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Isometric Immersions of Conformally Flat Riemannian Spaces with Negative Sectional Curvature

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Introduction.

One example of conformally flat Riemannian space is given by a warped product of 1-dimensional space M^1 and the space $M^n(c)$ of constant sectional curvature c ([6]). Especially, taking the function $f(t)=(t^2+1)/2$ for $t\in R^1$, the warped product space $M^n=R^1\times_f M^{n-1}(-1)$ is conformally flat and of non-constant negative sectional curvature.

In this paper we investigate the umbilic space of an isometric immersion of a conformally flat space M^n into the Euclidean space R^{n+p}. The umbilic spaces of an immersion were first studied by B. O'Neill [3] when M^n is of positive constant curvature, and M. Sekizawa [4] generalized the results due to O'Neill to the case of conformally flat spaces with positive curvature. We make use of the metric of Lorentz signature and the theory of flat bilinear forms introduced by J. D. Moore [1], [2]. Then Theorem 6 gives a characterization of an umbilic space of an isometric immersion of a conformally flat space into the Euclidean space. To determine the maximum dimension of the umbilic spaces, we apply one of the results of Moore [1] about the dimension of the nullity space of a flat bilinear form. Our main result (Theorem 8) is that the dimension of the umbilic space for such an immersion of the space of negative sectional curvature is not greater than 1.

1. Conformally flat spaces.

Let M^n be a Riemannian space which admits an isometric immersion into the Euclidean space R^{n+p}. Notations are as follows: For $m\in M^n$, $T_m(M)$ (or $T_m(M)^\perp$); the tangent (or normal) space of M^n at m, $\langle\,,\rangle$; the Euclidean metric of R^{n+p} and the induced metric on M^n, $\alpha: T_m(M)\times T_m(M)\to T_m(M)^\perp$; the second fundamental form of the immersion, ξ_λ ($\lambda=1,\cdots,p$); the orthonormal vectors of $T_m(M)^\perp$, A_λ: $T_m(M)\to T_m(M)$; the second fundamental tensor of type (1, 1) such that

$$\langle\alpha(x, y), \xi_\lambda\rangle=\langle A_\lambda(x), y\rangle \qquad \text{for} \quad x, y\in T_m(M).$$

Let $R(x, y)$ be the curvature tensor of M^n for $x, y\in T_m(M)$, Q: $T_m(M)\to$

$T_m(M)$ be the Ricci tensor of type (1, 1), and R be the scalar curvature of M^n. Then we have the Gauss equation

(1) $$-R(x, y)z=\sum_\lambda \langle A_\lambda x, z\rangle A_\lambda y-\sum_\lambda \langle A_\lambda y, z\rangle A_\lambda x,$$

$$-\langle R(x, y)z, w\rangle=\langle \alpha(x, z), \alpha(y, w)\rangle-\langle \alpha(y, z), \alpha(x, w)\rangle.$$

Suppose that M^n is conformally flat. Then the curvature tensor of M^n satisfies

(2) $$-\langle R(x, y)z, w\rangle=\phi(x, z)\langle y, w\rangle-\phi(y, z)\langle x, w\rangle$$
$$+\phi(y, w)\langle x, z\rangle-\phi(x, w)\langle y, z\rangle,$$

where

$$\phi(x, y)=\frac{1}{n-2}\{\langle Qx, y\rangle-R\langle x, y\rangle/2(n-1)\}.$$

We take a vector space $W_m=T_m(M)^\perp\oplus R^2$ in which we define the inner product by

$$\langle\!\langle(\xi, a, b), (\xi', a', b')\rangle\!\rangle=\langle \xi, \xi'\rangle+ab'+a'b$$

for $\xi, \xi'\in T_m(M)$ and $a, b, a', b'\in R$. Then W_m is a $(p+2)$-dimensional metric vector space of Lorentz signature. The W_m-valued symmetric bilinear form β on $T_m(M)$ is defined by

$$\beta(x, y)=(\alpha(x, y), \langle x, y\rangle, -\phi(x, y)).$$

From (1) and (2), we have

$$\langle \alpha(x, z), \alpha(y, w)\rangle-\langle x, z\rangle\phi\langle y, w\rangle-\langle y, w\rangle\phi(x, z)$$
$$=\langle \alpha(x, w), \alpha(y, z)\rangle-\langle x, w\rangle\phi(y, z)-\langle y, z\rangle\phi(x, w),$$

which implies

(3) $$\langle\!\langle\beta(x, z), \beta(y, w)\rangle\!\rangle-\langle\!\langle\beta(x, w), \beta(y, z)\rangle\!\rangle=0.$$

A bilinear form β satisfying (3) for any vectors x, y, z and w in $T_m(M)$ is called flat with respect to the metric $\langle\!\langle\ ,\ \rangle\!\rangle$ (see [1]). Then taking into consideration the more general work of J. D. Moore [1], we have the following results: Let $S(\beta)$ be the linear subspace of W_m spanned by the vectors $\beta(x, y)$ as x and y range over $T_m(M)$, and put $S(\beta)_0=\{\xi\in S(\beta)\,;\ \langle\!\langle\xi, S(\beta)\rangle\!\rangle=0\}$. Taking a linear complement subspace W_2 of $S(\beta)_0$ in $S(\beta)$, we let W_1 be the orthogonal complement to W_2 in W_m. Then we have $S(\beta)_0=S(\beta)\cap W_1$ and $W_m=W_1\oplus W_2$. We define the symmetric bilinear forms β_1 and β_2 on $T_m(M)$ as the W_1- and W_2-components of β. Then Corollary 3 of [1] says that

(4) (i) *the restriction of* $\langle\!\langle\ ,\ \rangle\!\rangle$ *to* W_1 *and* W_2 *are non-degenerate,*

(ii) β_1 *is null, that is,* $\langle\!\langle\beta_1(x, y), \beta_1(x, y)\rangle\!\rangle=0$ *for any* $x, y\in T_m(M)$,

(iii) $S(\beta_2)=W_2$ *and* $\dim N(\beta_2)\geqq n-\dim W_2$, *where*

$$N(\beta_2)=\{x\in T_m(M)\,;\ \beta_2(x, T_m(M))=0\}.$$

We will cite this result as (4) of J. D. Moore in the following.

LEMMA 1. *Let the codimension* $p \leqq n-1$. *Then the space* $N(\beta_2)=(0)$ *if and only if* $S(\beta)_0=(0)$.

PROOF. First we assume $S(\beta)_0=(0)$. Let $x \in N(\beta_2)$, then we have $\beta(x, y)=\beta_1(x, y) \in S(\beta)_0$ for any $y \in T_m(M)$. Hence $\langle x, y\rangle=0$ holds, from which $x=0$ follows. Conversely we suppose $S(\beta)_0$ is not (0). Since W_m has the Lorentz signature, the dimension of the subspace consisting of null vectors is one, and hence we have $\dim S(\beta)_0=1$. From the definition of $S(\beta)_0$, the metric $\ll , \gg_{S(\beta)}$ restricted to $S(\beta)$ is degenerate. On the other hand $\ll , \gg_{W_m}$ is non-degenerate and hence $S(\beta)$ is a proper subspace of W_m. Thus we have

$$\dim W_m \geqq \dim S(\beta)+1=\dim W_2+2\,,$$

from which $\dim W_2 \leqq p$ follows. Then by virtue of (4) of J. D. Moore, we have

$$\dim N(\beta_2) \geqq n-\dim W_2 \geqq n-p \geqq 1$$

by the assumption $p \leqq n-1$. Therefore $N(\beta_2) \neq (0)$ is proved.

2. Umbilic spaces $N(\beta_2)$ at m.

In this section we assume $N(\beta_2) \neq (0)$.

LEMMA 2. *Let* $x \in N(\beta_2)$, *then* x *satisfies*

$$|\alpha(x, y)|^2=2\langle x, y\rangle \phi(x, y)$$

for any $y \in T_m(M)$, *where* $|\ |$ *denotes the norm of* $\langle\ ,\ \rangle$.

PROOF. Let $x \in N(\beta_2)$. Since $\beta_2(x, y)=0$ for any $y \in T_m(M)$, we have $\beta_1(x, y)=\beta(x, y) \in S(\beta)_0$. Hence

$$\ll \beta(x, y), \beta(x, y) \gg=0$$

holds, and the lemma follows from the definition of the inner product $\ll , \gg$.

LEMMA 3. *Let* $x \in N(\beta_2)$, *then* x *satisfies for any* $y \in T_m(M)$

$$\alpha(x, y)=\langle x, y\rangle \alpha(x_0, x_0)\,,$$

$$\phi(x, y)=\langle x, y\rangle \phi(x_0, x_0)\,,$$

where x_0 *is any unit vector in* $N(\beta_2)$.

PROOF. Taking any unit vector x_0 in $N(\beta_2)$, we see that $\beta(x_0, x_0)$ is a non-zero vector in $S(\beta)_0$. As we mentioned in the proof of Lemma 1, $S(\beta)_0$ is necessarily one dimensional. Thus for $x \in N(\beta_2)$, we have $\beta(x, y)=\beta_1(x, y)=k\, \beta(x_0, x_0)$ for some k. It follows that

$$(\alpha(x, y), \langle x, y\rangle, -\phi(x, y))=k(\alpha(x_0, x_0), 1, -\phi(x_0, x_0))\,,$$

and the lemma is easily obtained.

REMARK. The unit vector x_0 in Lemma 3 is taken arbitrarily in $N(\beta_2)$. However, in the following we fix such a vector x_0 once, and use it all over the section.

THEOREM 4. *We have* $\ll\beta(x, x), \beta(x, x)\gg\geqq 0$ *for any* $x\in T_m(M)$. *Hence it holds*

$$|\alpha(x, x)|^2\geqq 2|x|^2\psi(x, x) \tag{5}$$

for any $x\in T_m(M)$.

PROOF. Let $N(\beta_2)^{\perp}=\{z\in T_m(M);\ \langle z, N(\beta_2)\rangle=0\}$. Any non-zero vector $x\in T_m(M)$ can be decomposed as $x=z+w$, $z\in N(\beta_2)$ and $w\in N(\beta_2)^{\perp}$. From Lemma 3, they satisfy

$$\alpha(x, x)=\alpha(z, z)+\alpha(w, w)\,,$$

$$\psi(x, x)=\psi(z, z)+\psi(w, w)\,.$$

Therefore using Lemmas 2, 3 and the flatness of β, we obtain

$$\begin{aligned}\langle\alpha(x, x), \alpha(x_0, x_0)\rangle&=|z|^2|\alpha(x_0, x_0)|^2+\langle\alpha(x_0, w), \alpha(x_0, w)\rangle\\&\quad+|w|^2\psi(x_0, x_0)+\psi(w, w)\\&=(2|z|^2+|w|^2)\psi(x_0, x_0)+\psi(x, x)-|z|^2\psi(x_0, x_0)\\&=|x|^2\psi(x_0, x_0)+\psi(x, x)\,.\end{aligned}$$

Making use of the Schwarz inequality, we get

$$|\alpha(x, x)|^2|\alpha(x_0, x_0)|^2\geqq(|x|^2\psi(x_0, x_0)+\psi(x, x))^2\,,$$

and hence we have

$$|x|^4\psi(x_0, x_0)^2+2(|x|^2\psi(x, x)-|\alpha(x, x)|^2)\psi(x_0, x_0)+\psi(x, x)^2\leqq 0\,.$$

Since this holds for some $\psi(x_0, x_0)$ which is non-negative by Lemma 2, we have to get

$$(|x|^2\psi(x, x)-|\alpha(x, x)|^2)^2-|x|^4\psi(x, x)^2\geqq 0\,,$$

and

$$|x|^2\psi(x, x)-|\alpha(x, x)|^2\leqq 0\,.$$

Then if $\alpha(x, x)=0$, we have $\psi(x, x)\leqq 0$ from which (5) holds. If $\alpha(x, x)\neq 0$, then the above

$$|\alpha(x, x)|^2(|\alpha(x, x)|^2-2|x|^2\psi(x, x))\geqq 0$$

means (5) again, and the theorem is proved.

In the next place, we consider the case in which the equality holds in Theorem 4, that is, we take a vector x satisfying

(6) $$|\alpha(x, x)|^2=2|x|^2\psi(x, x).$$

Then the above Schwarzian inequality becomes as

$$(|x|^2\psi(x_0, x_0)-\psi(x, x))^2\leqq 0,$$

and hence we have

(7) $$\psi(x, x)=|x|^2\psi(x_0, x_0).$$

As the vectors $\alpha(x, x)$ and $\alpha(x_0, x_0)$ are linearly dependent in this case, we put $\alpha(x, x)=k\alpha(x_0, x_0)$ for some k. Then it holds

$$\langle\alpha(x, x), \alpha(x_0, x_0)\rangle=2k\psi(x_0, x_0),$$

and by virtue of flatness of β, we have

$$\begin{aligned}\langle\alpha(x, x), \alpha(x_0, x_0)\rangle&=\langle\alpha(x, x_0), \alpha(x, x_0)\rangle-2\langle x, x_0\rangle\psi(x, x_0)\\&\quad+|x|^2\psi(x_0, x_0)+\psi(x, x)\\&=2|x|^2\psi(x_0, x_0)\end{aligned}$$

where we used Lemma 2 and (7). Therefore we have

$$\psi(x_0, x_0)(k-|x|^2)=0.$$

If $\psi(x_0, x_0)=0$, then from $|\alpha(x_0, x_0)|^2=2\psi(x_0, x_0)=0$, $\alpha(x_0, x_0)$ must be zero vector, and hence $\alpha(x, x)=0$. If $\psi(x_0, x_0)\neq 0$, then $k=|x|^2$ from which $\alpha(x, x)=|x|^2\alpha(x_0, x_0)$ is obtained. In any case, we have proved the next

LEMMA 5. *If $x\in T_m(M)$ satisfies $|\alpha(x, x)|^2=2|x|^2\psi(x, x)$, then*

$$\alpha(x, x)=|x|^2\alpha(x_0, x_0)$$

holds, where x_0 is a unit vector in $N(\beta_2)$.

Making use of Lemma 5, we show

THEOREM 6. *If $x\in T_m(M)$ satisfies $|\alpha(x, x)|^2=2|x|^2\psi(x, x)$, then x belongs to $N(\beta_2)$. Therefore it holds that*

$$N(\beta_2)=\{x\in T_m(M);\ |\alpha(x, x)|^2=2|x|^2\psi(x, x)\}.$$

PROOF. Since the vector in $N(\beta_2)$ satisfies the equation (6), it is sufficient for the proof of the last assertion of the theorem to show the first statement. Let x be a vector with (6), then it satisfies $\alpha(x, x)=|x|^2\alpha(x_0, x_0)$ and $\psi(x, x)=|x|^2\psi(x_0, x_0)$. Then for any vector $y\in T_m(M)$, we have

$$\begin{aligned}\langle\!\langle\beta(x, y), \beta(x, y)\rangle\!\rangle&=\langle\!\langle\beta(x, x), \beta(y, y)\rangle\!\rangle\\&=\langle\alpha(x, x), \alpha(y, y)\rangle-|x^2|\psi(y, y)-|y|^2\psi(x, x)\\&=|x|^2\{\langle\alpha(x_0, y), \alpha(x_0, y)\rangle-2\langle x_0, y\rangle\psi(x_0, y)\\&\quad+\psi(y, y)+|y|^2\psi(x_0, x_0)\}-|x|^2\psi(y, y)-|y|^2\psi(x, x)\end{aligned}$$

$$=|x|^2\{\langle x_0, y\rangle^2 2\phi(x_0, x_0)-2\langle x_0, y\rangle\langle x_0, y\rangle\phi(x_0, x_0)\}$$
$$=0$$

and hence x satisfies

$$|\alpha(x, y)|^2=2\langle x, y\rangle\phi(x, y) \tag{8}$$

for any $y\in T_m(M)$. Since $\langle\alpha(y, z), \alpha(x_0, x_0)\rangle=\phi(y, z)+\langle y, z\rangle\phi(x_0, x_0)$ is valid, it follows

$$0\geqq\langle\alpha(x, y), \alpha(x_0, x_0)\rangle^2-|\alpha(x, y)|^2|\alpha(x_0, x_0)|^2$$
$$=(\phi(x, y)-\langle x, y\rangle\phi(x_0, x_0))^2,$$

and therefore

$$\phi(x, y)=\langle x, y\rangle\phi(x_0, x_0) \tag{9}$$

is obtained for any y. Moreover the vectors $\alpha(x, y)$ and $\alpha(x_0, x_0)$ are linearly dependent. Hence by the similar way as the proof of Lemma 5, we get

$$\alpha(x, y)=\langle x, y\rangle\alpha(x_0, x_0). \tag{10}$$

Then we can easily deduce for any y, z and $w\in T_m(M)$

$$\begin{aligned}\langle\!\langle\beta(x, y), \beta(z, w)\rangle\!\rangle&=\langle\alpha(x, y), \alpha(z, w)\rangle-\langle x, y\rangle\phi(z, w)-\langle z, w\rangle\phi(x, y)\\ &=\langle x, y\rangle(\langle z, w\rangle\phi(x_0, x_0)+\phi(z, w))\\ &\quad-\langle x, y\rangle\phi(z, w)-\langle z, w\rangle\langle x, y\rangle\phi(x_0, x_0)\\ &=0.\end{aligned}$$

This means $x\in N(\beta_2)$, and the theorem is proved.

COROLLARY 7. *We have*

$$N(\beta_2)=\{x\in T_m(M)\,;\ \alpha(x, x)=|x|^2\alpha(x_0, x_0)\},$$

where x_0 is a unit vector in $N(\beta_2)$.

PROOF. Let x satisfy $\alpha(x, x)=|x|^2\alpha(x_0, x_0)$. By the same way as we get the equation (9), we have $\phi(x, x)=|x|^2\phi(x_0, x_0)$ easily. Hence it follows $|\alpha(x, x)|^2=2|x|^2\phi(x, x)$.

Summarizing the above theorems, we have proved the following relations about $N(\beta_2)$: Let x_0 be any unit vector in $N(\beta_2)$, then

$$\begin{aligned}N(\beta_2)&=\{x\in T_m(M)\,;\ \langle\!\langle\beta(x, y), \beta(z, w)\rangle\!\rangle=0 \text{ for any } y, z \text{ and } w\}\\ &=\{x\in T_m(M)\,;\ \langle\!\langle\beta(x, y), \beta(x, y)\rangle\!\rangle=0 \text{ for any } y\}\\ &=\{x\in T_m(M)\,;\ \langle\!\langle\beta(x, x), \beta(x, x)\rangle\!\rangle=0\}\\ &=\{x\in T_m(M)\,;\ \alpha(x, y)=\langle x, y\rangle\alpha(x_0, x_0) \text{ for any } y\}\end{aligned}$$

$$=\{x\in T_m(M)\,;\ \alpha(x, x)=|x|^2\alpha(x_0, x_0)\}.$$

REMARK. For an isometric immersion $M^n \rightarrow R^{n+p}$, we define the umbilic subspace at $m\in M^n$ by the set

$$U(m)=\{x\in T_m(M)\,;\ \alpha(x, y)=\langle x, y\rangle\ \xi, \text{ for any } y\in T_m(M)\}$$

where ξ is a certain normal vector. If $U(m)\neq(0)$, then taking a unit vector x_0 in $U(m)$, we have $\xi=\alpha(x_0, x_0)$. It follows then

$$U(m)=\{x\in T_m(M)\,;\ \alpha(x, y)=\langle x, y\rangle\alpha(x_0, x_0) \text{ for any } y\}$$

which shows that $N(\beta_2)$ coincides with an umbilic space at m.

3. Negative sectional curvature.

Let $M^n \rightarrow R^{n+p}$ be an isometric immersion of a conformally flat Riemannian space into the Euclidean space. Then the sectional curvature $k(x, y)$ of the plane spanned by the orthonormal vectors $x, y\in T_m(M)$ is given by

$$\begin{aligned} k(x, y)&=-\langle R(x, y)x, y\rangle \\ &=\langle\alpha(x, x), \alpha(y, y)\rangle-\langle\alpha(x, y), \alpha(x, y)\rangle \\ &=\phi(x, x)+\phi(y, y) \end{aligned}$$

making use of the flatness of β. We shall show

THEOREM 8. *Let M^n be conformally flat and of negative sectional curvature. If M^n is isometrically immersed into R^{n+p} ($p\leqq n-1$), then we have* $\dim N(\beta_2)\leqq 1$ *and* $p\geqq n-2$.

PROOF. Suppose that $\dim N(\beta_2)\geqq 2$, then we can take the orthonormal vectors x and y in $N(\beta_2)$. By virtue of Lemma 2 the sectional curvature $k(x, y)$ would satisfy

$$0>k(x, y)=\phi(x, x)+\phi(y, y)\geqq 0$$

which is a contradiction. Therefore we have $\dim N(\beta_2)\leqq 1$. Next from $W\supset S(\beta)=S(\beta)_0\oplus W_2$, the inequality $\dim W_2\leqq\dim W-\dim S(\beta)_0$ holds. If $\dim N(\beta_2)=1$, then $\dim S(\beta)_0=1$ by Lemma 1. Thus it follows $\dim W_2\leqq\dim W-1=p+1$. Therefore by virtue of (4) of J. D. Moore, we have

$$1=\dim N(\beta)_2\geqq n-(p+1)$$

and hence $p\geqq n-2$ holds. If $\dim N(\beta_2)=0$, then from $\dim W_2\leqq\dim W=p+2$, we get

$$0\geqq n-\dim W_2\geqq n-(p+2).$$

Thus $p\geqq n-2$ is obtained again.

THEOREM 9. *Let M^n be a space of constant sectional curvature $c<0$ which* *is immersed isometrically into* R^{n+p} $(p\leqq n-1)$. *Then we have* $N(\beta_2)=(0)$ *and* $p=n-1$.

PROOF. As M^n is of constant sectional curvature, the bilinear form ϕ has a constant eigenvalue λ, hence we can take the orthonormal basis $\{e_i\}$ in $T_m(M)$ such that

$$\phi(e_i, e_j)=\lambda\delta_{ij}$$

hold for $i, j=1, \cdots, n$. Then

$$\beta(e_i, e_j)=(\alpha(e_i, e_j), \delta_{ij}, -\lambda\delta_{ij})$$

are valid. If $N(\beta_2)\neq(0)$, then we may assume e_1 is a vector in $N(\beta_2)$ and it follows that

$$0\leqq|\alpha(e_1, e_1)|^2=2\phi(e_1, e_1)=2\lambda$$

and

$$0>k(e_1, e_2)=\phi(e_1, e_1)+\phi(e_2, e_2)=2\lambda.$$

This is impossible. Therefore $N(\beta_2)$ must be (0). Next we have

$$\begin{aligned} S(\beta)&=\{(\alpha_{ij}, 1, -\lambda), (\alpha_{ij}, 0, 0)\,;\ 1\leqq i\leqq j\leqq n\} \\ &=\{(\alpha_{ij}, 0, 0), (\alpha_{ii}, 0, 0)+(0, 1, -\lambda)\,;\ 1\leqq i\leqq j\leqq n\} \\ &\subset\{(\alpha_{ij}, 0, 0)\,;\ 1\leqq i\leqq j\leqq n\}\oplus\{(0, 1, -\lambda)\} \end{aligned}$$

where we put $\alpha(e_i, e_j)=\alpha_{ij}$. Then the space $\{(\alpha_{ij}, 0, 0)\,;\ 1\leqq i\leqq j\leqq n\}$ is at most p-dimensional, we see that

$$\dim S(\beta)\leqq p+1\leqq n.$$

On the other hand, from $N(\beta_2)=(0)$, $S(\beta)_0=(0)$ follows. Then $S(\beta)=W_2$ and

$$0\geqq n-\dim S(\beta)\geqq(p+1)-\dim S(\beta)$$

from which we conclude easily that

$$\dim S(\beta)=n=p+1.$$

REMARK. We state in more detail about the isometric immersion $M^n\rightarrow R^{n+p}(p\leqq n-1)$, where M^n is conformally flat and of negative sectional curvature.

CASE 1. $N(\beta_2)\neq(0)$. Then $S(\beta)_0$ is 1-dimensional, and

$$W_m\supsetneqq S(\beta)=S(\beta)_0\oplus W_2\supsetneqq W_2.$$

Thus $\dim W_2\leqq p$, and $1\geqq n-\dim W_2\geqq n-p$. Consequently, we have $p=n-1$ and

$$\dim S(\beta)=n,\ \dim W_2=n-1,\ \dim W_m=n+1.$$

CASE 2. $N(\beta_2)=(0)$. Then it follows $S(\beta)_0=(0)$ and $S(\beta)=W_2$. Hence we have $0\geqq n-\dim S(\beta)$ and $p\geqq n-2$.

(2-1) Let $p=n-2$. Then we have

$$\dim S(\beta)=\dim W_2=\dim W_m=\mathrm{n}\,.$$

(2-2) Let $p=n-1$. Then $n\leqq\dim S(\beta)\leqq n+1$.
(i) If $\dim S(\beta)=n+1$, then we have

$$\dim S(\beta)=\dim W_2=\dim W_m=n+1\,.$$

(ii) If $\dim S(\beta)=n$, then we have

$$\dim S(\beta)=\dim W_2=n,\ \dim W_m=n+1\,.$$

Theorem 9 treats of the case of (2-2-ii). Another remark is that the above cases are only necessary conditions of our immersions.

We show the following theorem in the case one.

THEOREM 10. *Let M^n be conformally flat and of negative sectional curvature which is immersed isometrically in R^{2n-1}. If $N(\beta_2)\neq(0)$ at each point on a domain D of M^n, then there exists a unit vector field in $N(\beta_2)$ which is differentiable on D.*

PROOF. Since $N(\beta_2)\neq(0)$, we have $\dim N(\beta_2)=1$, and we take a continuous unit vector field x on D which is in $N(\beta_2)$ at each point of D. Then x satisfies $\psi(x, y)=\langle x, y\rangle\psi(x, x)$ for any $y\in T_m(M)$ and hence $\psi(x, x)=\lambda_1$ is an eigenvalue of ψ and x is the corresponding eigenvector of ψ. From $2\lambda_1=|\alpha(x, x)|^2$, λ_1 is a non-negative function of M^n. Let $\lambda_2, \cdots, \lambda_n$ and $y_2, \cdots, y_n$ be the eigenvalues and corresponding unit eigenvectors of ψ. Then for any j, $2\leqq j\leqq n$,

$$k(x, y_j)=\psi(x, x)+\psi(y_j, y_j)=\lambda_1+\lambda_j$$

is negative, from which we have

$$\lambda_j<-\lambda_1\leqq 0\,.$$

It follows that $\lambda_2, \cdots, \lambda_n$ can not coincide with λ_1 and the multiplicity of λ_1 is necessarily one. Hence λ_1 is a differentiable function on M^n, and the eigenvector field x can be taken as differentiable on D.

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