j,$$

then at least one of the matrices A_i must be zero.

LEMMA 3.6. *For arbitrary real numbers $a_1, \cdots, a_n$, the following equality holds.*

$$-2\sum_{i\neq j}a_ia_j-\sum a_i^2=\frac{1}{n}\sum_{i<j}(a_i-a_j)^2-\left(2-\frac{1}{n}\right)(\sum a_i)^2 .$$

THEOREM 3.7. *Let M_0 be an α-submanifold in a conformally flat l. c. k. manifold M^{2n} satisfying $\tilde{\nabla}\alpha=0$, $|\alpha|=1$ and $n>1$. If M_0 is compact minimal and satisfies $|h|^2<\dfrac{(n-1)^2}{2n-3}$, then M_0 is totally geodesic.*

PROOF. Let $(H_{i,jk})$ be the local expression of H_i with respect to a local orthonormal basis $E_1, \cdots, E_{n-1}$ of $\mathfrak{X}(M_0)$. Then $(\mathrm{tr}\,(H_iH_j))$ is a symmetric matrix and independent of the choice of the basis. Hence we may assume that $\mathrm{tr}\,(H_iH_j)=0$ if $i\neq j$.

From lemma 3.4, 3.5 and 3.6, we have

$$\begin{aligned}\frac{1}{2}\Delta|h|^2&=(n-1)|h|^2+\sum_{i,j}\mathrm{tr}\,(H_iH_j-H_jH_i)^2-\sum_i(\mathrm{tr}\,H_i^2)^2+|'\nabla h|^2\\&\geqq(n-1)|h|^2-2\sum_{i\neq j}(\mathrm{tr}\,H_i^2)(\mathrm{tr}\,H_j^2)-\sum_i(\mathrm{tr}\,H_i^2)^2\\&=(n-1)|h|^2+\frac{1}{n-1}\sum_{i<j}(\mathrm{tr}\,H_i^2-\mathrm{tr}\,H_j^2)^2-\left(2-\frac{1}{n-1}\right)\sum_i(\mathrm{tr}\,H_i^2)^2\\&=\frac{2n-3}{n-1}|h|^2\left(\frac{(n-1)^2}{2n-3}-|h|^2\right).\end{aligned}$$

Therefore, if $|h|^2<\dfrac{(n-1)^2}{2n-3}$, then $\Delta|h|^2=0$ follows and $|h|=0$ by a well-known theorem of E. Hopf. Q.E.D.

THOREM 3.8. *Let M_0 be an α-submanifold in a conformally flat l. c. k. manifold M^{2n} satisfying $\tilde{\nabla}\alpha=0$, $|\alpha|=1$ and $n>1$. If M_0 is compact minimal and of constant curvature c, then either M_0 is totally geodesic or $c\leqq\dfrac{4-n}{2}$, where the equality holds if and only if $'\nabla\, h=0$.*

PROOF. Since M_0 is of constant curvature c, we have $k=(n-2)(n-1)c$ and from equation (3.6)

$$0\leqq|h|^2=(n-1)(n-2)(1-c)\,,$$

which means that if $n=2$ or $c=1$ then M_0 is totally geodesic.

On the other hand, equations (3.4) and (3.5) become

$$\sum\mathrm{tr}\,(H_iH_j)^2-\sum_{i,j}(\mathrm{tr}\,H_iH_j)^2=(c-1)|h|^2\,,$$

$$\sum_{i,j}(\mathrm{tr}\,H_iH_j)^2=|h|^2\,.$$

Then from lemma 3.4, we have

$$(n-4+2c)|h|^2+|'\nabla h|^2=0\,,$$

which means that $|h|=0$ or $n-4+2c\leqq0$. This completes the proof. Q.E.D.

§ 4. Minimal α-submanifold with $|\boldsymbol{h}|^2=\frac{(n-1)^2}{2n-3}$.

Let M^{2n} be a conformally flat l.c.k. manifold satisfying $\bar{\nabla}\alpha=0$ and $|\alpha|=1$, and M_0 be an $(n-1)$-dimensional α-submanifold with natural induced metric.

We shall make use of the following convention on the ranges of indices:

$$1\leqq a,\ b\leqq 2n\,,\qquad 1\leqq i,\ j\leqq n-1\,,\qquad n\leqq \alpha,\ \beta\leqq 2n\,.$$

With respect to the frame field of M^{2n} choosen in § 2, let $\omega_1, \cdots, \omega_{2n}$ be the field of dual frames. Then we have $\omega_{2n-1}=\alpha$ and $\omega_{2n}=\beta$, and the structure equations of M^{2n} are as follows:

$$d\omega_a=\sum \omega_{ab}\wedge\omega_b\,,\qquad \omega_{ab}+\omega_{ba}=0\,, \tag{4.1}$$

$$d\omega_{ab}=\sum \omega_{ac}\wedge\omega_{cb}+\bar{\Omega}_{ab}\,, \tag{4.2}$$

$$\bar{\Omega}_{ab}=-\frac{1}{2}\sum \bar{R}_{abcd}\omega_c\wedge\omega_d=\omega_a\wedge\omega_b-\alpha_a\alpha_c\omega_c\wedge\omega_b+\alpha_b\alpha_c\omega_c\wedge\omega_a\,,$$

where α_a are the components of α.

Restricting these forms to M_0, we have the structure equations of the immersion.

$$\omega_\alpha=0\,, \tag{4.3}$$

$$\omega_{\alpha i}=\sum h_{\alpha ij}\omega_j\,,\qquad h_{\alpha ij}=h_{\alpha ji}\,, \tag{4.4}$$

$$d\omega_i=\sum \omega_{ij}\wedge\omega_j\,,\qquad \omega_{ij}+\omega_{ji}=0\,, \tag{4.5}$$

$$d\omega_{ij}=\sum \omega_{ik}\wedge\omega_{kj}+\Omega_{ij}\,,\qquad \Omega_{ij}=-\frac{1}{2}\sum R_{ijkl}\omega_k\wedge\omega_l\,, \tag{4.6}$$

$$R_{ijkl}=\delta_{il}\delta_{jk}-\delta_{ik}\delta_{jl}+\sum(h_{\alpha il}h_{\alpha jk}-h_{\alpha ik}h_{\alpha jl})\,. \tag{4.7}$$

Let us define $h_{\alpha ijk}$ by

$$\sum h_{\alpha ijk}\omega_k=dh_{\alpha ij}+h_{\alpha ik}\omega_{kj}+h_{\alpha kj}\omega_{ki}+h_{\beta ij}\omega_{\beta\alpha}\,. \tag{4.8}$$

Then from (4.2), (4.3) and (4.4), we have

$$h_{\alpha ijk}=h_{\alpha ikj}\,. \tag{4.9}$$

Let H_α denote the matrix formed from $h_{\alpha ij}$, then by lemma 3.4 we have

$$\frac{1}{2}\Delta|h|^2=|'\nabla h|^2+\sum_{\alpha\neq\beta}\mathrm{tr}\,(H_\alpha H_\beta-H_\beta H_\alpha)-\sum_\alpha(\mathrm{tr}\,H_\alpha^2)^2\,. \tag{4.10}$$

In the following, we consider the case $|h|^2=\frac{(n-1)^2}{2n-3}$.

Applying the inequality in the lemma 3.5 to (4.10), we have

$$\frac{1}{2}\Delta|h|^2\geqq(n-1)|h|^2-2\sum_{\alpha\neq\beta}(\mathrm{tr}\,H_\alpha^2)(\mathrm{tr}\,H_\beta^2)-\sum_\alpha(\mathrm{tr}\,H_\alpha^2)^2+|'\nabla h|^2 \tag{4.11}$$

$$=(n-1)|h|^2+\frac{1}{n-1}\sum_{\alpha<\beta}(\operatorname{tr} H_\alpha^2-\operatorname{tr} H_\beta^2)^2-\left(2-\frac{1}{n-1}\right)|h|^4+|'\nabla h|^2$$

Since $|h|^2=\dfrac{(n-1)^2}{2n-3}$, we have the following from (4.11):

$$|'\nabla h|^2=0\,, \tag{4.12}$$

$$\operatorname{tr} H_\alpha^2-\operatorname{tr} H_\beta^2=0\,, \tag{4.13}$$

$$\operatorname{tr}(H_\alpha H_\beta-H_\beta H_\alpha)^2=-2\operatorname{tr} H_\alpha^2 \operatorname{tr} H_\beta^2\,,\qquad \alpha\neq\beta\,. \tag{4.14}$$

By lemma 3.5 and (4.13), (4.14), we have $n\leqq 3$. If $n=2$, then M_0 is 1-dimensional and has scalar curvature -1. Thus n must be 3. By lemma 3.5, we have

$$H_3=x\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\qquad H_4=y\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\qquad H_5=H_6=(0)\,,$$

where x and y are constants. From (4.11) and $|h|^2=\dfrac{4}{3}$, we have $x^2=y^2=\dfrac{1}{3}$. Thus we may assume that $x=-y=\dfrac{1}{\sqrt{3}}$.

From (4.4), we have

$$\omega_{13}=h_{31j}\omega_j=x\omega_2\,,\qquad \omega_{23}=h_{32j}\omega_j=x\omega_1\,,$$
$$\omega_{14}=h_{41j}\omega_j=y\omega_1\,,\qquad \omega_{24}=h_{42j}\omega_j=-y\omega_2\,,$$
$$\omega_{15}=\omega_{25}=\omega_{16}=\omega_{26}=0\,.$$

By (4.4) and (4.11), we have

$$h_{\alpha kj}\omega_{ki}+h_{\alpha ik}\omega_{kj}+h_{\beta ij}\omega_{\beta\alpha}=0\,.$$

Putting $\alpha=3$ and $i=j=1$, this becomes

$$0=h_{3k1}\omega_{k1}+h_{31k}\omega_{k1}+h_{\beta 11}\omega_{\beta 3}=x\omega_{21}+x\omega_{21}+y\omega_{43}\,.$$

That is $\omega_{43}=2\omega_{21}$. In the same way, we have $\omega_{45}=\omega_{25}=\omega_{46}=\omega_{56}=0$.

In summary, we have

THEOREM 4.1. *Let M_0 be a compact minimal α-submanifold of a conformally flat l. c. k. manifold M^{2n} satisfying $\bar\nabla\alpha=0$, $\alpha=1$, and $|h|^2=\dfrac{(n-1)^2}{2n-3}$. Then $n=3$, hence $|h|^2=\dfrac{4}{3}$ and M_0 is of constant curvature $\dfrac{1}{3}$. With respect to an adapted orthogonal frame field $E_1,\cdots,E_6$, the connection form (ω_{ab}) restricted to M_0 is given by*

$$\begin{pmatrix}
0 & \omega_{12} & x\omega_2 & -x\omega_1 & 0 & 0\\
-\omega_{12} & 0 & x\omega_1 & x\omega_2 & 0 & 0\\
-x\omega_2 & -x\omega_1 & 0 & 2\omega_{12} & 0 & 0\\
x\omega_1 & -x\omega_2 & -2\omega_{12} & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & \omega_{56}\\
0 & 0 & 0 & 0 & -\omega_{56} & 0
\end{pmatrix},\qquad x=\frac{1}{\sqrt{3}}$$

Therefore such a submanifold is locally unique.

From now on, we consider the case that the ambient space is the Hopf manifold. Let us consider $(2n-1)$-dimensional sphere

$$S^{2n-1}=\{z\in\mathbf{C}^n \mid |z|=1\}$$

and the circle $S^1\left(\frac{1}{\pi}\right)$ defined by $\left\{\frac{1}{\pi}e^{i\theta}\right\}$

Then one of the l.c.k. structures of $S^1\left(\frac{1}{\pi}\right)\times S^{2n-1}$ is given as follows.

$$ds^2=\frac{1}{\pi^2}d\theta^2+\sum_{k=1}^{n}dz^k d\bar{z}^k ,$$

$$\Omega=\frac{2}{\pi}i\sum_{k=1}^{n}\bar{z}^k d\theta\wedge dz^k-i\sum_{k=1}^{n}dz^k\wedge d\bar{z}^k ,$$

$$\alpha=-\frac{1}{\pi}d\theta .$$

Clearly $S^1\left(\frac{1}{\pi}\right)\times S^{2n-1}$ satisfies the condition of theorem 4.1, and M_0 is immersed is S^{2n-1}.

On the other hand, we know the following results. ([7])

THEOREM. *Let M be an n-dimensional compact orientable Riemannian manifold which is minimally immersed in an $(n+p)$-dimensional sphere of constant curvature c. If the immersion is full and the sectional curvature of M is not smaller than $\frac{nc}{2(n+1)}$, then M is a sphere of constant curvature c or M is a Veronese manifold.*

Thus we have

COROLLARY 4.2. *If M_0 is a compact orientable minimal α-submanifold in $S^1\left(\frac{1}{\pi}\right)\times S^{2n-1}$ satisfying $|h|^2=\frac{(n-1)^2}{2n-3}$, then $n=3$ and M_0 is a Veronese manifold.*

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