Natural Science Report, Ochanomizu University, Vol. 30, No. 1, 1979
お茶の水女子大学 自然科学報告 第30巻 第1号

A Reduction of a Positive Operator in an Arbitrary Banach Lattice to its Irreducible Components

Ikuko Sawashima

Department of Mathematics, Faculty of Science,
Ochanomizu University, Tokyo

and Fumio Niiro

Department of Mathematics, College of General Education,
University of Tokyo

(Received April 10, 1979)

Let T be a positive linear operator on a Banach lattice. A reduction of T to its irreducible components was investigated by the authors in the case of a sub-Markov operator in $C(X)$ [7]. Later S. Miyajima obtained the theory in the case of AM- or AL-space [3] [4]. In these papers, the relation between the spectrum of T and those of component operators in corresponding spaces was established. The purpose of this note is to prove a theorem which is a generalization of these results to the case of an arbitrary Banach lattice. Recently S. Miyajima found a reduction method of Banach lattices and applied it to reduction of operators in Banach lattices [5]. This reduction method works effectively throughout our note.

We shall begin with preliminaries which are taken from [5] and are fundamental in this note. Let E be a Banach lattice and T on E be uniformly ergodic. Then the uniform limit of $\left(\sum_{k=0}^{N-1} T^k\right)\Big/N$, denoted by P, is a positive projection whose range PE is the eigenspace of T for 1. We assume PE contains a quasi-interior element e of E. Let E_e be the order ideal in E generated by e, which ideal is lattice isomorphic to $C(\Omega)$ and let the image of x in E_e by this isomorphism be denoted by $\tilde{x}$. We shall define similary $(PE)_e$ and $C(\Lambda)$. To each $\lambda \in \Lambda$, corresponds a Banach lattice E_λ, to which there exists a lattice homomorphism from E_e. The image of x in E_e by the homomorphism is denoted simply by x_λ in place of $[x_\lambda]$ in [5]. Let T_λ and P_λ be the operators induced on E_λ from T and P respectively. The order ideal $I_\lambda = \{x_\lambda \in E_\lambda; P_\lambda |x_\lambda| = 0\}$ is T_λ-invariant. Let T_λ/I_λ and P_λ/I_λ be the operators induced on E_λ/I_λ from T_λ and P_λ respectively. The operators T_λ and P_λ (resp. T_λ/I_λ and P_λ/I_λ) have similar properties as T and P have, for example, positivity and uniform ergodicity. Moreover

This research was supported in part by grants from the Scientific Research Fund of the Ministry of Education, Science and Culture of Japan.

$P_\lambda E_\lambda$ is one dimensional and T_λ/I_λ is irreducible. Further informations on these matters are found in I. Sawashima and F. Niiro [7] and S. Miyajima [5].

THEOREM. *Let T be a linear operator on a Banach lattice E which satisfies the following conditions:*

1) *T is positive,*

2) *the mean of the operators $\{T^n\}_{1\leq n\leq N}$ converges uniformly to P as N tends to ∞,*

3) *E has quasi-interior element e which satisfies $Te=e$. Then the following equality holds:*

$$\sigma(T)\cap\Gamma=\Big(\bigcup_{\lambda\in\Lambda}\sigma(T_\lambda)\Big)^{-}\cap\Gamma=\Big(\bigcup_{\lambda\in\Lambda}\sigma(T_\lambda/I_\lambda)\Big)^{-}\cap\Gamma\,,$$

where Γ is the unit circle in the complex number plane.

Before proving the theorem, we start with the following:

LEMMA. *For each ν in an index set A, let T_ν and P_ν be positive linear operators in a Banach lattice E_ν which have the following properties;*

1) *the spectral radius of T_ν is equal to 1,*

2) *$P_\nu T_\nu=P_\nu$*

and let $\sup_{\nu\in A}\|P_\nu\|<\infty$ and $\sup_{\nu\in A}\|T_\nu\|<\infty$. Suppose a complex number $\alpha_0\neq 1$ with $|\alpha_0|=1$ in $\bigcap_{\nu\in A}\rho(T_\nu)$ satisfies the condition

3) $\sup_{\nu\in A}\|R(\alpha_0, T_\nu)\|<\infty$.

Then there exists a positive number c such that for every $\nu\in A$

4) $\|P_\nu|x_\nu|\,\|\leq c\|P_\nu|\alpha_0 x_\nu-T_\nu x_\nu|\,\|$

holds for each $x_\nu\in E_\nu$ satisfying

5) $\|x_\nu\|/2\leq\|P_\nu|x_\nu|\,\|$.

PROOF. Let the conclusion of this lemma be not true. Then for an arbitrary $n\in\mathbf{N}$, there exist $\nu_n\in A$ and $x_n\in E_{\nu_n}$ such that

$$\frac{1}{2}\|x_n\|\leq\|P_n|x_n|\,\|$$

and

$$\|P_n|x_n|\,\|>n\|P_n|\alpha_0 x_n-T_n x_n|\,\|$$

where T_{ν_n} and P_{ν_n} are denoted simply T_n and P_n respectively. (Note that $\nu_1, \nu_2, \cdots$ are not necessarily different from each other.)

Since x_n is not 0, we may assume $\|x_n\|=1$ without loss of generality. Then $1/2\leq\|P_n|x_n|\,\|$ follows immediately. On the other hand, it is clear that $\|P_n|\alpha_0 x_n-T_n x_n|\,\|$ converges to 0 as $n\to\infty$. Put $b=\sup_n\|R(\alpha_0, T_n)\|$. Let d

be a positive number satisfying $bd<1$ and let α be a complex number such that $|\alpha-\alpha_0|<d$. Then α is in $\bigcap_{n=1}^{\infty}\rho(T_n)$ and $\|R(\alpha_0, T_n)-R(\alpha, T_n)\|\leqq b^2d/(1-bd)$ for every $n\in\mathbf{N}$, by the expansion of $R(\alpha, T_n)$ at α_0: $R(\alpha, T_n)=\sum_{k=1}^{\infty}(\alpha-\alpha_0)^{k-1}R(\alpha_0, T_n)^k$. Since P_n is positive, it follows that, for $y_n\in E_{\nu_n}$

$$\|P_n|R(\alpha_0, T_n)y_n|\,\|\leqq\|P_n|R(\alpha, T_n)y_n|\,\|+\frac{b^2d}{1-bd}\|P_n\|\,\|y_n\|\,.$$

Further, if $|\alpha|>1$, then

$$\begin{aligned}P_n|R(\alpha, T_n)y_n|&\leqq P_nR(|\alpha|, T_n)|y_n|\\&=\frac{1}{|\alpha|-1}P_n|y_n|\,.\end{aligned}$$

Therefore,

$$\|P_n|R(\alpha_0, T_n)y_n|\,\|\leqq\frac{1}{|\alpha|-1}\|P_n|y_n|\,\|+\frac{b^2d}{1-bd}\|P_n\|\,\|y_n\|\,.$$

From this together with the uniform boundedness of $\{P_n\}$, it is easily proved that, if $\|P_n|y_n|\,\|$ converges to 0 as $n\to\infty$ and if $\sup_n\|y_n\|<\infty$, then $\|P_n|R(\alpha_0, T_n)y_n|\,\|$ also converges to 0 as $n\to\infty$. Putting $y_n=\alpha_0x_n-T_nx_n$, we have

$$\lim_{n\to\infty}\|P_n|x_n|\,\|=0$$

which is a contradiction.

Clearly we have

Corollary. *Under the same condition of Lemma, assume further that $P_\nu E_\nu$ is one dimensional for each $\nu\in A$, then there exists a positive number c such that for every $\nu\in A$*

$$P_\nu|x_\nu|\leqq cP_\nu|\alpha_0x_\nu-T_\nu x_\nu|$$

holds for every $x_\nu\in E_\nu$ which satisfy

$$P_\nu|x_\nu|\geqq\frac{1}{2}|x_\nu|\,.$$

The following remarks for the lemma are easily seen from the proof:

Remark 1. If A is a finite set, then the assumption $P_\nu T_\nu=P_\nu$ may be replaced by $P_\nu T_\nu=T_\nu P_\nu$.

Remark 2. In the inequality 5), the number 1/2 does not have any special meaning but only to be a positive constant independent of $\nu\in A$ and $x_\nu\in E_\nu$.

REMARK 3. For T_ν, $\nu \in A$, in the lemma, the assumption of positivity, 1) and 2) may be replaced by the ones that there exists a positive operator S_ν in E_ν with $r(S_\nu)=1$ and $P_\nu S_\nu = P_\nu$ such that $|T_\nu x_\nu| \leqq S_\nu |x_\nu|$ for every $x_\nu \in E_\nu$.

With above preparations we shall prove the theorem.

PROOF OF THEOREM. It was already proved that

$$\Big(\bigcup_{\lambda \in A} \sigma(T_\nu)\Big)^{-} \cap \Gamma = \Big(\bigcup_{\lambda \in A} \sigma(T_\lambda / I_\lambda)\Big)^{-} \cap \Gamma$$

and

$$\sigma(T) \cap \Gamma \supset \Big(\bigcup_{\lambda \in A} \sigma(T_\lambda)\Big)^{-} \cap \Gamma$$

in Theorem 3 in [5]. So, to prove the remaining equality, we assume

$$\alpha_0 \in \sigma(T) \cap \Gamma \cap \Big(\bigcap_{\lambda \in A} \rho(T_\lambda)\Big)^{\circ}$$

and will derive a contradiction. If $\sup_{\lambda \in A} \|R(\alpha_0, T_\lambda)\| = \infty$, then the discussion in the proof of Theorem 6 in [7] leads to a contradiction. Therefore we assume further

$$(*) \qquad \sup_{\lambda \in A} \|R(\alpha_0, T_\lambda)\| < \infty .$$

Since $\alpha_0 \in \sigma(T) \cap \Gamma$ and E_e is dense in E by Condition 3), there exists a sequence $x^{(n)}$ in E_e such that

$$\|x^{(n)}\| = 1 \qquad \text{and} \qquad \lim_{n \to \infty} \|\alpha_0 x^{(n)} - T x^{(n)}\| = 0 .$$

$T|x^{(n)}| \geqq |x^{(n)}| - |\alpha_0 x^{(n)} - T x^{(n)}|$ is clear. Then the following inequality

$$((I-P)|x^{(n)}|) \vee 0 \leqq R(\alpha, T)|\alpha_0 x^{(n)} - T x^{(n)}| + (\alpha - 1)\{(R(\alpha, T)(I-P)|x^{(n)}|) \vee 0\}$$

holds for $\alpha > 1$, as in the proof of Lemma 2 in [6] or Lemma 5 in [7]. Since Condition 2 is equivalent to the one that $\alpha = 1$ is a pole of the resolvent $R(\alpha, T)$ of order 1 and P is the leading coefficient of its Laurent's expansion by Theorems 4 and 5 in [1], $\sup_{\alpha > 1} \|R(\alpha, T)(I-P)\|$ is finite. Therefore $\|((I-P)|x^{(n)}|) \vee 0\|$ converges to 0 as $n \to \infty$. Put

$$w^{(n)} = ((I-P)|x^{(n)}|) \vee 0 .$$

Then it is an immediate consequence that

$$|x^{(n)}| \geqq w^{(n)} \geqq 0$$

and

$$\lim_{n \to \infty} \|w^{(n)}\| = 0 .$$

Let $f_n(\omega)$ be the complex valued function defined on Ω by

$$\begin{cases} \arg f_n(\omega)=\arg \tilde{x}^{(n)}(\omega) & \text{if } \tilde{x}^{(n)}(\omega)\neq 0 \\ |f_n(\omega)|=|\tilde{x}^{(n)}(\omega)|-\tilde{w}^{(n)}(\omega)\,. \end{cases}$$

Since f_n is clearly in $C(\Omega)$, we can define a new sequence $y^{(n)}$ in E_e which corresponds to f_n, i.e., $\tilde{y}^{(n)}=f_n$. Then,

$$|x^{(n)}-y^{(n)}|=|\,|x^{(n)}|-|y^{(n)}|\,|=w^{(n)}$$

and

$$P|x^{(n)}|\geqq|y^{(n)}|=|x^{(n)}|-w^{(n)}\geqq 0$$

are clear by the definition. Therefore this new sequence $y^{(n)}$ in E_e has the following properties;

$$\lim_{n\to\infty}\|y^{(n)}\|=1\,,$$

$$\lim_{n\to\infty}\|\alpha_0 y^{(n)}-Ty^{(n)}\|=0$$

and

$$(\ast_{\ast}) \qquad |y^{(n)}|\leqq P|y^{(n)}|+Pw^{(n)}\,.$$

Hence, for every $\lambda\in\Lambda$ and every $n\in\mathbf{N}$

$$|y^{(n)}{}_\lambda|\leqq P_\lambda|y^{(n)}{}_\lambda|+P_\lambda w^{(n)}{}_\lambda$$

holds. If $P_\lambda w^{(n)}{}_\lambda\leqq P_\lambda|y^{(n)}{}_\lambda|$, then $\|y^{(n)}{}_\lambda\|/2\leqq\|P_\lambda|y^{(n)}|\,\|$ is clear. By Theorem 2 in [5] and Assumption (*), all the hypotheses in Corollary of Lemma for T_λ, P_λ, $\lambda\in\Lambda$ and α_0 are satisfied. Then, there exists a positive number c such that for every λ

$$P_\lambda|y^{(n)}{}_\lambda|\leqq cP_\lambda|\alpha_0 y^{(n)}{}_\lambda-T_\lambda y^{(n)}{}_\lambda|$$

holds for $y^{(n)}{}_\lambda\in E_\lambda$ satisfying $P_\lambda w^{(n)}{}_\lambda\leqq P_\lambda|y^{(n)}{}_\lambda|$. Since $P_\lambda E_\lambda$, is one dimensional, $P_\lambda w^{(n)}{}_\lambda\nleqq P_\lambda|y^{(n)}|$ means $P_\lambda w^{(n)}{}_\lambda>P_\lambda|y^{(n)}{}_\lambda|$. Therefore, the following inequality

$$P_\lambda|y^{(n)}{}_\lambda|\leqq P_\lambda w^{(n)}{}_\lambda+cP_\lambda|\alpha_0 y^{(n)}{}_\lambda-T_\lambda y^{(n)}{}_\lambda|$$

holds for every λ and for every n. Applying Corollary 3 in [5], we have

$$P|y^{(n)}|\leqq Pw^{(n)}+cP|\alpha_0 y^{(n)}-Ty^{(n)}|\,.$$

Hence,

$$|y^{(n)}|\leqq 2Pw^{(n)}+cP|\alpha_0 y^{(n)}-Ty^{(n)}|$$

follows from the inequality $(\ast_{\ast})$. Thus, we have

$$\|y^{(n)}\|\leqq 2\|P\|\,\|w^{(n)}\|+c\|P\|\,\|\alpha_0 y^{(n)}-Ty^{(n)}\|\,.$$

Since, the right side of the inequality tends to 0 as $n\to\infty$, $\|y^{(n)}\|$ also converges to 0 as $n\to\infty$. This contradicts

$$\lim_{n\to\infty}\|y^{(n)}\|=1 .$$

References

[1] S. Karlin: Positive operators. J. Math. Mech., **8** (1959), 907–937.
[2] S. Miyajima: A note on reduction of positive operators. J. Fac. Sci. Univ. Tokyo, Sec. I A, **21** (1974), 289–298.
[3] ———: A note on reduction of positive operators, II. J. Fac. Sci. Univ. Tokyo, Sec. I A, **23** (1976), 245–256.
[4] ———: On a reduction of positive operators. J. Fac. Sci. Univ. Tokyo, Sec. I A, **24** (1977), 405–424.
[5] ———: A reduction of Banach lattices and its applications to positive ergodic operators. J. Fac. Sci. Univ. Tokyo, Sec. I A, to appear.
[6] F. Niiro: On indecomposable operators in l_p $(1<p<\infty)$ and a problem of H. H. Schaefer. Sci. Pap. Coll. Gen. Educ. Univ. Tokyo, **14** (1964), 165–179.
[7] I. Sawashima and F. Niiro: Reduction of a sub-Markov operator to its irreducible components. Nat. Sci. Rep. Ochanomizu Univ., 24 (1973), 35–59.