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The Absolute Value of an Element of the Complexification of a Simplex Space

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§ 1. Introduction.

For an element $\tilde{f}$ of the complexification $\tilde{E}$ of a simplex space E, its absolute value $|\tilde{f}|$ can be considered in the second dual E'', which is an AM space. In this paper, we shall investigate the property of this absolute value. § 2 is devoted to the study of the Riesz separation property concerning the absolute values. In § 3, we examine the absolute value $|\tilde{f}|$ in case of a separable simplex space and show that $|\tilde{f}|$ satisfies the barycentric calculus and is an upper semi-continuous function, if it is considered as a function on the stump of the positive cone of the dual space endowed with the weak*-topology. These results will be applied to the study of spectral theory of positive operators in a simplex space [8].

§ 2. Complexification of a simplex space and Riesz separation property.

Let E be a simplex space over the real field, i. e. an ordered Banach space whose dual is a Banach lattice of type L [3, 5]. The stump X of the positive cone of the dual space E' is defined as the set

$$\{x\in E'\,;\ x\geqq 0,\ \|x\|\leqq 1\},$$

endowed with the weak*-topology. Then X is a simplex [2, § 28]. $A_0(X)$ [resp. $A_{C,0}(X)$] denote the space of real-valued [resp. complex-valued] continuous affine functions on X vanishing at 0. Then E may be identified with $A_0(X)$ [5, Th. 2.2].

For any $f_1, f_2\in E$, there exists in E'' the element

$$\bigvee_{0\leqq\theta<2\pi}((\cos\theta)f_1+(\sin\theta)f_2),$$

since a simplex space E may be considered as a subspace of the second dual space E'', which is an order complete Banach lattice [3]. Therefore, for any $\tilde{f}=f_1+if_2$, we can define the absolute value $|\tilde{f}|$ in E'' by

$$|\tilde{f}|=|f_1+if_2|=\bigvee_{0\leqq\theta<2\pi}((\cos\theta)f_1+(\sin\theta)f_2).$$

We also define the norm as usual

$$\|\tilde{f}\|_1=\sup_{x\in X}|\tilde{f}(x)|\,. \tag{1}$$

The complexification $\tilde{E}=E+iE$ of a simplex space E with the norm (1) is a complex Banach space, which may be identified with $A_{C,0}(X)$.

PROPOSITION. *For $\tilde{f}\in\tilde{E}$, the norm $\|\tilde{f}\|_1$ defined by (1) is equal to the norm $\||\tilde{f}|\|$ considered in the second dual E''.*

PROOF. It is clear that $\|\tilde{f}\|_1\leqq\||\tilde{f}|\|$ holds by the definition. Since the second dual E'' has an order unit $\mathbf{1}$ such as $\mathbf{1}(x)=\|x\|$ for $x\in E'$ with $x\geqq 0$, we have

$$|((\cos\theta)f_1+(\sin\theta)f_2)(x)|\leqq|\tilde{f}(x)|\leqq\|\tilde{f}\|_1\,\|x\|=\|\tilde{f}\|_1\cdot\mathbf{1}(x)$$

for $\tilde{f}=f_1+if_2$ and $x\in E'$ with $x\geqq 0$. This implies $|\tilde{f}|\leqq\|\tilde{f}\|_1\cdot 1$ in E''. Therefore $\||\tilde{f}|\|\leqq\|\tilde{f}\|_1$ holds //

For the absolute value, we have the following extension of the Riesz separation property.

THEOREM 1. *Let $\tilde{E}$ be the complexification of a simplex space E. If h_1, $h_2\geqq|\tilde{f}|$, $|\tilde{g}|$ in E'', for h_1, $h_2\in E$ and $\tilde{f}$, $\tilde{g}\in\tilde{E}$, then we can find an element $k\in E$ such that*

$$h_1,\ h_2\geqq k\geqq|\tilde{f}|,\ |\tilde{g}|\ \text{in } E''.$$

PROOF. Define $h_0(x)=\min\{h_1(x),\ h_2(x)\}$ and $f_0(x)=\max\{|f_1(x)+if_2(x)|,\ |g_1(x)+ig_2(x)|\}$ as functions on X, where $\tilde{f}=f_1+if_2$ and $\tilde{g}=g_1+ig_2$. Then f_0 [resp. h_0] is a continuous convex [resp. concave] function on X with $f_0(0)=h_0(0)=0$. Therefore, there exists $k\in A_0(X)$ such that $h_0\geqq k\geqq f_0$, since X is a simplex [2, Th. 28.6]. Then k is a desired one. //

A face of a convex set S in a vector space is a convex subset F such that if $\alpha x+(1-\alpha)y\in F$ with x, $y\in S$ and $0<\alpha<1$, then x, $y\in F$. The refinement of the above theorem is the following, which is the extension of [5, Th. 2.4].

THEOREM 2. *Let E be a simplex space, $\tilde{E}$ be its complexification, X be the stump of the positive cone of E' and F be a closed face of X. Suppose $\tilde{f}\in\tilde{E}$ and $f_0\in E$ satisfy $|\langle\tilde{f},x\rangle|\leqq\langle f_0,x\rangle$ for any $x\in F$. Then there exists $h\in E$ such that*

$$h\geqq f_0,\ |\tilde{f}|$$

$$\|h\|\leqq\max\{\|f_0\|,\ \|\tilde{f}\|\}$$

and

$$\langle h,x\rangle=\langle f_0,x\rangle\quad\text{for any } x\in F.$$

And moreover if we suppose in addition that $g\geqq|\tilde{f}|$, f_0 in E'' for some $g\in E$, then there exists $h_1\in E$ such that

$$g\geqq h_1\geqq|\tilde{f}|,\ \ f_0\ \text{in } E''$$

and

$$\langle h_1, x\rangle = \langle f_0, x\rangle \quad \textit{for any } x \in F.$$

PROOF. Put $\phi(x)=\max\{|\langle\tilde{f}, x\rangle|, \langle f_0, x\rangle\}$ for $x\in X$, $\psi(x)=\max\{\|\tilde{f}\|, \|f_0\|\}$ for $x\in X\backslash(F\cup\{0\})$ and $\psi(x)=\langle f_0, x\rangle$ for $x\in F\cup\{0\}$. Since F is a closed face of X, ϕ [resp. ψ] is convex continuous [resp. concave lower semi-continuous] with $\phi\leqq\psi$. Therefore, there is a continuous affine function h with $\phi\leqq h\leqq\psi$ since X is a simplex, and h is a desired one.

Next, put $f'(x)=|\langle\tilde{f}, x\rangle|$. Then the function ϕ_1 [resp. ψ_1] equal to f_0 on F and to f' [resp. g] on the complement is convex and upper semi-continuous [resp. concave lower semi-continuous] with $\phi_1\leqq\psi_1$. So in the same way, we can find h_1. //

Let $\partial_e X$ be the set of extreme points of X. Then any element x of $\partial_e X$ is a one-point face of X, so we have

COROLLARY 1. *For any $\tilde{f}\in\tilde{E}$ and $x\in\partial_e X$, there exists $h\in E$ such that $h\geqq|\tilde{f}|$ and $\langle h, x\rangle=|\langle\tilde{f}, x\rangle|$.*

The absolute value $|\tilde{f}|$ of $\tilde{f}\in\tilde{E}$ is affine but not continuous in general, while $|\langle\tilde{f}, x\rangle|$ is a continuous function on X but not affine and moreover,

$$|\langle\tilde{f}, x\rangle|\leqq\langle x, |\tilde{f}|\rangle.$$

For $x\in\partial_e X$, we have the following.

COROLLARY 2. *For any $\tilde{f}\in\tilde{E}$ and $x\in\partial_e X$, the equality*

$$|\langle\tilde{f}, x\rangle|=\langle x, |\tilde{f}|\rangle$$

holds.

§ 3. The absolute value.

By using the unique maximal probability measure μ_x on a simplex X with resultant $x\in X$, we know [6, Cor. 2.5] that a simplex space E is isomorphic to the space $\{f\in C(X);\ f(x)=\mu_x(f)$ for all $x\in X$ and $f(0)=0\}$.

We say a function f on X satisfies the barycentric calculus if $f(x)=\mu_x(f)$ for all $x\in X$.

An element of E'' can be considered as a function on X, but does not necessarily satisfy the barycentric calculus as Choquet's example shows [1, Example I. 2.10], since the topology on X is the weak*-topology. But if E is separable, the following theorem holds for the absolute value $|\tilde{f}|$ of $\tilde{f}\in\tilde{E}$, which is an element of E''.

THEOREM 3. *Let E be a separable simplex space and $\tilde{E}$ be its complexification. Then for any $\tilde{f}\in\tilde{E}$, the absolute value $|\tilde{f}|$ satisfies the barycentric calculus.*

In the proof of the theorem, we shall apply the following notation for the oscillation of a real valued function f over a subset Y of its domain X;

$$Of(Y)=\sup_{x,\, y\in Y}|f(x)-f(y)|.$$

In case of a separable simplex space, X is metrizable [4, Th. V. 5.1], and so the maximal probability measure μ_x with resultant $x\in X$ is supported by the set of extreme points $\partial_e X$ of X (Choquet's theorem [7, Sec. 3]). The absolute value $|\tilde{f}|$ of $\tilde{f}\in\tilde{E}$ in E'' is not continuous, but as for the oscillation of $|\tilde{f}|$ and the maximal measure μ_x with resultant $x\in X$, we have the following.

LEMMA. *For any $\varepsilon>0$ and every Borel subset B of X with $\mu_x(B)>0$, there exists a Borel set $D\subset B$ such that $\mu_x(D)>0$ and $O|\tilde{f}|(\overline{\mathrm{co}}\, D)\leqq\varepsilon$, where $\overline{\mathrm{co}}\, D$ means the convex closure of D.*

PROOF. Since X is metrizable, $\partial_e X$ is a Borel set and the maximal probability measure μ_x is supported by $\partial_e X$. So we have

$$\begin{aligned}\mu_x(B)&=\mu_x(B\cap\partial_e X)\\ &=\sup\{\mu_x(K)\,;\ K \text{ is compact and } K\subset B\cap\partial_e X\},\end{aligned}$$

by the regularity of μ_x. Therefore there exists a compact set $K\subset\partial_e X\cap B$ such that $\mu_x(K)>0$. Since $\tilde{f}\in\tilde{E}$ is a complex-valued, continuous affine function, the range of $\tilde{f}$ is a bounded set of a complex plane. So the range of $\tilde{f}$ may be included in a finite union of disjoint rectangles Q_j, each of the diagonal length less than ε. This induces a decomposition of K into a finite union of disjoint Borel sets

$$D_j=\tilde{f}^{-1}(Q_j)\cap K.$$

Since Q_j is convex and $\tilde{f}$ is affine, $\tilde{f}^{-1}(Q_j)$ is convex and so $\mathrm{co}\, D_j$ (convex hull of D_j) is contained in $\tilde{f}^{-1}(Q_j)$, that is, $\tilde{f}(\mathrm{co}\, D_j)$ is contained in Q_j. By the continuity of $\tilde{f}$, $\tilde{f}(\overline{\mathrm{co}}\, D_j)$ is contained in $\overline{Q_j}$ (closure of Q_j). Put

$$I_j=\{|\xi+i\eta|\,;\ (\xi,\,\eta)\in Q_j\}.$$

Then $\overline{I_j}$ is a closed interval of length less than ε, i. e.

$$\overline{I_j}=[c_j,\,d_j] \quad\text{with}\quad c_j\leqq d_j<c_j+\varepsilon. \tag{4.1}$$

For any $x\in\overline{\mathrm{co}}\, D_j$, $|\tilde{f}(x)|\in\overline{I_j}$, which implies $|\tilde{f}(x)|\geqq c_j$. By the definition of $|\tilde{f}|$, $|\tilde{f}|(x)\geqq|\tilde{f}(x)|$ for any $x\in X$. Therefore

$$|\tilde{f}|(x)\geqq c_j \quad\text{for any } x\in\overline{\mathrm{co}}\, D_j. \tag{4.2}$$

Put $g_j(x)=|\tilde{f}(x)|$ for $x\in D_j$. Then there exists a continuous affine extension g on X of g_j such that

$$g(x)\leqq d_j \quad\text{for any } x\in X,$$

as

$$d_j\geqq|\tilde{f}(x)|=g_j(x) \quad\text{for } x\in D_j.$$

Since $\overline{\mathrm{co}}\, D_j$ is a closed face [2, Prob. 27.7], we get $h\in E$ such that

$$h \geqq |\tilde{f}| \quad \text{and} \quad h(x)=g(x) \leqq d_j \quad \text{for } x \in \overline{\text{co}}\, D_j \tag{4.3}$$

by applying Theorem 2 with g and $\tilde{f}$. From (4.1), (4.2) and (4.3), we have $O|\tilde{f}|(\overline{\text{co}}\, D_j) \leqq \varepsilon$. Now $\sum_j \mu_x(D_j)=\mu_x(K) \neq 0$ and so there is at least one index k such that $\mu_x(D_k) \neq 0$. By defining $D=D_k$, we have the desired result. //

PROOF OF THEOREM 3. For $\tilde{f} \in \tilde{E}$ and a maximal measure μ_x on X, the above lemma shows that the assumption of Lemma I. 2.5 in [1] is satisfied. So there exists a sequence $\{D_n\}$ of pairwise disjoint Borel subsets of X such that

$$O|\tilde{f}|(\overline{\text{co}}\, D_n) \leqq \varepsilon \quad n=1, 2, \cdots \tag{4.4}$$

$$\sum_n \mu_x(D_n)=1.$$

Choose a natural number N such that

$$\lambda_0=\sum_{n>N} \mu_x(D_n) \leqq \varepsilon \tag{4.5}$$

and define

$$\lambda_n=\mu_x(D_n) \quad \text{for } n=1, 2, \cdots, N.$$

Define

$$\mu_n=\lambda_n^{-1}\mu_x | D_n \quad \text{for } n=1, 2, \cdots, N$$

and

$$\mu_0=\lambda_0^{-1}\mu_x | K \backslash (D_1 \cup \cdots \cup D_N)) \quad \text{if } \lambda_0 \neq 0$$

$$\mu_0=0 \quad \text{if } \lambda_0=0.$$

Then we can express μ_x as a convex combination $\mu_x=\sum_{n=0}^{N} \lambda_n \mu_n$. Letting x_n be the resultant of μ_n and using the fact that μ_n is concentrated on D_n, we obtain $x_n \in \overline{\text{co}}\, D_n$ for $n=1, 2, \cdots, N$. By (4.4), this gives

$$|\tilde{f}(x_n)-\mu_n(\tilde{f})| \leqq \varepsilon, \quad n=1, 2, \cdots, N. \tag{4.6}$$

Clearly the resultant x of μ_x is $\sum_{n=0}^{N} \lambda_n x_n$ and so by (4.5), (4.6) and by the affine nature of $\tilde{f}$, we have

$$|\tilde{f}(x)-\mu_x(\tilde{f})| = |\sum_{n=0}^{N} \lambda_n \tilde{f}(x_n) - \sum_{n=0}^{N} \lambda_n \mu_n(\tilde{f})|$$

$$\leqq \sum_{n=0}^{N} \lambda_n |\tilde{f}(x_n)-\mu_n(\tilde{f})| \leqq \varepsilon(2\|\tilde{f}\|+1).$$

This completes the proof since $\varepsilon>0$ was arbitrary. //

Using the above theorem, we have

THEOREM 4. *Let E be a separable simplex space and $\tilde{E}$ be its complexification. For any $\tilde{f} \in \tilde{E}$, the absolute value $|\tilde{f}|$ is an upper semi-continuous function on X with the weak*-topology and $|\tilde{f}|(x)=\inf\{g(x)\,;\, g \in E,\ g \geqq |\tilde{f}| \text{ in } E''\}$.*

PROOF. Let $h(x)=\inf\{g(x)\,;\, g \in E,\ g \geqq |\tilde{f}| \text{ in } E''\}$. It is clear that

$$h(x) \geqq |\tilde{f}|(x). \tag{4.7}$$

For $x \in \partial_e X$, there exists $g \in E$ such that

$$g \geqq |\tilde{f}| \quad \text{and} \quad g(x) = |\tilde{f}(x)|$$

by Corollary 1 of Theorem 2. Therefore

$$h(x) \leqq g(x) = |\tilde{f}(x)| \leqq |\tilde{f}|(x),$$

which implies $h(x) = |\tilde{f}|(x)$ by (4.7). So

$$h | \partial_e X = |\tilde{f}| \, | \partial_e X. \tag{4.8}$$

For any $x \in X$, there corresponds a maximal probability measure μ_x with resultant x supported by $\partial_e X$. By the definition of h and Theorem 1, h is an upper semi-continuous, affine function on X and so satisfies the barycentric calculus [1, Th. I. 2.6], i.e. $h(x) = \mu_x(h)$. Therefore we have by Theorem 3 and (4.8),

$$h(x) = \mu_x(h) = \int_{\partial_e X} h \, d\mu_x = \int_{\partial_e X} |\tilde{f}| \, d\mu_x = \mu_x(|\tilde{f}|) = |\tilde{f}|(x),$$

which completes the proof. //

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