Motion of a Charged Particle in an Electromagnetic Field

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Equations of motion of a charged particle in the Coulomb field of an electric charge combined with the magnetic field of a magnetic monopole at the same point are integrable by separation of variables in classical dynamics as well as in relativistic dynamics. The Schrödinger equation and the Dirac equation of the particle are also soluble by separation of variables. While eigenvalues and eigenfunctions of the angular operator are different from those of the pure Coulomb field, the energy formula is very similar to that of the pure Coulomb field.

§1. Motion in classical dynamics.

A magnetic monopole at the origin of a cartesian coordinate system produces a magnetic field with flux density

$$B_x=-\frac{\beta x}{r^3}, \qquad B_y=-\frac{\beta y}{r^3}, \qquad B_z=-\frac{\beta z}{r^3}, \qquad r=\sqrt{(x^2+y^2+z^2)}$$

in the cartesian coordinates, $-4\pi\beta$ denoting the strength of the magnetic monopole. The field may be represented by a vector potential A with components

$$A_x=-\frac{\beta yz}{r(x^2+y^2)}, \qquad A_y=\frac{\beta zx}{r(x^2+y^2)}, \qquad A_z=0\,.$$

We note here that

$$\begin{aligned}A_x dx+A_y dy+A_z dz&=-\frac{\beta z}{r(x^2+y^2)}(y dx-x dy)=\frac{\beta z}{r}\,d\tan^{-1}\frac{y}{x}\\&=\beta\cos\theta d\phi\end{aligned}$$

r, θ, ϕ being polar coordinates. Hence the magnetic field of the monopole may be represented by the vector potential with components $A_r=0$, $A_\theta=0$, $A_\phi=\beta\cos\theta$. An electric charge $-4\pi\alpha$ produces an electric field with its potential $V=-\alpha/r$.

A particle with the unit mass and the unit electric charge has the Lagrangian

$$L=\frac{1}{2}(\dot{r}^2+r^2\dot{\theta}^2+r^2\sin^2\theta\cdot\dot{\phi}^2)+\beta\cos\theta\cdot\dot{\phi}+\frac{\alpha}{r}, \tag{1}$$

canonical momenta

$$p_r=\frac{\partial L}{\partial\dot{r}}=\dot{r}, \qquad p_\theta=\frac{\partial L}{\partial\dot{\theta}}=r^2\dot{\theta}, \qquad p_\phi=\frac{\partial L}{\partial\dot{\phi}}=r^2\sin^2\theta\cdot\dot{\phi}+\beta\cos\theta,$$

and the Hamiltonian

$$H=\frac{1}{2}p_r^2+\frac{1}{2r^2}p_\theta^2+\frac{1}{2r^2\sin^2\theta}(p_\phi-\beta\cos\theta)^2-\frac{\alpha}{r}.$$

The partial differential equation of Hamilton-Jacobi

$$\frac{\partial S}{\partial t}+\frac{1}{2}\left(\frac{\partial S}{\partial r}\right)^2+\frac{1}{2r^2}\left(\frac{\partial S}{\partial\theta}\right)^2+\frac{1}{2r^2\sin^2\theta}\left(\frac{\partial S}{\partial\phi}-\beta\cos\theta\right)^2-\frac{\alpha}{r}=0 \tag{2}$$

is integrable by separation of variables, by putting

$$S=-Et+f(r)+g(\theta)+p\phi, \qquad E,\ p:\ \text{constants}$$

and getting

$$[g'(\theta)]^2+\frac{(p-\beta\cos\theta)^2}{\sin^2\theta}=q, \qquad q:\ \text{a constant}$$

$$[f'(r)]^2-2E-\frac{2\alpha}{r}+\frac{q}{r^2}=0$$

And we have

$$S=-Et+\int^r\sqrt{\left(2E+\frac{2\alpha}{r}-\frac{q}{r^2}\right)}dr+\int^\theta\sqrt{\left(q-\frac{(p-\beta\cos\theta)^2}{\sin^2\theta}\right)}d\theta+p\phi.$$

The period in t for r varying between two limits is equal to

$$\frac{2\pi\alpha}{(-2E)^{3/2}}$$

the same as that in the pure Coulomb field.

§2. Motion in relativistic dynamics.

From (1) we have the relativistic Lagrangian

$$L=-R+A_\phi\dot{\phi}+A_0\dot{t}, \qquad R=\sqrt{(\dot{t}^2-\dot{r}^2-(r\dot{\theta})^2-(r\sin\theta\dot{\phi})^2)} \tag{3}$$

where we put the light velocity$=1$ and $A_\phi=\beta\cos\theta$, $A_0=\alpha/r$, $\dot{r}=dr/ds$, s being an appropriate parameter. We have

$$p_r=\frac{\dot{r}}{R}, \qquad p_\theta=\frac{r^2\dot{\theta}}{R}, \qquad p_\phi=\frac{r^2\sin^2\theta}{R}\dot{\phi}+A_\phi, \qquad p_0=-\frac{\dot{t}}{R}+A.$$

and $$p_r^2+\frac{p_\theta^2}{r^2}+\frac{(p_\phi-A_\phi)^2}{r^2\sin^2\theta}-(p_0-A_0)^2=-1,$$

hence the Hamilton-Jacobi equation

$$\left(\frac{\partial S}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial S}{\partial \theta}\right)^2+\frac{1}{r^2\sin^2\theta}\left(\frac{\partial S}{\partial \phi}-A_\phi\right)^2-\left(\frac{\partial S}{\partial t}-A_0\right)^2=-1\,. \tag{4}$$

This equation is also integrable by separation of variables, by putting

$$S=f(r)+g(\theta)+p\phi-Et\,.$$

$g(\theta)$ turns out to be the same as in classical dynamics. We have

$$f(r)=\int^r\sqrt{\left\{\left(E+\frac{\alpha}{r}\right)^2-1-\frac{q}{r^2}\right\}}dr\,.$$

The period in t becomes $2\pi\alpha(1-E^2)^{-3/2}$.

§ 3. Schrödinger equation.

The Schrödinger equation runs as

$$\frac{1}{i}\frac{\partial\psi}{\partial t}+\left[\frac{1}{2}(p_x-A_x)^2+\frac{1}{2}(p_y-A_y)^2+\frac{1}{2}(p_z-A_z)^2-A_0\right]\psi=0$$

or, in the polar coordinates

$$\frac{1}{i}\frac{\partial\psi}{\partial t}+\left[-\frac{1}{2}\left(\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}+\frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)\right.$$
$$\left.+\frac{i\beta\cos\theta}{r^2\sin^2\theta}\frac{\partial}{\partial\phi}+\frac{\beta^2\cos^2\theta}{2r^2\sin^2\theta}-\frac{\alpha}{r}\right]\psi=0\,. \tag{5}$$

This equation is soluble by separation of variables. By putting

$$r\psi=R(r)\Theta(\theta)\,e^{im\phi}\,e^{-iEt}$$

we have

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\sin\theta\frac{d\Theta}{d\theta}+\left[\eta-\frac{(m-\beta\cos\theta)^2}{\sin^2\theta}\right]\Theta=0 \tag{6}$$

$$\left(\frac{d}{dr^2}-\frac{\eta}{r^2}+\frac{2\alpha}{r}+2E\right)R=0 \tag{7}$$

η being a constant of separation. When the magnetic field is absent, we know that

$$\eta=l(l+1)\,,\qquad \Theta=P_l^m(\cos\theta)\,,$$

from the text books on quantum mechanics.

The equation (6) may be changed into

$$\frac{d}{du}(1-u^2)\frac{d\Theta}{du}+\left[\eta-\frac{(m-\beta u)^2}{1-u^2}\right]\Theta=0\,,\qquad u=\cos\theta \tag{8}$$

which is a Riemann's P-equation[1] with a solution of the type

$$\Theta=P\left\{\begin{array}{cccc} 1 & -1 & \infty & \\ \dfrac{m-\beta}{2} & \dfrac{m+\beta}{2} & \dfrac{1}{2}+\sqrt{\dfrac{1}{4}+\eta+\beta^2} & u \\ -\dfrac{m-\beta}{2} & -\dfrac{m+\beta}{2} & \dfrac{1}{2}-\sqrt{\dfrac{1}{4}+\eta+\beta^2} & \end{array}\right\}$$

We classify solutions according as indices at $u=1$ and $u=-1$ are positive or negative, to ensure Θ to be finite.

1° $m-\beta>0$, $m+\beta>0$.

$$\Theta=(1-u)^{(m-\beta)/2}(1+u)^{(m+\beta)/2}F\left\{m+\frac{1}{2}+\sqrt{\frac{1}{4}+\eta+\beta^2},\right.$$

$$\left.m+\frac{1}{2}-\sqrt{\frac{1}{4}+\eta+\beta^2},\ 1+m+\beta\,;\ \frac{u+1}{2}\right\}.$$

For F, hypergeometric function, to be finite at $u=1$, the second argument of F must be zero or a negative integer, since m is positive there. Hence

$$m+\frac{1}{2}-\sqrt{\frac{1}{4}+\eta+\beta^2}=-k\,, \qquad k=0,\ 1,\ 2,\ \cdots$$

$$\eta=\left(m+k+\frac{1}{2}\right)^2-\frac{1}{4}-\beta^2=(m+k)(m+k+1)-\beta^2\,.$$

If we put $\beta=0$, we get the well-known relation $\eta=l(l+1)$, $l=m+k$.

2° $m-\beta<0$, $m+\beta>0$.

$$\Theta=(1-u)^{(\beta-m)/2}(1+u)^{(m+\beta)/2}F\left\{\beta+\frac{1}{2}+\sqrt{\frac{1}{4}+\eta+\beta^2},\right.$$

$$\left.\beta+\frac{1}{2}-\sqrt{\frac{1}{4}+\eta+\beta^2},\ 1+m+\beta\,;\ \frac{u+1}{2}\right\}.$$

Since β is positive, we have

$$\beta+\frac{1}{2}-\sqrt{\frac{1}{4}+\eta+\beta^2}=-k\,, \qquad k=0,\ 1,\ 2,\ \cdots$$

$$\eta=(k+\beta)(k+\beta+1)-\beta^2\,.$$

3° $m-\beta<0$, $m+\beta<0$, $(m<0)$,

$$\Theta=(1-u)^{(\beta-m)/2}(1+u)^{-(m+\beta)/2}F\left\{-m+\frac{1}{2}+\sqrt{\frac{1}{4}+\eta+\beta^2},\right.$$

$$\left.-m+\frac{1}{2}-\sqrt{\frac{1}{4}+\eta+\beta^2},\ 1-m-\beta\,;\ \frac{u+1}{2}\right\}$$

$$\eta=(k-m)(k-m+1)-\beta^2\,, \qquad k=0,\ 1,\ 2,\ \cdots.$$

4° $m-\beta>0$. $m+\beta<0$, $(\beta<0)$,

$$\Theta=(1-u)^{(m-\beta)/2}(1+u)^{-(m+\beta)/2}F\left\{-\beta+\frac{1}{2}+\sqrt{\frac{1}{4}+\eta+\beta^2},\right.$$

$$-\beta+\frac{1}{2}-\sqrt{\frac{1}{4}+\eta+\beta^2},\ 1-m-\beta\,;\ \frac{u+1}{2}\Big\},$$

$$\eta=(k-\beta)(k-\beta+1)-\beta^2, \qquad k=0,\ 1,\ 2,\ \cdots.$$

In short we have

$$\eta=(k+s)(k+s+1)-\beta^2, \qquad k=0,\ 1,\ 2,\ \cdots,$$

$$s=\max(|m|,\ |\beta|),$$

$$\Theta=(1-u)^{|m-\beta|/2}(1+u)^{|m+\beta|/2}F\Big\{2s+1+k,\ -k,\ 1+|m+\beta|\,;\frac{u+1}{2}\Big\}.$$

A change of variable $r\sqrt{(-8E)}=\rho$ brings the equation (7) of the radial part into the following form

$$\left[\frac{d^2}{d\rho^2}+\frac{1/4-p^2}{\rho^2}+\frac{q}{\rho}-\frac{1}{4}\right]R=0,$$

$$p^2=\eta+\frac{1}{4}=\left(k+\frac{1}{2}+s\right)^2-\beta^2, \qquad q=\alpha/\sqrt{(-2E)},$$

and gives us a solution expressed by Whittaker's function[2)]

$$R=M_{q,\,p}(\rho)$$

$$=\rho^{1/2+p}\,e^{-(1/2)\rho}\Big\{1+\frac{1/2+p-q}{1!(2p+1)}\rho+\frac{(1/2+p-q)(3/2+p-q)}{2!(2p+1)(2p+2)}\rho^2+\cdots\Big\}.$$

For R be finite at $\rho=\infty$, we have the condition

$$\frac{1}{2}+p-q=-n, \qquad n=0,\ 1,\ 2,\ \cdots,$$

hence

$$q=\frac{1}{2}+p+n=\frac{1}{2}+\sqrt{\left(k+\frac{1}{2}+s\right)^2-\beta^2}+n.$$

If we put $\beta=0$, $s=m$, we have the relation $q=k+|m|+n+1$, which is well known.

§ 4. Dirac equation.

We put the Dirac equation in the following form

$$F\psi=0, \qquad F=p_0-A_0+\alpha_1(p_x-A_x)+\alpha_2(p_y-A_y)+\alpha_3(p_z-A_z) \tag{9}$$

in cartesian coordinates with the condition that ψ be one-valued and square-integrable. Symbols used here mean as follows

$$p_0=\frac{1}{i}\frac{\partial}{\partial t}, \qquad p_x=\frac{1}{i}\frac{\partial}{\partial x}, \qquad p_y=\frac{1}{i}\frac{\partial}{\partial y}, \qquad p_z=\frac{1}{i}\frac{\partial}{\partial z},$$

$$A_0=\frac{\alpha}{r}, \qquad A_x=\frac{-\beta yz}{r(x^2+y^2)}, \qquad A_y=\frac{\beta zx}{r(x^2+y^2)}, \qquad A_z=0$$

with the agreement $m=1$, $c=1$, $h/2\pi=1$, $e=1$.

In polar coordinates ($x=r\sin\theta\cos\phi$, $y=r\sin\theta\sin\phi$, $z=r\cos\theta$), the operator F may be written

$$F=\alpha_r p_r+\alpha_\theta\frac{1}{r}p_\theta+\alpha_\phi\frac{1}{r\sin\theta}(p_\phi-A_\phi)+p_0-A_0-\alpha_0$$

where

$$p_r=\frac{1}{i}\frac{\partial}{\partial r},\qquad p_\theta=\frac{1}{i}\frac{\partial}{\partial\theta},\qquad p_\phi=\frac{1}{i}\frac{\partial}{\partial\phi}$$

$$\alpha_r=\alpha_1\sin\theta\cos\phi+\alpha_2\sin\theta\sin\phi+\alpha_3\cos\theta$$

$$\alpha_\theta=\alpha_1\cos\theta\cos\phi+\alpha_2\cos\theta\sin\phi-\alpha_3\sin\theta$$

$$\alpha_\phi=-\alpha_1\sin\phi+\alpha_2\cos\phi\,.$$

Two matrices S and T defined by

$$S=\exp\left(-\frac{1}{2}\phi\alpha_1\alpha_2\right)=\cos\frac{\phi}{2}-\alpha_1\alpha_2\sin\frac{\phi}{2}$$

$$T=\exp\left(\frac{1}{2}\theta\alpha_1\alpha_3\right)=\cos\frac{\theta}{2}+\alpha_1\alpha_3\sin\frac{\theta}{2}$$

transform the matrices α_1, α_2, α_3 as follows

$$S\alpha_1S^{-1}=\alpha_1\cos\phi+\alpha_2\sin\phi\,,\qquad T\alpha_1T^{-1}=\alpha_1\cos\theta-\alpha_3\sin\theta$$

$$S\alpha_2S^{-1}=-\alpha_1\sin\phi+\alpha_2\cos\phi\,,\qquad T\alpha_2T^{-1}=\alpha_2$$

$$S\alpha_3S^{-1}=\alpha_3\,,\qquad T\alpha_3T^{-1}=\alpha_1\sin\theta+\alpha_3\cos\theta\,.$$

Hence the matrix $U=ST$ allows us to see that

$$\alpha_r=U\alpha_3U^{-1}\,,\quad \alpha_\theta=U\alpha_1U^{-1}\,,\quad \alpha_\phi=U\alpha_2U^{-1}\,,\quad \alpha_0=U\alpha_0U^{-1}\,,$$

$$U^{-1}p_rU=p_r\,,\quad U^{-1}p_\theta U=p_\theta-\frac{i}{2}\alpha_1\alpha_3\,,\quad U^{-1}p_\phi U=p_\phi-\frac{i}{2}\alpha_2(\alpha_1\cos\theta+\alpha_3\sin\theta)$$

and finally

$$U^{-1}FU=\alpha_3\left(p_r-\frac{i}{r}\right)+\alpha_1\frac{1}{r}\left(p_\theta-\frac{i}{2}\cot\theta\right)+\alpha_2\frac{p_\phi-\beta\cos\theta}{r\sin\theta}+p_0-A_0-\alpha_0$$

$$=\alpha_3\frac{1}{r}p_r r+\alpha_1\frac{1}{r\sqrt{\sin\theta}}p_\theta\sqrt{\sin\theta}+\alpha_2\frac{p_\phi-\beta\cos\theta}{r\sin\theta}+p_0-A_0-\alpha_0$$

Therefore the equation $F\psi=0$ may be changed into

$$\mathfrak{F}\omega=0$$

where the operator $\mathfrak{F}$ and the spinor ω are defined by

$$\mathfrak{F}=\alpha_3p_r+\alpha_1\frac{1}{r}p_\theta+\alpha_2\frac{1}{r\sin\theta}(p_\phi-\beta\cos\theta)+p_0-\frac{\alpha}{r}-1\,,$$

$$\psi=U\frac{1}{r\sqrt{\sin\theta}}\omega\,.$$

The angle ϕ is an ignorable coordinate, so the operator p_ϕ commutes with $\mathscr{F}$. Since U involves the factor $S=\cos\frac{\phi}{2}-\alpha_1\alpha_2\sin\frac{\phi}{2}$, the one-valuedness of ψ requires that

$$p_\phi\omega=m\omega$$

m being any half-integer.

An operator Q defined by

$$Q=i\alpha_3\alpha_0\left[\alpha_1 p_\theta+\alpha_2\frac{1}{\sin\theta}(p_\phi-\beta\cos\theta)\right] \tag{11}$$

commutes with α_0, α_3, p_r, p_0-A_0, hence with $\mathscr{F}$,

$$\mathscr{F}Q-Q\mathscr{F}=0.$$

Therefore the spinor ω may be labeled by the eigenvalues of p_ϕ and Q. The operator p_0 also commutes with $\mathscr{F}$. We denote the eigenvalue of p_0 by $-E$.

§ 5. Eigenvalues of the operator L.

With matric representation of α_0, α_1, α_2 and α_3

$$\alpha_0=1\times\sigma_3\,,\quad \alpha_1=\sigma_1\times\sigma_1\,,\quad \alpha_2=\sigma_2\times\sigma_1\,,\quad \alpha_3=\sigma_3\times\sigma_1\,,$$

we have

$$Q=L\times\sigma_3=\begin{pmatrix} L & 0\\ 0 & -L\end{pmatrix}$$

$$L=\sigma_2 p_\theta-\sigma_1\frac{1}{\sin\theta}(p_\phi-\beta\cos\theta).$$

If we denote the eigenvalue of the operator L by λ, the eigenvalues of the operator Q will be λ and $-\lambda$. Eigenfunctions $\chi=(\chi_1, \chi_2)$ corresponding to the eigenvalue λ are to be determined by the equation $L\chi=\lambda\chi$ or

$$\left(-\frac{\partial}{\partial\theta}-\frac{m-\beta\cos\theta}{\sin\theta}\right)\chi_2=\lambda\chi_1$$

$$\left(\frac{\partial}{\partial\theta}-\frac{m-\beta\cos\theta}{\sin\theta}\right)\chi_1=\lambda\chi_2$$

which may be written in the following form

$$\frac{d\chi_1}{du}=-\frac{m-\beta u}{1-u^2}\chi_1-\lambda\frac{\chi_2}{\sqrt{1-u^2}}$$

$$\frac{d\chi_2}{du}=\frac{m-\beta u}{1-u^2}\chi_2+\lambda\frac{\chi_1}{\sqrt{1-u^2}}$$

where the independent variable θ is replaced by $u=\cos\theta$.

Since the original spinor ψ is replaced by $\omega=U^{-1}\psi\cdot r\sqrt{\sin\theta}$, it seems reasonable to require that $\omega/\sqrt{\sin\theta}$ be finite. Therefore we impose the boundary condition that $\chi_1/\sqrt{\sin\theta}\equiv w_1$ and $\chi_2/\sqrt{\sin\theta}\equiv w_2$ be finite at $u=1$ and $u=-1$. Equations to be satisfied by χ_1 and χ_2 individually turn out to be

$$\left[(1-u^2)\frac{d^2}{du^2}-u\frac{d}{du}+\frac{mu-\beta-(m-\beta u)^2}{1-u^2}+\lambda^2\right]\chi_1=0$$

$$\left[(1-u^2)\frac{d^2}{du^2}-u\frac{d}{du}+\frac{\beta-mu-(m-\beta u)^2}{1-u^2}+\lambda^2\right]\chi_2=0\,.$$

These equations are Riemann's P-equations. Referring to the formulas on Riemann's P-function[1], we have

$$w_1=\frac{\chi_1}{\sqrt{\sin\theta}}=P\left\{\begin{matrix} 1 & -1 & \infty & \\ \frac{m-\beta}{2}-\frac{1}{4} & \frac{m+\beta}{2}+\frac{1}{4} & \frac{1}{2}+\sqrt{\lambda^2+\beta^2} & u \\ \frac{\beta-m}{2}+\frac{1}{4} & -\frac{m+\beta}{2}-\frac{1}{4} & \frac{1}{2}-\sqrt{\lambda^2+\beta^2} & \end{matrix}\right\}$$

$$w_2=\frac{\chi_2}{\sqrt{\sin\theta}}=P\left\{\begin{matrix} 1 & -1 & \infty & \\ \frac{m-\beta}{2}+\frac{1}{4} & \frac{m+\beta}{2}-\frac{1}{4} & \frac{1}{2}+\sqrt{\lambda^2+\beta^2} & u \\ \frac{\beta-m}{2}-\frac{1}{4} & -\frac{m+\beta}{2}+\frac{1}{4} & \frac{1}{2}-\sqrt{\lambda^2+\beta^2} & \end{matrix}\right\}.$$

Said boundary condition requires that each of w_1 and w_2 have at least one positive index at $u=1$ and $u=-1$, therefore there arise four cases where both w_1 and w_2 have positive indices at $u=1$ and $u=-1$.

1° $m+\beta>\frac{1}{2}$, $m-\beta>\frac{1}{2}$,

$$w_1=(1-u)^{(m-\beta)/2-1/4}(1+u)^{(m+\beta)/2+1/4}F\left\{m+\frac{1}{2}+\sqrt{\lambda^2+\beta^2},\right.$$
$$\left.m+\frac{1}{2}-\sqrt{\lambda^2+\beta^2},\ \frac{3}{2}+m+\beta\,;\ \frac{u+1}{2}\right\}.$$

The condition that the hypergeometric function F be finite at $u=1$ leads to the condition

$$m+\frac{1}{2}-\sqrt{\lambda^2+\beta^2}=-k\,,\qquad k=0,\ 1,\ 2,\ \cdots,$$

or

$$\lambda^2=\left(m+\frac{1}{2}+k\right)^2-\beta^2\,.$$

2° $m+\beta>\frac{1}{2}$, $\beta-m>\frac{1}{2}$,

$$w_1=(1-u)^{(\beta-m)/2+1/4}(1+u)^{(m+\beta)/2+1/4}F\Big\{\beta+1+\sqrt{\lambda^2+\beta^2},$$

$$\beta+1-\sqrt{\lambda^2+\beta^2},\ \frac{3}{2}+m+\beta\,;\ \frac{u+1}{2}\Big\}$$

$$\lambda^2=(\beta+1+k)^2-\beta^2, \qquad k=0,\ 1,\ 2,\ \cdots.$$

3° $m+\beta<-\dfrac{1}{2}, \qquad \beta-m>\dfrac{1}{2},$

$$w_1=(1-u)^{(\beta-m)/2+1/4}(1+u)^{-(m+\beta)/2-1/4}F\Big\{-m+\frac{1}{2}+\sqrt{\lambda^2+\beta^2},$$

$$-m+\frac{1}{2}-\sqrt{\lambda^2+\beta^2},\ \frac{1}{2}-m-\beta\,;\ \frac{u+1}{2}\Big\}$$

$$\lambda^2=\left(-m+\frac{1}{2}+k\right)^2-\beta^2, \qquad k=0,\ 1,\ 2,\ \cdots.$$

4° $m+\beta<-\dfrac{1}{2}, \qquad m-\beta>\dfrac{1}{2},$

$$w_1=(1-u)^{(m-\beta)/2-1/4}(1+u)^{-(m+\beta)/2-1/4}F\Big\{-\beta+\sqrt{\lambda^2+\beta^2},$$

$$-\beta-\sqrt{\lambda^2+\beta^2},\ \frac{1}{2}-m-\beta\,;\ \frac{u+1}{2}\Big\}$$

$$\lambda^2=(-\beta+k)^2-\beta^2, \qquad k=0,\ 1,\ 2,\ \cdots.$$

In short we have

$$\lambda^2=\left(s+\frac{1}{2}+k\right)^2-\beta^2, \qquad k=0,\ 1,\ 2,\ \cdots,$$

$$s=\max\left\{|m|,\ \left|\beta+\frac{1}{2}\right|\right\}.$$

§ 6. Energy formula.

Replacing the operator Q by one of its eigenvalues, λ, we put

$$\mathscr{F}=-E-\frac{\alpha}{r}-\alpha_0+\alpha_3 p_r+i\alpha_3\alpha_0\frac{\lambda}{r}$$

$$=\alpha_3\Big\{-\alpha_3 E-\alpha_3\alpha_0+p_r+\frac{i\lambda\alpha_0-\alpha\alpha_3}{r}\Big\}.$$

We transform the coefficient of $1/r$ in the bracket into a diagonal form by the matrix V defined by

$$V=\lambda+\rho-i\alpha\alpha_0\alpha_3, \quad V^{-1}=(\lambda+\rho+i\alpha\alpha_0\alpha_3)/2\rho(\rho+\lambda), \quad \rho=\sqrt{\lambda^2-\alpha^2}$$

getting
$$V^{-1}(i\lambda\alpha_0-\alpha\alpha_3)V=i\rho\alpha_0.$$

Further we have

$$V^{-1}(\alpha_3 E+\alpha_3\alpha_0)V=\Big(\frac{\lambda}{\rho}\alpha_3+\frac{i\alpha}{\rho}\alpha_0\Big)E+\alpha_3\alpha_0\,,$$

consequently

$$V^{-1}\alpha_3\mathscr{T}V=p_r+\frac{i}{r}\rho\alpha_0-\Big(\frac{\lambda}{\rho}\alpha_3+\frac{i\alpha}{\rho}\alpha_0\Big)E-\alpha_3\alpha_0$$

$$=\begin{pmatrix} p_r+\dfrac{i\rho}{r}-\dfrac{i\alpha E}{\rho} & \Big(-\dfrac{\lambda}{\rho}E+1\Big)\sigma_3 \\ \Big(-\dfrac{\lambda}{\rho}E-1\Big)\sigma_3 & p_r-\dfrac{i\rho}{r}+\dfrac{i\alpha E}{\rho}\end{pmatrix}$$

where σ_3 is one of Pauli matrices. Radial eigenfunctions $\phi=V^{-1}\omega$ are to satisfy

$$\Big(p_r+\frac{i\rho}{r}-\frac{i\alpha E}{\rho}\Big)\phi_1+\Big(-\frac{\lambda}{\rho}E+1\Big)\sigma_3\phi_2=0$$

$$\Big(-\frac{\lambda E}{\rho}-1\Big)\sigma_3\phi_1+\Big(p_r-\frac{i\rho}{r}+\frac{i\alpha E}{\rho}\Big)\phi_2=0\,.$$

Elimination of ϕ_2 leads to a Whittaker equation

$$\Big[\frac{d^2}{dr^2}+\frac{\rho-\rho^2}{r^2}+\frac{2\alpha E}{r}+E^2-1\Big]\phi_1=0\,.$$

A change of variable $2r\sqrt{1-E^2}=x$ brings this equation into the standard form

$$\Big[\frac{d^2}{dx^2}-\frac{1}{4}+\frac{\kappa}{x}+\frac{1/4-\mu^2}{x^2}\Big]\phi_1=0$$

$$\kappa=\frac{\alpha E}{\sqrt{1-E^2}},\qquad \mu=\Big|\rho-\frac{1}{2}\Big|$$

and gives us the solution

$$\phi_1=M_{\kappa,\mu}(x)$$
$$=x^{1/2+\mu}e^{-x/2}\Big\{1+\frac{1/2+\mu-\kappa}{1!(2\mu+1)}x+\frac{(1/2+\mu-\kappa)(3/2+\mu-\kappa)}{2!(2\mu+1)(2\mu+2)}x^2+\cdots\Big\}$$

The condition that ϕ_1 be finite for $x\to\infty$ leads to the energy formula

$$\kappa=\frac{1}{2}+\mu+n\,,\qquad n=0,\,1,\,2,\,\cdots,$$

or
$$E=\Big[1+\frac{\alpha^2}{(1/2+\mu+n)^2}\Big]^{-1/2}$$

which is very similar to that of the pure Coulomb field.

The existence of a magnetic monopole has been proposed by Dirac[3] but has not been confirmed to date, despite boundless efforts of many researchers. It seems remarkable that, in a field produced by a magnetic monopole and a point charge, all of the Hamilton-Jacobi equations, the Schrödinger equation and the Dirac equation of a charged particle are soluble by separation of variables.

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