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Theorems of Korovkin Type in an Ordered Vector Space with a Locally Convex Topology

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§ 1. Introduction

Let E be an ordered vector space. Denote by $\boldsymbol{R}^Y$ the ordered vector space of all real-valued functions defined on a set Y. Given a linear subspace F of E such that E is F^+-bounded, i. e. for each $f\in E$ there is $g\in F^+$ satisfying $-g\leqq f\leqq g$. Let A be a positive linear map from F into $\boldsymbol{R}^Y$. In a preceding paper [3] the author has proved that for a net $(L_i)_{i\in I}$ of monotone maps from E into $\boldsymbol{R}^Y$ pointwise convergence of $(L_i g)_{i\in I}$ for all $g\in F$ implies pointwise convergence of $(L_i f)_{i\in I}$ for all elements $f\in E$ which are $A(F)$-affine in the sence of [3]. The set of $A(F)$-affine elements is a linear subspace of E containing F and coincides with E if and only if the $A(F)$-boundary of Y equals Y. In the proof, the assumption that E is F^+-bounded is essential.

In this paper, using a Hahn-Banach type theorem of Anger and Lembcke ([1]) where sublinear functionals are replaced by hypolinear ones and endowing E with a locally convex topology, we shall obtain the analogous theorems without the assumption that E is F^+-bounded.

Moreover, using these results, we shall define an integral with respect to a finitely additive, positive real-valued set function on a ring of sets.

Further, in case Y is a compact Hausdorff space and A is a continuous positive linear map from E into $C(Y)$, the analogous theorems with uniform convergence instead of pointwise convergence will be obtained.

§ 2. The $A(C)$-boundary and $A(C)$-affine elements

Let E be an ordered vector space with a locally convex topology and $\mathscr{B}$ a fundamental system of convex symmetric neighborhoods of 0. Suppose that C is a convex cone in E. An element $f\in E$ is called almost upper C-bounded if it satisfies the following condition:

(B_1) For every $V\in\mathscr{B}$ there are $u\in V$ and $g\in C$ with $f\leqq g+u$. The set of all almost upper C-bounded elements in E is denoted by C_u^*. Let Y be a set and A be a monotone sublinear map from C into $\boldsymbol{R}^Y$. For $f\in E$ and $y\in Y$, define

$$\overline{Af}(y):=\sup_{V\in\mathscr{B}}\inf\{Ag(y):\ u\in V,\ g\in C,\ f\leqq g+u\}$$

if $f\in C_u^*$ and $\overline{A}f(y):=+\infty$ elsewhere. Denote $\overline{A}f$ by the function: $y\mapsto\overline{A}f(y)$.

In the sequel we shall assume that

(B_2) $\overline{A}f(y)>-\infty$ for every $f\in E$ and every $y\in Y$. Easily we have the following properties of the envelopes.

PROPOSITION 2.1. *The map: $f\mapsto\overline{A}f$ from E into $\bar{\boldsymbol{R}}^Y$ with $\bar{\boldsymbol{R}}=(-\infty, +\infty]$ has the following properties:*

(i) $\overline{A}(f+g)\leqq\overline{A}f+\overline{A}g$,

(ii) $\overline{A}(\lambda f)=\lambda\overline{A}f$ $(\lambda\in\boldsymbol{R}^+)$,

(iii) *$f\leqq g$ implies $\overline{A}f\leqq\overline{A}g$,*

(iv) *$\overline{A}g\leqq Ag$ for every $g\in C$.*

Consequently, the map: $f\mapsto\overline{A}f(y)$ is a monotone hypolinear functional on E. Recall that a hypolinear functional on E means a sublinear functional on E which may attain the value $+\infty$.

REMARK 2.1. If A is a monotone sublinear map from E into $\boldsymbol{R}^Y$ and $u\mapsto Au(y)$ is locally upper bounded, $\overline{A}f(y)>-\infty$ for every $f\in E$. Indeed, for $\varepsilon>0$ there exist $M>0$ and $W\in\mathcal{B}$ satisfying $Au(y)\leqq M$ for every $u\in W$. Assume that $f\leqq g+u$ where $g\in C$ and $u\in W$. Since $Af(y)\leqq Ag(y)+Au(y)\leqq Ag(y)+M$, it holds that $Af(y)-M\leqq\inf\{Ag(y): f\leqq g+u,\ g\in C,\ u\in W\}$ and hence $\overline{A}f(y)\geqq Af(y)-M>-\infty$.

PROPOSITION 2.2. *The hypolinear functional: $f\mapsto\overline{A}f(y)$ is lower semicontinuous on E for each $y\in Y$.*

PROOF. Given $f\in E$ and an arbitrary real number α. Assume that $\overline{A}f(y)>\alpha$. We shall show the existence of an open set G containing f such that $\overline{A}g(y)>\alpha$ for all $g\in G$. If $f\in E\backslash C_u^*$, we may put $G=E\backslash C_u^*$. Indeed, $E\backslash C_u^*$ is open and $\overline{A}g(y)=+\infty$ at $g\in E\backslash C_u^*$. If $f\in C_u^*$, there is $V\in\mathcal{B}$ such that

$$\inf\{Ag(y): g\in C,\ u\in V,\ f\leqq g+u\}>\alpha$$

Take $W\in\mathcal{B}$ satisfying $W+W\subset V$. Then

$$\inf\{Ah(y): h\in C,\ v\in W,\ f+w\leqq h+v\}$$

$$\geqq\inf\{Ah(y): h\in C,\ u\in V,\ f\leqq u+h\}$$

and hence $\overline{A}(f+w)(y)>\alpha$ for each $w\in W$. It suffices to take $G=f+W$.

Let y be an element of Y. The set of all positive continuous linear functional μ on E satisfying $\mu(g)\leqq Ag(y)$ for every $g\in C$ is denoted by $M_y(C)$.

LEMMA 2.1. *For each $f\in E$ and each $y\in Y$ it holds that $(-\overline{A}(-f)(y), \overline{A}f(y))\subset\{\mu(f): \mu\in M_y(C)\}\subset[-\overline{A}(-f)(y), \overline{A}f(y)]$. Here, if $-\overline{A}(-f)(y)=\overline{A}f(y)$, we use the convention $(-\overline{A}(-f)(y), \overline{A}f(y))=\{\overline{A}f(y)\}$.*

PROOF. By Propositions 2.1 and 2.2, the map: $h \mapsto \overline{A}h(y)$ is a lower semi-continuous hypolinear functional on E. Using Proposition 3.2 in [1] there exists, for every $\alpha \in (-\overline{A}(-f)(y), \overline{A}f(y))$, $\mu \in E'$ such that

$$\mu(f)=\alpha \quad \text{and} \quad \mu(h) \leqq \overline{A}h(y) \quad \text{for each} \quad h \in E.$$

Since $g \leqq 0$ implies $\mu(h) \leqq \overline{A}h(y) \leqq \overline{A}0(y)=0$, μ is positive. If $g \in C$, it holds that $\mu(g) \leqq \overline{A}g(y) \leqq Ag(y)$. Consequently $\mu \in M_y(C)$. Further, let $\mu \in M_y(C)$. If $f \in E \backslash C_u^*$, then $\mu(f) \leqq +\infty = \overline{A}f(y)$. Assume that $f \in C_u^*$. For each $\varepsilon > 0$ there exists $V \in \mathscr{B}$ such that $\mu(u) < \varepsilon$ for every $u \in V$ since μ is continuous. The inequality $f \leqq g+u (g \in C,\ u \in V)$ implies $\mu(f) \leqq \mu(g)+\mu(u) \leqq Ag(y)+\varepsilon$ and hence

$$\mu(f) \leqq \inf\{Ag(y) : f \leqq g+u,\ g \in C,\ u \in V\}+\varepsilon \leqq \overline{A}f(y)+\varepsilon .$$

Consequently $\mu(f) \leqq \overline{A}f(y)$ for every $f \in E$. Replacing f by $-f$, it follows that $\mu(f) \geqq -\overline{A}(-f)(y)$.

An element f of E is called $A(C)$-affine if $\overline{A}f = -\overline{A}(-f)$ on Y. Immediately the following corollary follows from the definition and Lemma 2.1.

COROLLARY 2.1. *An element f of E is $A(C)$-affine if and only if the function: $\mu \mapsto \mu(f)$ is constant on $M_y(C)$ for each $y \in Y$.*

The set of all $y \in Y$ for which $M_y(C)$ consists of one element is denoted by $\partial(A(C))$ and called the $A(C)$-boundary. By the definition and Lemma 2.1 we have the following Proposition 2.3 and Corollary 2.2.

PROPOSITION 2.3. *A point y in Y belongs to $\partial(A(C))$ if and only if $\overline{A}f(y) = -\overline{A}(-f)(y)$ for every $f \in E$.*

COROLLARY 2.2. *$\partial(A(C))=Y$ if and only if every element f of E is $A(C)$-affine.*

EXAMPLE 1. Let X be a locally compact Hausdorff space and put $E=C_0(X)$ (with the uniform norm) and $Y=X$. Then it is obvious that the $I(C)$-boundary with respect to the identity map I on E is equal to the Choquet boundary of X with respect to C: the set of all points $x \in X$ for which $M_x(C)=\{\varepsilon_x\}$.

EXAMPLE 2. Let X and Y be locally compact Hausdorff spaces. Assume that X has at least $n+1$ points and F is a linear subspace of $C_0(Y)$ satisfying the following assumption:

Given any n distinct points of X, there exists $g \in F$ such that $g(x) \geqq 0$ and $g(x)=0$ exactly when $x=x_i$ for $i=1, 2, \cdots, n$.

Denote by A a positive linear map from $C_0(X)$ into $C_0(Y)$ of the form

$$(Ag)(y) := \sum_{i=1}^{n} \phi_i(y) g(\varphi_i(y)) \qquad (g \in C_0(X),\ y \in Y),$$

where $\psi_i \in C_0^+(Y)$ and φ_i is a continuous map from Y into X for $i=1, \cdots, n$. Since A is a positive linear map form $C_0(X)$ into $C_0(Y)$ and for every $y \in Y$ the function $g \mapsto (Ag)(y)$ is continuous at 0 in $C_0(X)$, it follows from Remark 2.1 that $\overline{Af(y)} > -\infty$ for every $f \in E$ and every $y \in Y$. Further we have $\delta(A(F)) = Y$ (cf. Proposition 2.3 in [3]).

EXAMPLE 3. Let E be an ordered vector space with a locally convex topology and F a subspace of E. A positive continuous linear functional A on F is considered as a positive continuous linear map from E into $\boldsymbol{R}^Y$ where Y consists of one point y. The point y is contained in $\delta(A(F))$ if and only if the linear functional A can be uniquely extended to a positive continuous linear functional on E.

§ 3. Pointwise convergence

Let E be an ordered vector space with a locally convex topology and Y be a set. A net $(L_i)_{i \in I}$ of maps from E into $\boldsymbol{R}^Y$ is said to satisfy the condition (I) (resp. (II)) if it has the following properties (s) and (p):

(s) L_i is monotone, subadditive (resp. superadditive) and satisfies $L_i(0)=0$ for all $i \in I$,

(p) for each $\varepsilon > 0$ and for each $y \in Y$ there exists $V \in \mathscr{B}$ such that $L_i u(y) < \varepsilon$ (resp. $L_i u(y) > -\varepsilon$) for all $u \in V$ and all $i \in I$.

THEOREM 3.1. *Let $(L_i)_{i \in I}$ be a net satisfying the condition (I) and C be a convex cone in E. Suppose that $\overline{\lim_i} L_i g(y) \leqq Ag(y)$ for every $g \in C$ and every $y \in Y$. Then the net $(L_i f(y))_{i \in I}$ in $\boldsymbol{R}$ converges to $\overline{Af}(y)$ for every affine element $f \in E$ and every $y \in Y$.*

PROOF. Let f be an affine element in E. Since $\overline{Af}(y) = -\overline{A(-f)}(y)$ for all $y \in Y$, it holds that $\overline{Af}(y) < \infty$. Let y be a point of Y and given $\varepsilon > 0$. Then, there is $V \in \mathscr{B}$ satisfying $L_i u(y) < \varepsilon$ for every $u \in V$ by the assumption. Since $\inf\{\overline{Ag}(y) : f \leqq g+u,\ u \in V,\ g \in C\} < \overline{Af(y)} + \varepsilon$, there are $g \in C$ and $u \in V$ such that $f \leqq g+u$ and $Ag(y) < \overline{Af}(y) + \varepsilon$. From the assumption it follows that $L_i g(y) < Ag(y) + \varepsilon$ for all $i \geqq i_0$ for sufficiently great i_0. Consequently

$$L_i f(y) \leqq L_i g(y) + L_i u(y) < Ag(y) + \varepsilon + \varepsilon < \overline{Af}(y) + 3\varepsilon,$$

whence

$$\overline{\lim_i} L_i f(y) \leqq \overline{Af}(y). \tag{3.1}$$

Replacing f by $-f$ we have also

$$\overline{\lim_i} L_i(-f)(y) \leqq \overline{A(-f)}(y).$$

By the subadditivity of L_i and $L_i(0)=0$, we have

$$-\overline{A(-f)}(y) \leqq -\overline{\lim_i} L_i(-f)(y) \leqq -\overline{\lim_i}(-L_i f(y)) = \underline{\lim_i} L_i f(y). \tag{3.2}$$

From (3.1), (3.2) and $\overline{A}f(y)=-\overline{A}(-f)(y)$, it follows that $\lim_i L_i f(y)=\overline{A}f(y)$.

Let F be a linear subspace of E and A a positive linear map from F into $\boldsymbol{R}^Y$. Then it holds that

(3.3) $$-\overline{A}(-f)=\inf_{V\in\mathfrak{B}} \sup\{Ag : f\geqq g-u,\ g\in F,\ u\in V\} \quad \text{for every } f\in E.$$

THEOREM 3.2. *F be a linear subspace of E and A a positive linear map from F into $\boldsymbol{R}^Y$. Assnme that a net $(L_i)_{i\in I}$ satisfies the condition (I) or (II). If $\lim_i L_i g(y)=Ag(y)$ for all $g\in F$ and all $y\in Y$, then $\lim_i L_i f(y)=\overline{A}f(y)$ for every affine element f and every $y\in Y$.*

PROOF. If a net $(L_i)_{i\in I}$ satisfies (I), it follows from Theorem 3.1 that

$$\lim_i L_i f(y)=\overline{A}f(y) \quad \text{for every affine element } f.$$

Next, assume that a net $(L_i)_{i\in I}$ satisfies (II). Using the superadditivity of L_i, the condition (p) and (3.3) we have

(3.4) $$\underline{\lim}_i L_i f(y)\geqq=-\overline{A}(-f)(y) \quad \text{for every } f\in E.$$

Replacing f by $-f$,

$$\underline{\lim}_i L_i(-f)(y)\geqq-\overline{A}f(y)\,,$$

whence

(3.5) $$\overline{A}f(y)\geqq-\underline{\lim}_i L_i(-f)(y)\geqq\overline{\lim}_i L_i f(y)\,.$$

From (3.4) and (3.5) it follows that

$$\lim_i L_i f(y)=\overline{A}f(y) \quad \text{for every affine element } f.$$

COROLLARY 3.1. *Suppose that $\delta(A(C))=Y$ in addition to the assumptions of Theorem 3.2. Then $(L_i f(y))_{i\in I}$ converges to $\overline{A}f(y)$ for every $f\in E$ and every $y\in Y$.*

PROOF. This is an immediate consequence of Theorem 3.2 and Corollary 2.2.

§4. Uniform convergence

Let E be an ordered vector space with a locally convex topology and Y a set. Suppose that a positive linear map A from E into $\boldsymbol{R}^Y$ satisfies the following condition (B_3):

(B_3) for each $y\in Y$ the function: $f\mapsto Af(y)$ from E into $\boldsymbol{R}$ is continuous at 0 in E.

Then, we have

PROPOSITION 4.1. For the envelop $\overline{A}f$ with respect to a linear subspace F of E it holds that

$$-\overline{A(-f)}(y) \leqq Af(y) \leqq \overline{Af}(y) \quad \text{for each } f \in E \text{ and each } y \in Y.$$

PROOF. Let y be a point of Y and f an element of E. For $\varepsilon>0$, there exists $V \in \mathcal{B}$ such that $Au(y)<\varepsilon$ for all $u \in V$. The inequality $f \leqq g+u (g \in F, u \in V)$ implies

$$Ag(y) \geqq Af(y)-Au(y)>Af(y)-\varepsilon$$

and hence

$$\inf\{Ag(y): f \leqq g+u,\ g \in F,\ u \in V\} \geqq Af(y)-\varepsilon.$$

Consequently $\overline{Af}(y) \geqq Af(y)-\varepsilon$. Since ε is an arbitrary positive number, it holds that $\overline{Af}(y) \geqq Af(y)$ for all $f \in E$. Replacing f by $-f$, we have

$$A(-f)(y) \leqq \overline{A(-f)}(y),$$

whence

$$-\overline{A(-f)}(y) \leqq Af(y) \leqq \overline{Af}(y).$$

In this section we assume that Y is a compact Hausdorff space and a positive linear map A from E into $C(Y)$ satisfies the condition (B_3). Further, a net $(L_i)_{i \in I}$ of maps from E into $C(Y)$ is said to satisfy the condition (III) (resp. (IV)) if it has the following properties (s) and (u):

(s) L_i is monotone, subadditive (resp. superadditive) and satisfies $L_i(0)=0$ for all $i \in I$,

(u) for each $\varepsilon>0$ there exists $V \in \mathcal{B}$ such that $L_i u(y)<\varepsilon$ (resp. $L_i u(y)>-\varepsilon$) for all $i \in I$, for all $u \in V$ and for all $y \in Y$.

THEOREM 4.1. *Suppose that a net $(L_i)_{i \in I}$ of maps from E into $C(Y)$ satisfies the condition* (III) *or* (IV) *and F is a linear subspace of E. If $(L_i g)_{i \in I}$ converges uniformly to Ag for all $g \in F$, then $(L_i f)_{i \in I}$ also converges uniformly to Af for each $A(F)$-affine element $f \in E$.*

PROOF. Let f be an $A(F)$-affine element. Then, since the map A satisfies the condition (B_3), it holds that $\overline{Af}=-\overline{A(-f)}=Af$. As a similar method in the proof of Theorem 3.2, it suffices to prove in case that $(L_i)_{i \in I}$ satisfies condition (III). By (u) for each $\varepsilon>0$ there exists $V \in \mathcal{B}$ such that

$$L_i u(y)<\varepsilon \quad (i \in I,\ u \in V,\ y \in Y). \tag{4.1}$$

Let y be a point of Y. For $\varepsilon>0$, we can find $g_y \in F$ and $u_y \in V$ satisfying

$$f \leqq g_y+u_y \quad \text{and} \quad Ag_y(y)<\overline{Af}(y)+\varepsilon=Af(y)+\varepsilon$$

and hence, by continuity, find a neighborhood U_y of y with $Ag_y<Af+\varepsilon$ on U_y. Since Y is compact, there are finite points $y_1, \cdots, y_n$ in Y with $Y \subset \bigcup_{i=1}^{n} U_{y_i}$. Put $g_{y_i}=g_i$. Then it holds that

$$\min_{1 \leqq i \leqq n} Ag_i<Af+\varepsilon \quad \text{on } Y. \tag{4.2}$$

From the assumption, there exists an index i_0 such that for all $i \geqq i_0$

$$L_i g_j(y) < A g_j(y) + \varepsilon \quad (y \in Y,\ j=1, \cdots, n). \tag{4.3}$$

The relations (4.1), (4.2) and (4.3) imply

$$L_i f \leqq \min_{1 \leqq j \leqq n} L_i(g_j + u_j) \leqq \min_{1 \leqq j \leqq n} (L_i g_j + L_i u_j)$$
$$\leqq \min_{1 \leqq j \leqq n} L_i g_j + \varepsilon \leqq \min_{1 \leqq j \leqq n} A g_j + 2\varepsilon \leqq Af + 3\varepsilon .$$

Replacing f by $-f$, we have

$$L_i(-f) \leqq \overline{A(-f)} + 3\varepsilon \quad \text{for all } i \geqq i_1$$

and hence

$$L_i f \geqq -L_i(-f) \geqq -\overline{A(-f)} - 3\varepsilon = Af - 3\varepsilon.$$

Therefore $(L_i f)_{i \in I}$ converges uniformly to Af.

Let F be a linear subspace of E. We denote by Kor(F, A) the set of all $f \in E$ satisfying the following assertion:

For every net $(L_i)_{i \in I}$ from E into $C(Y)$ which satisfies the condition (III), $(L_i f)_{i \in I}$ converges uniformly to Af if $(L_i f)_{i \in I}$ converges uniformly to Ag for all $g \in F$.

THEOREM 4.2. *An element $f \in E$ belongs to* Kor(F, A) *if and only if it is $A(F)$-affine.*

PROOF. Let f be $A(F)$-affine. Then, from Theorem 4.1 and the definition of Kor(F, A) it follows that $f \in$ Kor(F, A). Conversely, we can prove by the same method as Theorem 3.4 in [3] that $f \in$ Kor(F, A) is $A(F)$-affine.

§ 5. An integration with respect to a finitely additive set function

Let $\mathcal{A}$ be a ring of subsets of X and m be a finitely additive set function on $\mathcal{A}$ with $0 \leqq m(B) < \infty$ for each $B \in \mathcal{A}$. Further, assume that there is a constant $M > 0$ such that $m(B) \leqq M$ for every $B \in \mathcal{A}$. In this section we shall consider an integration with respect to m. Denote by E the set of all bounded real-valued function f on X satisfying the following condition (i_0):

(i_0) for each $\varepsilon > 0$ there exists $B \in \mathcal{A}$ such that $|f| \leqq \varepsilon$ on B^c, where B^c is the complement of B.

Then E is an ordered vector space with the usual order and also a normed space with the sup-norm. Put

$$F := \left\{ \sum_{i=1}^{n} \alpha_i 1_{A_i}^{1)} : \alpha_i \in \boldsymbol{R},\ A_i \in \mathcal{A},\ n \in \boldsymbol{N} \right\},$$

$$Ag := \sum_{i=1}^{n} \alpha_i m(A_i) \quad \text{for} \quad g = \sum_{i=1}^{n} \alpha_i 1_{A_i} \in F .$$

Then A is a positive linear functional on a linear subspace F of E and satisfies

1) We denote by 1_B the characteristic function of a set B.

$$|Ag|\leqq M\|g\| \quad \text{for every } g\in F.$$

Put

$$\overline{A}f:=\sup_{\varepsilon>0}\inf\{Ag:\ f\leqq g+u,\ g\in F,\ u\in V_\varepsilon\}$$

for each $f\in E$, where $V_\varepsilon=\{u\in E:\ \|u\|<\varepsilon\}$.

PROPOSITION 5.1. *$\overline{A}f>-\infty$ for every $f\in E$.*

PROOF. Let f be an element of E. From the assumption there is, for $\varepsilon>0$, $A_\varepsilon\in\mathcal{A}$ such that $|f|\leqq\varepsilon$ on A_ε^c. Put $h:=\max(\min(f,\varepsilon),-\varepsilon)$. Then $h\in V_\varepsilon$ and $f\leqq\|f\|1_A+h$. Suppose that $f\leqq g+u$ with $g\in F$ and $u\in V_\varepsilon$. Then

$$f\leqq\min(\|f\|1_{A_\varepsilon},g)+\max(u,h)$$

and $\max(u,h)\in V_\varepsilon$. Put

$$g_1:=\min(\|f\|1_{A_\varepsilon},g).$$

Then $g_1\in F$ and $Ag_1\leqq A(\|f\|1_A)$. Hence

$$\inf\{Ag:\ f\leqq g+u,\ g\in F,\ u\in V_\varepsilon\}$$
$$=\inf\{Ag:\ f\leqq g+u,\ g\in F,\ g\leqq\|f\|1_{A_\varepsilon},\ u\in V_\varepsilon\}.$$

Suppose that $f\leqq g+u$ with $g\in F$, $g\leqq\|f\|1_{A_\varepsilon}$ and $u\in V_\varepsilon$. If $\|g\|=\sup\{g(x):x\in X\}$, it holds that $\|g\|\leqq\|f\|$ and hence $Ag\geqq-M\|g\|\geqq-M\|f\|$. If $\|g\|=-\inf\{g(x):x\in X\}$, the relation

$$g\geqq f-u\geqq f-\varepsilon\geqq-\|f\|-\varepsilon$$

implies

$$Ag\geqq-M\|g\|\geqq-M(\|f\|+\varepsilon).$$

Therefore $Ag\geqq-M(\|f\|+\varepsilon)$ and hence

$$\overline{A}f\geqq\sup\{-M(\|f\|+\varepsilon):\ \varepsilon>0\}=-M\|f\|>-\infty. \tag{5.1}$$

REMARK 5.1. $-\overline{A}(-f)\leqq M\|f\|$ for every $f\in E$ by (5.1).

If $f\in E$ is $A(F)$-affine, we call f integrable with respect to m and write

$$m(f):=\overline{A}f=-\overline{A}(-f).$$

Then $f\mapsto m(f)$ is a positive linear functional on the linear space of integrable functions and the relation (5.1) and Remark 5.1 imply

$$|m(f)|\leqq M\|f\| \quad \text{for every integrable function } f.$$

We consider a partition Δ of $B\in\mathcal{A}$:

$$B=\bigcup_{i=1}^n B_i,\quad B_i\cap B_j=\varnothing\quad(i\neq j),\quad B_i\in\mathcal{A}. \tag{5.2}$$

Denote by $\mathcal{G}$ the set of all pairs (B,Δ) of $B\in\mathcal{A}$ and a partition Δ of B. For two pairs (A_1,Δ_1) and (A_2,Δ_2) in $\mathcal{G}$. We write $(A_1,\Delta_1)\leqq(A_2,\Delta_2)$ if $A_1\subset A_2$ and Δ_2 is a refinement of Δ_1. $\mathcal{G}$ is directed by the order relation. For a pair $(\Delta,B)=v$ given by (5.2) and for $f\in E$, define

$$M_v f := \sum_{i=1}^{n} (\sup_{x\in B_i} f(x))m(B_i)$$

and

$$N_v f := \sum_{i=1}^{n} (\inf_{x\in B_i} f(x))m(B_i) .$$

Then M_v (resp. N_v) is monotone and sublinear (resp. superlinear) functional on E. Further it holds that

$$|M_v g| \leqq M\|g\| \quad \text{and} \quad |N_v g| \leqq M\|g\| \quad \text{for every } g\in E.$$

Immediately we have

$$\lim_v M_v g = Ag \quad \text{and} \quad \lim_v N_v g = Ag \quad \text{for every } g\in F.$$

From Theorem 3.2 it follows that

$$\lim_v M_v f = \overline{Af} = \lim_v N_v f \quad \text{for each } A(F)\text{-affine element } f.$$

Thus we have

PROPOSITION 5.2. *For every $A(F)$-affine element f $\lim_v M_v f$ and $\lim_v N_v f$ exist and both of them equal to $\overline{Af}$.*

Conversely, we have

PROPOSITION 5.3. *If $\lim_v M_v f = \lim_v N_v f$, f is $A(F)$-affine.*

PROOF. Put $\lim_v M_v f = \lim_v N_v f = k$. For $\varepsilon>0$, there is $v\in\mathcal{G}$ with $v=(A_0, \Delta_0)$ such that

$$k-\varepsilon < N_v f \leqq M_v f < k+\varepsilon .$$

Since f is an element of E, there exists, for each $\delta>0$, $B_\delta\in\mathcal{A}$ satisfying $|f|\leqq\delta$ on B_δ^c. Put $u=\min(|f|, \delta)\in V_\delta$. Take a partition Δ_1 of B_δ and $(C, \Delta_2)\in\mathcal{G}$ satisfying $(A_0, \Delta_0)\leqq(C, \Delta_2)$ and $(B_\delta, \Delta_1)\leqq(C, \Delta_2)$. For $(C, \Delta_2)=v_2$ given by

$$C=\sum_{i=1}^{n} C_i, \quad C_i\cap C_j=\varnothing \quad (i\neq j), \quad C_i\in\mathcal{A} .$$

Put $\alpha_i=\sup_{x\in C_i} f(x)$ and $\beta_i=\inf_{x\in C_i} f(x)$. Then

$$f\leqq \sum_{i=1}^{n} \alpha_i 1_{C_i}+u, \quad A\Big(\sum_{i=1}^{n} \alpha_i 1_{C_i}\Big)=M_{v_2} f\leqq M_v f \tag{5.3}$$

and

$$f\geqq \sum_{i=1}^{n} \beta_i 1_{C_i}-u, \quad A\Big(\sum_{i=1}^{n} \beta_i 1_{C_i}\Big)=N_{v_2} f\geqq N_v f . \tag{5.4}$$

Suppose that $f\leqq g+w$ where $g\in F$, and $0\leqq w\in V_\delta$. Further, suppose that $g_1-w_1\leqq f$ where $g_1\in F$ and $0\leqq w_1\in V_\delta$. Then $g_1-w_1\leqq g+w$ and hence

$$g_1\leqq g+2\delta 1_{B_1} \quad \text{where} \quad \operatorname{Supp} g_1\subset B_1\in\mathcal{A} .$$

This implies $Ag_1\leqq Ag+A(2\delta 1_{B_1})\leqq Ag+2\delta M$. Using (5.3)

$$Ag_1 \leqq \inf\{Ag : f \leqq g+w,\ g \in F,\ w \in V_\delta\} + 2\delta M$$
$$\leqq M_{v_2} f + 2\delta M \leqq M_v f + 2\delta M\,.$$

From (5.4) and the previous inequality it follows that

$$N_v f \leqq N_{v_2} f \leqq \sup\{Ag_1 : f_1 \geqq g_1 - w_1,\ g_1 \in F,\ w_1 \in V_\delta\}$$
$$\leqq \inf\{Ag : f \leqq g+w,\ g \in F,\ w \in V_\delta\} + 2\delta M$$
$$\leqq M_v f + 2\delta M\,.$$

Converging δ to zero, we have

$$k-\varepsilon \leqq N_v f \leqq -\overline{A(-f)} \leqq \overline{Af} \leqq M_v f < k+\varepsilon\,.$$

Since ε is arbitrary, it holds that $\overline{Af} = -\overline{A(-f)}$.

References

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