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The Choquet Boundary and the Integral Representation

Hisako Watanabe

Department of Mathematics, Faculty of Science,
Ochanomizu University
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§1. Introduction.

Let X be a compact Hausdorff space and C be a minstable convex cone in $C(X)$ which contains a strictly positive function. N. Boboc and A. Cornea proved in [3] that for each x there is a C-minimal measure μ on X such that

$$\mu(g) \leqq g(x) \qquad \text{for every } g \in C,$$

and proved under the assumption that X is metrizable and $C \not\subset C^+(X)$ that each C-minimal measure is supported by the Choquet boundary which is not empty. The assumption that C has a strictly positive function is essential in the proof.

H. Bauer and K. Donner considered the Choquet boundary $\partial_{\mathscr{H}} X$ with respect to an arbitrary linear subspace $\mathscr{H}$ of $C_0(X)$, the set of all continuous functions on a locally compact Hausdorff space X which tend to zero at infinity. Under their definition the Choquet boundary is empty in case all functions in $\mathscr{H}$ take zero at a common point in X, or in case $\mathscr{H}$ is not linearly separating.

We shall define the Choquet boundary $\delta(C)$ with respect to an arbitrary convex cone C in $C(X)$ where X is a compact Hausdorff space and study the sufficient condition for $\delta(C)$ to be not empty. We shall also discuss the existence of minimal measures with respect to the preoder $\prec_C$ on positive measures and show that minimal measures are supported by the union of $\delta(C)$ and $X_0(C) := \{x \in X : g(x) = 0 \text{ for all } g \in C\}$. Further we shall show, under the additional assumptions, that $\delta(C)$ is the union of all minimal C-stable sets disjoint to $X_0(C)$.

§2. The preoder on positive measures.

Let C be a convex cone of lower semicontinuous functions on a compact Hausdorff space X. Remark that in this paper a lower semicontinuous function on X means a lower semicontinuous function from X into $\boldsymbol{R} \cup \{+\infty\}$. Denote by M^+ the set of all positive measures on X. For two measures $\mu, \nu \in M^+$, we write

$$\mu \prec_C \nu \qquad \text{or simply } \mu \prec \nu$$

if $\mu(g) \leqq \nu(g)$ for every $g \in C$. The relation $\prec_C$ is a preoder on M^+.

A lower semicontinuous function f on X is called C-concave if for each

$x \in X$ and for each μ with $\mu \prec \varepsilon_x$ it holds that

$$\mu(f) \leqq f(x).$$

The set of all lower semicontinuous C-concave functions on X is denoted by $\hat{C}$. Obviously $\hat{C}$ is a min-stable convex cone. Recall that a convex cons S is called min-stable if $f, g \in S$ implies min $(f, g) \in S$.

The following theorem with respect to a hypolinear functional is important. Recall that a sublinear map from a vector space E into $\boldsymbol{R} := \boldsymbol{R} \cup \{+\infty\}$ is called a hypolinear functional on E.

THEOREM 2.1. (*Anger- Lembcke*) *Let q be a lower semicontinuous hypolinear functional on a locally convex space E. Then, for each $h \in E$ and for each $\lambda \in (-q(-h), q(h))$ there is a continuous linear functional μ on E such that*

$$\mu(h) = \lambda \quad \text{and } \mu(f) \leqq q(f) \quad \text{for all } f \in E.$$

Let C be a convex cone of lower semicontinuous functions on X. A function f in $C(X)$ is called C-almost bounded if for each $\varepsilon > 0$ there is $g \in C$ satisfying $f \leqq g + \varepsilon$. The set of all C-almost upper bounded continuous functions is denoted by C_u^*. We remark that C_u^* is closed in $C(X)$ with the sup-norm.

PROPOSITION 2.1. *Let C be a convex cone of lower semicontinuous functions on X and P be a monotone sublinear map from C into R. Assume that for each $f \in C(X)$*

$$\hat{p}(f) := \sup_{\varepsilon > 0} \mathit{inf} \ \{p(g) : g + \varepsilon \geqq f\} > -\infty.$$

Here we regard inf ϕ as $+\infty$. Then the map $f \mapsto \hat{p}(f)$ is hypolinear, monotone and lower semicontinuous on $C(X)$ with the sup-norm.

PROOF. (Subadditivity) Let f and g be two elements of $C(X)$ For each $\varepsilon > 0$ there are $g_1 \in C$ and $g_2 \in C$ satisfying

$$f_1 \leqq g_1 + \varepsilon, \qquad f_2 \leqq g_2 + \varepsilon.$$

Then $g_1 + g_2 \in C$ and $f_1 + f_2 \leqq g_1 + g_2 + 2\varepsilon$. From the monotonity of C it follows that $p(g_1 + g_2) \leqq p(g_1) + p(g_2)$ and hence

$$\begin{aligned} \inf\{p(g) : f_1 + f_2 \leqq g + 2\varepsilon\} &\leqq \inf\{p(g) : f_1 \leqq g + \varepsilon\} \\ &\quad + \inf\{p(g) : f_2 \leqq g + \varepsilon\} \\ &\leqq \hat{p}(f_1) + \hat{p}(f_2). \end{aligned}$$

Therefore, $\hat{p}(f_1 + f_2) \leqq \hat{p}(f_1) + \hat{p}(f_2)$.
(Positively homogenuous) Assume that $\alpha > 0$. It holds that

$$\begin{aligned} \alpha \hat{p}(f) &= \alpha \sup_{\varepsilon > 0} \inf\{p(g) : g + \varepsilon \geqq f, \ g \in C\} \\ &= \sup_{\varepsilon > 0} \inf\{p(\alpha g) : \alpha g + \alpha\varepsilon \geqq \alpha f, \ g \in C\} \end{aligned}$$

$$=\sup_{\varepsilon>0}\inf\{p(h):\ h+\alpha\varepsilon\geqq f,\ h\in C\}$$

$$=\hat{p}(\alpha f)\,.$$

In case $\alpha=0$, it follows from the definition and the subadditivity that $\hat{p}(\alpha f)=\hat{p}(0)=0=\alpha\hat{p}(f)$.
(Monotonity) Obviously the inequality $f\leqq g$ implies $\hat{p}(f)\leqq\hat{p}(g)$.
(Lower semicontinuity) We shall show that the set $\{f\in C(X):\ \hat{p}(f)>\lambda\}$ is open for each $\lambda\in\boldsymbol{R}$. Assume that $\hat{p}(f_0)>\lambda$. If $f_0\notin C_u^*$, $U(f_0):=C(X)\backslash C_u^*$[1] is open and $U(f_0)\subset\{f:\ \hat{p}(f)=+\infty\}\subset\{f:\ \hat{p}(f)>\lambda\}$. Secondly, we consider in case $f_0\in C_u^*$. Since $\hat{p}(f_0)>\lambda$, there is $\varepsilon>0$ such that $\inf\{p(g):\ g+\varepsilon\geqq f_0,\ g\in C\}>\lambda$. Let h be an arbitrary element of $U\left(f_0,\ \frac{\varepsilon}{2}\right)$. For all $g\in C$ satisfying $g+\frac{\varepsilon}{2}\geqq h$, it holds that $g+\frac{\varepsilon}{2}\geqq h\geqq f_0-\frac{\varepsilon}{2}$ and hence $g+\varepsilon\geqq f_0$. Consequently,

$$\inf\left\{p(g):\ g+\frac{\varepsilon}{2}+h\right\}\geqq\inf\{p(g):\ g+\varepsilon\geqq f_0\}\ .$$

By the definition of $\hat{p}(h)$ we have $\hat{p}(h)>\lambda$. Therefore $U\left(f_0,\ \frac{\varepsilon}{2}\right)\subset\{f:\ \hat{p}(f)>\lambda\}$.

Let μ be a positive measures on X. For $f\in C(X)$ we define

$$Q_\mu^c(f):=\sup_{\varepsilon>0}\inf\{\mu(g):\ g+\varepsilon\geqq f,\ g\in C\}\,.$$

For $x\in X$ we write $Q_x^c(f)$ instead of $Q_{\varepsilon_x}^c(f)$. The function $x\mapsto Q_x^c(f)$ is denoted by Q^cf. By Proposition 2.1 the map $f\mapsto Q_\mu^c(f)$ is monotone, hypolinear and lower semicontinuous.

PROPOSITIOH 2.2. *The following relations are verified for two functions $f,\ g\in C(X)$, for $\alpha\in\boldsymbol{R}^+$ and for two measures $\mu,\ \nu\in M^+$.*

(i) $Q_\mu^c(f+g)\leqq Q_\mu^c(f)+Q_\mu^c(g)$,

(ii) $Q^c(\alpha f)=\alpha Q^c(f)$,

(iii) $\mu(f)\leqq Q_\mu^c(f)$,

(iv) $f\leqq g$ *implies* $Q_\mu^c(f)\leqq Q_\mu^c(g)$,

(v) $\mu\prec\nu$ *implies* $Q_\mu^c(f)\leqq Q_\nu^c(f)$,

(vi) $f\mapsto Q_\mu^c(f)$ *is lower semicontinuous.*

PROOF. (i) (ii) (iv) (vi) Immediately these relations and properties are obtained by Proposition 2.1.
(iii) Suppose that $\varepsilon>0$ and $g+\varepsilon\geqq f$ with $g\in C$. Since μ is positive, it holds that $\mu(g)\geqq\mu(f)-\varepsilon\mu(1)$ and hence

1) For two subsets A, B of X with $A\subset B$ we denote by $A\backslash B$ the complementary set of B with respect to A.

$$\hat{\mu}(f) \geqq \sup_{\varepsilon>0} \mu(f) - \varepsilon\mu(1)) = \mu(f).$$

(v) is obvious.

THEOREM 2.2. *For $f \in C(X)$ and $\mu \in M^+$, it holds that*

$$Q^c_\mu(f) = \sup\{\nu(f) : \nu \prec \mu\}.$$

Here the relation $\prec$ is the preoder with respect to C.

PROOF. By Proposition 2.2 the map $h \mapsto Q^c_\mu(h)$ is monotone, hypolinear and lower semicontinuous. Let α be an arbitrary real number satisfying $Q^c_\mu(f) > \alpha > -Q^c_\mu(-f)$. By Theorem 2.1 there is a continuous linear functional ν on $C(X)$ such that

$$\nu(f) = \alpha \quad \text{and} \quad \nu(h) \leqq Q^c_\mu(h) \quad \text{for all} \quad h \in C(X).$$

If $h \leqq 0$, it holds that $\nu(h) \leqq Q^c_\mu(h) \leqq Q^c_\mu(0)$. Consequently ν is positive. For every $g \in C$ it holds that

$$\nu(g) = \sup_{\substack{f \leqq g \\ f \in C(X)}} \nu(f) \leqq \sup_{\substack{f \leqq g \\ f \in C(X)}} Q^c_\mu(f) \leqq \mu(g)$$

and hence $\nu \prec \mu$. Therefore $Q^c_\mu(f) \leqq \sup\{\nu(f) : \nu \prec \mu\}$. On the other hand, let $\lambda \prec \mu$. For $\varepsilon > 0$ and $g \in C$ satisfying $f \leqq g + \varepsilon$, it follows that

$$\lambda(f) \leqq \lambda(g) + \varepsilon\lambda(1) \leqq \mu(g) + \varepsilon\lambda(1).$$

Consequently

$$\lambda(f) - \varepsilon\lambda(1) \leqq \inf\{\mu(g) : g + \varepsilon \geqq f,\ g \in C\}$$

and hence $\lambda(f) \leqq Q^c_\mu(f)$. Therefore we have the conclusion.

§ 3. Minemal measures.

In this section we shall assume that C is a min-stable convex cone in $C(X)$.

PROPOSITIO 3.1. *Let μ, ν be two positive measures. Then $\mu \prec_C \nu$ if and only if $\mu \prec_{\hat{C}} \nu$.*

PROOF. Since $C \subset \hat{C}$, it is obvious that $\mu \prec_{\hat{C}} \nu$ implies $\mu \prec_C \nu$. Conversely, suppose that $\mu \prec_C \nu$. From the definition of concave functions it follows that $\lambda \prec_C \varepsilon_x$ for $\lambda \in M^+$ *and* $x \in X$ if any only if $\lambda \prec_{\hat{C}} \varepsilon_x$.
By Theorem 2.2 we have, for each $x \in X$ and for each $f \in C(X)$,

$$Q^c_x(f) = \sup\{\lambda(f) : \lambda \prec_C \varepsilon_x\} \sup\{\lambda(f) : \lambda \prec_{\hat{C}} \varepsilon_x\} = Q^{\hat{c}}_x(f).$$

Since C and $\hat{C}$ are min-stable, we have

$$Q^c_\nu(f) = \sup_{\varepsilon>0} \inf\{\nu(g) : g + \varepsilon \geqq f,\ g \in C\} = \nu(Q^C f)$$
$$= \nu(Q^{\hat{C}} f) = Q^{\hat{C}}_\nu(f).$$

Using Proposition 2.2, we have, for every $g\in\hat{C}$,

$$\begin{aligned}\mu(g)&=\sup\{\mu(f): f\leqq g,\ f\in C(X)\}\\ &\leqq\sup\{Q_\mu^c(f): f\leqq g,\ f\in C(X)\\ &\leqq\sup\{Q_\nu^c(f): f\leqq g,\ f\in C(X)\}\\ &=\sup\{Q_\nu^{\hat{C}}(f): f\leqq g,\ f\in C(X)\}\leqq\nu(g)\,.\end{aligned}$$

Therefore we have $\mu\prec_{\hat{C}}\nu$.

We denote by $C(X, C)$ the set of all $f\in C(X)$ for which the equality $f(x)=\alpha f(y)$ holds for any two points $x, y\in X$ and every $\alpha\in\boldsymbol{R}$ for which the equality $g(x)=\alpha g(y)$ holds for all $g\in C$.

PROPOSITION 3.2. $C(X, C)=\overline{C-C}$.

PROOF. Obviously we have $\overline{C-C}\subset C(X, C)$. Let f be arbitrary function in $C(X, C)$ and x, y be arbitrary two points in X. For every $\varepsilon>0$ there is $g\in C-C$ such that

$$|f(x)-g(x)|<\varepsilon\,,\qquad |f(y)-g(y)|<\varepsilon\,.$$

By the Kakutani-Stone theorem (cf. [5, p. 39]) we have $f\in\overline{C-C}$.

PROPOSITION 3.3. *Let ν be a positive bounded linear functional on a linear sublattice F of $C(X)$. Then ν can be extended to a positive measure on X.*

PROOF. For $f\in C(X)$, put

$$Q_\nu^F(f)=\sup_{\varepsilon>0}\ \inf\{\nu(g): g+\varepsilon\geqq f,\ g\in F\}\,.$$

Assume that $g\in F$ satisfies $g+\varepsilon\geqq f$ for a real number $\varepsilon>0$. Put $g^+:=\max(g, 0)$ and $g^-:=\max(-g, 0)$. Then $g^+\in F$, $g^-\in F$ and $g=g^+-g^-$. If $g^-\equiv 0$, it holds that $\nu(g)=\nu(g^+)\geqq 0$. If $g^-\not\equiv 0$, it follows that $g^+(x)=0$ at a point x satisfying $g^-(x)=\|g^-\|>0$. Consequently

$$-\|g^-\|+\varepsilon=g^+(x)-g^-(x)+\varepsilon\geqq f(x)\geqq\min_{y\in X} f(y)\,.$$

Hence

$$\nu(g)=\nu(g^+)-\nu(g^-)\geqq-\nu(g^-)\geqq-\|\nu\|\,\|g^-\|\geqq\|\nu\|(\min_{y\in X} f(y)-\varepsilon)\,.$$

Therefore $Q_\nu^F(f)\geqq\|\nu\|\min_{y\in X} f(y)>-\infty$. By proposition 3.1 the map $f\mapsto Q_\nu^F(f)$ is monotone, hypolinear and lower semicontinuous. By theorem 3.1 there is a continuous linear μ on $C(X)$ such that $\mu(f)\leqq Q_\nu^F(f)$ for all $f\in C(X)$. If $f\leqq 0$, we have $\mu(f)\leqq Q_\nu^F(f)\leqq Q_\nu^F(0)=0$. Hence μ is positive. Especially, it follows that

$$\mu(g)\leqq Q_\nu^F(g)\leqq\nu(g)\quad\text{and}\quad\mu(-g)\leqq Q_\nu^F(-g)\leqq\nu(-g)$$

for every $g\in F$ and hence $\mu(g)=\nu(g)$.

A positive measure ν on X is called C-minimal if $\mu\prec_C\nu$ for $\mu\in M^+$ implies $\nu\prec_C\mu$. The following properties (i) (ii) (iii) are equivalent by Proposition 3.2:

(i) $\mu \prec_C \nu$ and $\nu \prec_C \mu$,

(ii) $\mu(g)=\nu(g)$ for all $g \in C$,

(iii) $\mu(g)=\nu(g)$ for all $g \in C(X, C)$.

The restriction $\mu | C-C$ of a positive measure μ to the sublattics $C-C$ of $C(X)$ is a positive bounded linear functional on $C-C$. On the other hand, from Proposition 3.3 it follows that a positive bounded linear functional on $C-C$ is extended to a positive measure on X.

THEOREM 3.1. *If $\mu \in M^+$ satisfies $Q^c_\mu(f)<\infty$ for every $f \in -C$, there is a C-minimal measure $\nu \in M^+$ satisfying $\nu \prec \mu$.*

PROOF. For every $g \in C$ we have $Q^c_\mu(g) \leqq \mu(g)<\infty$ and $-Q^c_\mu(-g)>-\infty$ by the assumption. Consequently we have $-\infty<-Q^c_\mu(-f) \leqq Q^c_\mu(f)<\infty$. Suppose that $\nu \prec \mu$ and $f \in C(X)$. It holds that, for $g \in C$ with $g+\varepsilon \geqq f$, $\mu(g) \geqq \nu(g) \geqq \nu(f)-\varepsilon\nu(1)$ and hence

$$Q^c_\mu(f) \geqq \nu(f). \tag{3.1}$$

Since the inequality holds for $-f$, we have $-Q^c_\mu(-f) \geqq \nu(f)$. Accordingly

$$-\infty<-Q^c_\mu(f) \leqq \nu(f) \leqq Q^c_\mu(f)<\infty \quad \text{for every } f \in C-C. \tag{3.2}$$

We consider the conjugate space $(C-C)'$ of the normed space $C-C$ which is endowed with the topology $\sigma((C-C)', C-C)$. Put

$$M_\mu := \{\nu_{|C-C} : \nu \in M^+,\ \nu \prec \mu\}.$$

Then M_μ is a subset of $(C-C)'$. Since the enequality (3.2) holds for every $f \in C-C$, $\bar{M}_\mu$ is compact in $(C-C)'$ (cf. [4, Theorem 23.11]). On the other hand M_μ is closed. In fact, let ν be an arbitrary element of $\bar{M}_\mu$. Since ν is positive, ν can be extended to a positive measure on X. It is easy to see $\nu(g) \leqq \mu(g)$ for all $g \in C$. Consequently $\nu \in M_\mu$. Therefore M_μ is compact. Using the compactness of M_μ, we see that the preoder $\prec_C$ is inductive. By Zorn's lemma there is a C-minimal measure ν_1 with $\nu_1 \prec_C \mu$.

Especially, we have

COROLLARY 3.1. *If $Q^c_x(f)<\infty$ for all $f \in -C$ and for $x \in X$, there is a C-miniml measure $\nu \in M^+$ with $\nu \prec_C \varepsilon_x$.*

PROPOSITION 3.4. *If $\nu \in M^+$ is a C-minimal, it holds that $\nu(f)=Q^c_\nu(f)=\nu(Q^C f)$ for $f \in C(X, C)$.*

PROOF. Since ν is C-minimal, we have $\mu(f)=\nu(f)$ for every $\mu \in M^+$ with $\mu \prec \nu$ and for every $f \in C(X, C)$. By Theorem 2.2, we have

$$Q^c_\nu(f)=\sup\{\mu(f) : \mu \prec \nu\}=\nu(f)$$

for every $f \in C(X, C)$. Since C is min-stable and the function $\varepsilon \mapsto \inf\{g(x) : g+\varepsilon \geqq f,\ g \in C\}$ decreases, we have $Q^c_\mu(f)=\nu(Q^C f)$.

§ 4. The Choquet boundary.

Let S be a convex cone in $C(X)$. We denote by $X_0(S)$ or simply X_0 the set $\{x \in X: g(x(=0 \text{ for all } g \in S\}$. Put

$$C_S := \{\min(g_1, g_2, \cdots, g_n) : g_i \in S, n \in \boldsymbol{N}\}$$

Then C_S is a min-stable convex cone and the relation $\mu \prec_S \varepsilon_x$ is equivalent to the relation $\mu \prec_{C_S} \varepsilon_x$. Consequently we have $Q_x^S(f) = \sup_{\varepsilon > 0} \inf \{g(x) : g + \varepsilon \geqq f, g \in C_S\}$ for all $f \in C(X)$ by Theorem 2.2.

We denote by $\delta(S)$ the set of all points $x \in X \backslash X_0(S)$ such that ε_x is C_S-minimal and call the Choquet boundary with respect to S.

PROPOSITION 4.1. *Let x be a point in X. Then $x \in \delta(S)$ if and only if $x \notin X_0(S)$ and $Q_x^S(f) = f(x)$ for all $f \in -C_S$.*

PROOF. If a point $x \notin X_0(S)$ belongs to $\delta(S)$, we have $Q_x^S(f) = Q_x^{C_S}(f) = f(x)$ for all $f \in C_S$ by Proposition 3.4. Conversely, suppose that $x \notin X_0(S)$ and $Q_x^S(f) = f(x)$ for all $f \in -C_S$. Let μ be a positive measure on X with $\mu \prec_{C_S} \varepsilon_x$. Using Theorem 2.2, we have

$f(x) = Q_x^S(f) = Q_x^{C_S}(f) = \sup\{\mu(f) : \mu \prec_{C_S} \varepsilon_x\{ \geqq \mu(f) \geqq f(x)$. Consequently $\mu(f) = f(x)$ for all $f \in -C_S$. Therefore μ is C_S-minimal.

COROLLORY 4.1. *$Q_x^S(f) = f(x)$ for all $f \in -C_S$ if and only if $x \in \delta(S)$ or $x \in X_0(S)$.*

PROOF. This is an immediate consequence from Proposition 4.1.

PROPOSITION 4.2. *Let x be a point contained in the complement of $X_0(S)$. Assume that there are $u \in S$ and $v \in S$ such that $u(x_0) > 0$ and $v(x_0) < 0$. Further, assume that there is $w \in C_S$ such that $w \geqq 0$ and the set $\{x \in X: w(x) = 0\}$ is equal to $X_0(S) \cup \{x_0\}$.*

PROOF. For each $\mu \in M^+$ with $\mu \prec_{C_S} \varepsilon_x$, it follows that $0 \leqq \mu(w) \leqq w(x_0) = 0$ and hence $\mu(w) = 0$. Since w is non-negative, the support μ is included by the set $\{x: w(x) = 0\}$. By the assumption we can write

$$\mu = \mu_1 + \alpha\, \varepsilon_{x_0}$$

where μ_1 is a positive measure of which suoport is contained in $X_0(S)$ and α is a non-negative real number. Accordingly

$$v(x_0) \geqq \mu(v) = \mu_1(v) + \alpha v(x_0) = \alpha v(x_0).$$

From the inequality $v(x_0) < 0$ it follows that $\alpha \geqq 1$. Similarly, considering u, we have $\alpha \leqq 1$, and hence $\alpha = 1$. Since $f = 0$ on $X_0(S)$ for all $f \in -C_S$, we have $\mu(f) = f(x_0)$. Therefore x_0 is a point of the Choquet boundary.

EXAMPLE 1. Let X be the closed interval $[0, 1]$ in $\boldsymbol{R}$ and S be the linear space generated the function $f(x) = x$. Then we have $C_S = S$, $C(X, C_S) = S$ and $\delta(S) = (0, 1]$.

EXAMPLE 2. Let $X=[0, 1]$ and $S=\{ax+bx^2: a, b\in \boldsymbol{R}\}$. Then we have $X_0(S)=\{0\}$, $C(X, C_S)=\{f\in C([0, 1]): f(0)=0\}$ and $\delta(S)=\{1\}$.

EXAMPLE 3. Let $X=[0, 1]$ and $S=\{ax+bx^2+cx^3: a, b, c\in \boldsymbol{R}\}$. Then we have $X_0(S)=\{0\}$, $C(X, C_S)=\{f\in C([0, 1]): f(0)=0\}$ and $\delta(S)=(0, 1]$.

Hearafter, assume that C is a min-stable convex cone in $C(X)$.

PROPOSITION 4.3. *Let X be metrizable. Then there is $f_0\in -\bar{C}$ such that*

$$\{x\in X: Q_x^c(f_0)=f_0(x)\}=\bigcap_{f\in -C}\{x\in X: Q_x^c(f)=f(x)\}=\delta(C)\cup X_0\,. \tag{4.1}$$

and $\delta(C)$ is a G_δ-set.

PROOF. Since X is compact and metrizable, there is a countable set $\{f_n\}\subset -C$ which is total in $C(X, C)$. Put

$$f_0=\sum_{n=1}^{\infty}\frac{1}{2^n}\frac{f_n}{\|f_n\|}.$$

Then $f_0\in -\bar{C}$ and it holds that $\{x: Q_x^c(f_0)=f_0(x)\}=\bigcap_{f\in -C}\{x: Q_x^c(f)=f(x)\}$ Further, it follows that

$$\{x\in X: Q_x^c(f_0)=f_0(x)\}=\bigcap_{n\in N}\left\{x\in X: Q_x^c(f_0)-f_0(x)<\frac{1}{n}\right\}$$

$$=\bigcap_{n\in N}\bigcup_{m\in N}\left\{x\in X: h_m(x)-f_0(x)<\frac{1}{n}\right\},$$

where $h_m(x)=\inf\left\{g(x): g\in C,\ g+\frac{1}{m}\geqq f\right\}$. Since h_m is upper semicontinuous, $\{x: Q_x^c(f_0)=f_0(x)\}$ is a G_δ-set. By Proposition 4.1 and (4.1) we have

$$\{x\in X: Q_x^c(f_0)=f_0(x)\}=\delta(C)\cup X_0(C)\,.$$

Since $X_0(C)$ is closed, $\delta(C)$ is a G_δ-set.

THEOREM 4.1. *Let X be a compact metrizable set and C be a min-stable convex cone in $C(X)$. Assume that there are $x_0\in X$ and $v\in C$ satisfying $v(x_0)<0$ and $Q_{x_0}^c(f)<\infty$ for every $f\in -C$. Then the Choquet boundary $\delta(C)$ is not empty and a G_δ-set. For every C-minimal measure $\mu\in M^+$, it follows that*

$$\mu(X\backslash(X_0(C)\cup\delta(C)))=0\,.$$

PROOF. From Theorem 3.1 it follows that there is a C-minimal measure ν with $\nu\prec\varepsilon_x$. Put $w=min\ (v, 0)$. Then $w\in C$ and it holds that

$$\nu(w)\leqq w(x_0)=v(x_0)<0\,.$$

Since $w\leqq 0$, we have $\nu(X\backslash X_0)>0$. Since X is metrizable, there is $f_0\in -\bar{C}$ satisfying (4.1) and it follows that $\nu(Q^c f_0-f_0)=0$. The inequality $Q^c f_0-f_0\geqq 0$ implies $\nu\{x: Q_x^c(f_0)-f_0(x)>0\})=0$. Consequently $\nu(X\backslash X_0\cup\delta(C)))=0$. Since $\nu(X\backslash X_0)>0$, $\delta(C)$ is not empty. Further, since $\mu(Q^c f_0-f_0)=0$ for every C-minimal measure μ, we obtain $\mu(X\backslash(X_0\cup\delta(C)))=0$.

§ 5. Minimal stable sets.

Let C be a min-stable convex cone in $C(X)$. A closed subset S of X is called C-stable or simply stable if for each $x \in X$ and for each measure μ with $\mu \prec_C \varepsilon_x$ the support S_μ of μ is contained in the set $S \cup X_0$. We denote by $\boldsymbol{S}$ the set of all C-stable closed subsets of X. Since $\boldsymbol{S}$ is inductive with respect to the oder $\subset$, there is, for each $S \in \boldsymbol{S}$, a minimal C-stable set included by S.

In this section we shall assume the following condition (p):

(p) for each $x \in X \backslash X_0$ there is $w \in C$ satisfying $w \geqq 0$ and $w(x) > 0$.

Let v be a function in C satisfying $v(x_0) < 0$ for $x_0 \in X \backslash X_0$. The function v is said to satisfy (c) at x_0 if there is a non-negative function $u \in C$ such that $u(x_0) > 0$ and for some real number b with $\dfrac{-v(x_0)}{u(x_0)} > b > 0$, it holds that $bu + v > 0$ on $U \backslash X_0$ where U is an open set containing X_0.

PROPOSITION 5.1. *Assume that* $v \in \bar{C}$ *with* $v(x_0) < 0$ *satisfies* (c) *at* x_0. *Then there is a C-stable set S inclvded by the set* $\{x \in X: v(x) < 0\}$.

PROOF. By the assumption there are a non-negative function $u \in \bar{C}$ with $u(x_0) > 0$, an open set U containing X_0 and a real number b with $\dfrac{-v(x_0)}{u(x_0)} > b > 0$ such that

$$bu + v > 0 \qquad \text{on } U \backslash X_0. \tag{5.1}$$

Since $X \backslash U$ is compact, by the condition (p) there is a non-negative function $w \in C$ with $w > 0$ on $X \backslash U$. We may suppose that w satisfies

$$\frac{-v(x_0)}{u(x_0)} \geqq \frac{-v(x_0)}{u(x_0) + w(x_0)} > b. \tag{5.2}$$

Let α_0 be the supremum of positive real numbers α satisfying

$$\{x \in X: \alpha(u(x) + w(x)) + v(x) \leqq 0\} \cap (X \backslash X_0) \neq \phi.$$

Then
$$\alpha_0 \geqq \frac{-v(x_0)}{u(x_0) + w(x_0)}.$$

From (5.1) it follows that

$$b(u(x) + w(x)) + v(x) \geqq bu(x) + v(x) > 0$$

on $U \backslash X_0$. Using (5.2), we have, for $\alpha \geqq \dfrac{-v(x_0)}{u(x_0) + w(x_0)}$,

$$\alpha(u + w) + v > 0 \qquad \text{on } U \backslash X_0.$$

On the other hand, put

$$\min_{x \in X \backslash U} (u(x) + w(x)) = \beta \qquad \text{and } \min_{x \in X} v(x) = \gamma.$$

Then we have $\beta > 0$ and $\gamma < 0$. Consequently

$$\alpha(u+w)+v>\frac{-\gamma}{\beta}(u+w)+\ v\geqq-\gamma+v\geqq 0 \text{ on } X\backslash U \qquad \text{for } \alpha>-\frac{\gamma}{\beta}.$$

Hence $\alpha_0=\sup\alpha\leqq\frac{-\gamma}{\beta}<\infty$. Immediately, by the definition of α_0, we have $\alpha_0(u+w)+v\geqq 0$ on $X\backslash X_0$ and hence on X. Further, there is $x_1\in X\backslash X_0$ such that

$$\alpha_0(u(x_1)+w(x_1))+v(x_1)=0\,. \tag{5.3}$$

In fact, suppose that no point $x_1\in X\backslash X_0$ satisfies (5.3). Since $\alpha_0(u+w)+v$ is strictly positive on $X\backslash U$, there is $\delta\in\boldsymbol{R}$ such that $\alpha_0>\delta>0$ and $\delta(u+w)+v>0$ on $X\backslash U$. Put $\alpha_1=\max(\delta, b)$. Then we have $\alpha_0>\alpha_1$ and for $\alpha\geqq\alpha_1$

$$\alpha(u+w)+v\geqq\alpha_1(u+w)+v>0 \qquad \text{on } X\backslash X_0\,.$$

This is a contradiction to the definition of α_0. Therefore there is $x_1\in X\backslash X_0$ satisfying (5.3). Put

$$S_0:=\{x\in X\backslash X_0:\ \alpha_0(u(x)+w(x))+v(x)=0\}\,.$$

Then $S_0\neq\phi$ and

$$S_0=(X\backslash U)\cap\{x\in X:\ \alpha_0(u(x)+w(x))+v(x)=0\}\,,$$

since $\alpha_0(u+w)+v>0$ on $U\backslash X_0$. For each $x\in S_0$ and for each $\mu\in M^+$ satisfying $\mu\prec\varepsilon_x$, it holds that

$$0\leqq\mu(\alpha_0(u+w)+v)\leqq\alpha_0(u(x)+w(x))+v(x)=0\,.$$

Consequently

$$S_\mu\subset\{y\in X:\ \alpha_0(u(y)+w(y)+v(y)=0\}=S_0\cup X_0\,.$$

Hence S_0 is C-stable. Since $u+w>0$ on $X\backslash U$, we have $v<0$ on S_0.

Immediatly we have

COROLLARY 5.1. *Under the same conditions as Proposition* 5.1., *there is a minimal C-stable set inclvded by* $\{x\in X:\ v(x)<0\}$.

The following proposition (resp. the corollary) can be proved by the same method as the assertion a)⇒b) (resp. e)⇒f)) of Theorem 2.1 in [3].

PROPOSITION 5.2. *Let S be a minimal C-stable set included by $X\backslash X_0$ and u be a function in C satisfying $u>0$ on S an $u(x_0)=1$ for some $x_0\in S$. Further assume that there is $v_1\in C$ such that $v_1(x_0)<0$. Then we have*

$$v=v(x_0)u \qquad \text{on } S \qquad \text{for every } v\in C\,.$$

COROLLARY 5.2. *Let S be a minimal C-stable set and for $x_0\in S$ there is $v_1\in C$ such that $v_1(x_0)<0$. Then x_0 is a point of $\delta(C)$.*

THEOREM 5.1. *Let v be an element of C with $v(x_0)<0$ for some $x_0\in X$ Assume that v satisfies (c) at x_0. Then we have*

$$\{x \in X : v(x) < 0\} \cap \delta(C) \neq \phi .$$

PROOF. By Proposition 5.1 there is a C-stable set S in $\{x : v(x) < 0\}$. From Zorn's lemma there is a minimal C-stable set S_0 in S. Each point of S_0 is one of $\delta(C)$ by Proposition 5.2. Hence we have the conclusion.

THEOREM 5.2. *Assume that for a point $x_0 \in \delta(C)$ there is $v \in C$ such that $v(x_0) < 0$ and v satisfies (c) at x_0. Then there is a minimal C-stable set S disjoint to X_0 and containing x_0.*

PROOF. This theorem can be proved by the same method as the assertion g)⇒a) in Theorem 2.1 in [3].

References

1) B. Anger and J. Lembche: Hahn-Banach type theorem for hypolinear functiolals, Math. Ann. **209** (1974), 121-151.
2) H. Bauer and K. Donner: Korovkin approximation in $C_0(X)$, Math. Ann. **236** (1978), 225-237.
3) N. Boboc and A. Cornea: Convex cones of lower semicontinuous functions on compact spaces‘ Rev. Roum. Math. Pures et Appl., 12 (1967), 471-525.
4) G. Choquet: Lectures on analysis, vol. 2, Benjamin, New York (1969).
5) L. Nachbin: Elements of approximation theory, Van Norstrand mathematical studies **14** (1967).