Simple Cubic Lattice Green Functions

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A simple cubic lattice Green function and its associate functions are grouped into avector. A differential equation for the vector is derived and studied. The same process is repeated on anisotropic lattice Green functions.

§ 1. Isotropic lattice Green functions

A lattice Green function $u(z)$ defined by

$$u(z)=\frac{1}{\pi^3}\iiint_0^\pi \frac{d\theta_1 d\theta_2 d\theta_3}{z-\cos\theta_1-\cos\theta_2-\cos\theta_3}, \qquad z>3,$$

may be transformed into an integral

$$u(z)=\frac{1}{\pi^3}\iiint_0^\pi d\theta_1 d\theta_2 d\theta_3 \int_0^\infty e^{-t(z-\cos\theta_1-\cos\theta_2-\cos\theta_3)}\,dt$$
$$=\int_0^\infty e^{-zt}[I_0(t)]^3\,dt$$

with the aid of a formula on modified Besessl functions[1]

$$\frac{1}{\pi}\int_0^\pi e^{z\cos\theta}\cos n\theta\,d\theta=I_n(z), \quad n=0, 1, 2, \cdots . \tag{1}$$

There exist tables of cubic lattice Green functions[3),4)] and studies of their analytic property.[5)] This paper is a trial to comprehend lattice Green functions together with their associate functions. We introduce a set of four integrals $u_k(z)$, $k=0, 1, 2, 3$, defined by

$$u_0(z)=\frac{1}{\pi^3}\iiint_0^\pi \frac{d\theta_1 d\theta_2 d\theta_3}{z-\cos\theta_1-\cos\theta_2-\cos\theta_3}=\int_0^\infty e^{-zt}[I_0(t)]^3\,dt \tag{2}$$

$$u_1(z)=\frac{1}{\pi^3}\iiint_0^\pi \frac{\cos\theta_1\,d\theta_1 d\theta_2 d\theta_3}{z-\cos\theta_1-\cos\theta_2-\cos\theta_3}=\int_0^\infty e^{-zt}[I_0(t)]^2 I_1(t)\,dt \tag{3}$$

$$u_2(z)=\frac{1}{\pi^3}\iiint_0^\pi \frac{\cos\theta_1\cos\theta_2\, d\theta_1 d\theta_2 d\theta_3}{z-\cos\theta_1-\cos\theta_2-\cos\theta_3}=\int_0^\infty e^{-zt}I_0(t)[I_1(t)]^2\, dt \tag{4}$$

$$u_3(z)=\frac{1}{\pi^3}\iiint_0^\pi \frac{\cos\theta_1\cos\theta_2\cos\theta_3\, d\theta_1 d\theta_2 d\theta_3}{z-\cos\theta_1-\cos\theta_2-\cos\theta_3}=\int_0^\infty e^{-zt}[I_1(t)]^3\, dt\,. \tag{5}$$

The last three integrals (3), (4), (5) may be shown to have their respective Laplacian integral representations by virtue of the formula (1) for $n=0, 1$. We seek differential equations to be satisfied by $u_k(z)$, $k=0, 1, 2, 3$. Partial integration of the Laplacian integral representation of $u_0(z)$ leads to

$$zu_0'=-z\int_0^\infty e^{-zt}tI_0^3\, dt=e^{-zt}tI_0^3|_0^\infty-\int_0^\infty e^{-zt}\frac{d}{dt}(tI_0^3)\, dt$$
$$=-\int_0^\infty e^{-zt}(I_0^3+3tI_0^2I_1)\, dt=-u_0+3u_1'\,,$$

where the abbreviation $u_0'=du_0/dz$ is used.

Similarly

$$zu_1'=-\int_0^\infty e^{-zt}\frac{d}{dt}(tI_0^2I_1)\, dt=-\int_0^\infty e^{-zt}(I_0^2I_1+2tI_0I_1^2+tI_0^2I_1')\, dt$$
$$=-\int_0^\infty e^{-zt}(2tI_0I_1^2+tI_0^3)\, dt=u_0'+2u_2'$$
$$zu_2'=2u_1'+u_3'+u_2$$
$$zu_3'=3u_2'+2u_3\,.$$

A formula on modified Bessel functions[2)]

$$tI_1'=tI_0-I_1 \tag{6}$$

was used in the above derivation.

It is to be noted that four functions u_0, u_1, u_2, u_3 form a closed set with respect to differentiation. These equations may be cast into a vector form as

$$\begin{bmatrix} z & -3 & 0 & 0 \\ -1 & z & -2 & 0 \\ 0 & -2 & z & -1 \\ 0 & 0 & -3 & z \end{bmatrix}\frac{d}{dz}\begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{bmatrix}=\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{bmatrix}, \tag{7}$$

or

$$(z-P)\frac{d\boldsymbol{u}}{dz}=Q\boldsymbol{u}\,, \tag{8}$$

$$P=\begin{bmatrix} 0 & 3 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 3 & 0 \end{bmatrix},\quad Q=\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix},\quad \boldsymbol{u}=\begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{bmatrix}.$$

If we eliminate u_1, u_2, u_3 from (7), we have an equation of the third order

$$(z^4-10z^2+9)u_0'''+(6z^3-30z)u_0''+(7z^2-12)u_0'+zu_0=0\,.$$

Solutions of this equation may be constructed from solutions of an equation of the second order

$$(z^4-10z^2+9)w''+(2z^3-10z)w'+(z^2-2)/4\cdot w=0$$

as has been shown by G. S. Joyce.[6)]

A change of variable $z^2=\zeta$ leads to a Lamé equation[7)]

$$(\zeta^3-10\zeta^2+9\zeta)\frac{d^2w}{d\zeta^2}+\frac{1}{2}(3\zeta^2-20\zeta+9)\frac{dw}{d\zeta}+\frac{\zeta-2}{16}w=0\,.$$

A further change of variable $\zeta-10/3=\wp(\eta)$, $\wp$: Weierstrassian elliptic function, leads to another form of Lamé equation[8)] of order $-1/2$.

$$\frac{d^2w}{d\eta^2}+\left(\frac{1}{4}\wp(\eta)+\frac{1}{3}\right)w=0$$

which is also difficult to solve. A possible solution of this Lamé equation might be found in the spirit of Halphen.[9)]

§ 2. The expansion at $z=\infty$

The integral representation of $u_0(z)$, (2) and the expansion of $1/(z-\cos\theta_1-\cos\theta_2-\cos\theta_3)$ in powers of $1/z$ give the expansion of $u_0(z)$ at $z=\infty$

$$u_0=\frac{1}{z}+\frac{3}{2}\frac{1}{z^3}+\frac{45}{8}\frac{1}{z^5}+\cdots\,. \tag{9}$$

The equation (8) may furnish another approach. An assumed expansion

$$\boldsymbol{u}=\frac{\boldsymbol{c}_0}{z}+\frac{\boldsymbol{c}_1}{z^2}+\frac{\boldsymbol{c}_2}{z^3}+\cdots$$

leads to the conditions

$$\left.\begin{aligned}(Q+1)\boldsymbol{c}_0&=0\\(Q+2)\boldsymbol{c}_1&=P\boldsymbol{c}_0\\(Q+3)\boldsymbol{c}_2&=2P\boldsymbol{c}_1\\(Q+4)\boldsymbol{c}_3&=3P\boldsymbol{c}_2\end{aligned}\right\}. \tag{10}$$

The first condition requires that $\boldsymbol{c}_0$ be an eigenvector of the matrix Q corresponding to its eigenvalue -1.

Then one gets, in view of (9)

$$c_0=\begin{bmatrix}1\\0\\0\\0\end{bmatrix}.$$

Successive coefficients c_k, $k=1, 2, 3, \cdots$ are obtained from (10) without difficulty.

$$c_1=\begin{bmatrix}0\\1/2\\0\\0\end{bmatrix},\quad c_2=\begin{bmatrix}3/2\\0\\1/2\\0\end{bmatrix},\quad c_3=\begin{bmatrix}0\\15/8\\0\\3/4\end{bmatrix},\quad c_4=\begin{bmatrix}45/8\\0\\3\\0\end{bmatrix},\quad c_5=\begin{bmatrix}0\\155/16\\0\\45/8\end{bmatrix},\quad \cdots.$$

§ 3. The transformation of the differential equation

The matrix P has four eigenvalues 3, 1, -1, -3 and takes a diagonal form when transformed by a matrix S

$$S=\frac{1}{2\sqrt{2}}\begin{bmatrix}1 & 3 & 3 & 1\\1 & 1 & -1 & -1\\1 & -1 & -1 & 1\\1 & -3 & 3 & -1\end{bmatrix},\qquad S^{-1}=S, \tag{11}$$

constructed from four eigenvectors of the matrix P. In fact we have

$$S^{-1}PS\equiv A=\begin{bmatrix}3 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & -1 & 0\\0 & 0 & 0 & -3\end{bmatrix}$$

and

$$S^{-1}QS\equiv B=\frac{1}{2}\begin{bmatrix}1 & -3 & 0 & 0\\-1 & 1 & -2 & 0\\0 & -2 & 1 & -1\\0 & 0 & -3 & 1\end{bmatrix}.$$

A change of variable

$$u=Sv, \qquad v=Su \tag{12}$$

leads to an equation for v

$$(z-A)\frac{dv}{dz}=Bv \tag{13}$$

or

$$(z-3)v_0' = \frac{1}{2}(v_0 - 3v_1)$$

$$(z-1)v_1' = \frac{1}{2}(-v_0 + v_1 - 2v_2)$$

$$(z+1)v_2' = \frac{1}{2}(-2v_1 + v_2 - v_3)$$

$$(z+3)v_3' = \frac{1}{2}(-3v_2 + v_3)\,.$$

Therefore, the differential equation (13) has four singular points $3 \equiv \alpha_0$, $1 \equiv \alpha_1$, $-1 \equiv \alpha_2$, $-3 \equiv \alpha_3$, indices at each singular point being 1/2, 0, 0, 0.

Hence a solution v_k at each singular point is a linear combination of a singular solution corresponding to the index 1/2 and a regular solution having three adjustable parameters. The lowest singular term of the singular solution may be obtained directly from Laplacian integral representations of v_k, $k=0, 1, 2, 3$, derived from (12) with the aid of (11), (2), (3), (4), (5),

$$\left.\begin{aligned} v_0 &= \frac{1}{2\sqrt{2}} \int_0^\infty e^{-zt}(I_0+I_1)^3\,dt \\ v_1 &= \frac{1}{2\sqrt{2}} \int_0^\infty e^{-zt}(I_0+I_1)^2(I_0-I_1)\,dt \\ v_2 &= \frac{1}{2\sqrt{2}} \int_0^\infty e^{-zt}(I_0+I_1)(I_0-I_1)^2\,dt \\ v_3 &= \frac{1}{2\sqrt{2}} \int_0^\infty e^{-zt}(I_0-I_1)^3\,dt \end{aligned}\right\}\,. \tag{14}$$

If the variable z is supposed to have a negative imaginary part, the variable t in (14) may be replaced by it since the positive real part of iz gurantees the convergence of integrals with respect to real t. Replacement of t by it in (14) gives

$$v_0 = \frac{i}{2\sqrt{2}} \int_0^\infty e^{-izt}(J_0(t)+iJ_1(t))^3\,dt$$

$$v_1 = \frac{i}{2\sqrt{2}} \int_0^\infty e^{-izt}(J_0(t)+iJ_1(t))^2(J_0(t)-iJ_1(t))\,dt$$

$$v_2 = \frac{i}{2\sqrt{2}} \int_0^\infty e^{-izt}(J_0(t)+iJ_1(t))(J_0(t)-iJ_1(t))^2\,dt$$

$$v_3 = \frac{i}{2\sqrt{2}} \int_0^\infty e^{-izt}(J_0(t)-iJ_1(t))^3\,dt\,.$$

$J_0(t)$ and $J_1(t)$ are Bessel functions of order 0 and 1, which may be expressed in terms of Hankel functions as

$$J_0(t)=\frac{1}{2}(H_0^1(t)+H_0^2(t)) , \qquad J_1(t)=\frac{1}{2}(H_1^1(t)+H_1^2(t)) .$$

The Hankel functions have their respective asymptotic expansions[10)]

$$H_\nu^1(t)=\left(\frac{2}{\pi t}\right)^{1/2} e^{i(t-(\nu/2)\pi-\pi/4)} \sum_{m=0}^{\infty} \frac{(-)^m(\nu, m)}{(2it)^m}$$

$$H_\nu^2(t)=\left(\frac{2}{\pi t}\right)^{1/2} e^{-i(t-(\nu/2)\pi-\pi/4)} \sum_{m=0}^{\infty} \frac{(\nu, m)}{(2it)^m}$$

from which one gets

$$h_+(t)=\frac{1}{2}(H_0^1(t)+iH_1^1(t))=\left(\frac{2}{\pi t}\right)^{1/2} e^{i(t-\pi/4)} \sum_{m=0}^{\infty} \frac{(-)^m a_m}{(2it)^m}$$

$$h_-(t)=\frac{1}{2}(H_0^1(t)-iH_1^1(t))=\left(\frac{2}{\pi t}\right)^{1/2} e^{i(t-\pi/4)} \sum_{m=0}^{\infty} \frac{(-)^m b_m}{(2it)^m}$$

$$a_m=\frac{1}{2}(0, m)+\frac{1}{2}(1, m) , \quad a_0=1$$

$$b_m=\frac{1}{2}(0, m)-\frac{1}{2}(1, m) , \quad b_0=0$$

and

$$k_+(t)=\frac{1}{2}(H_0^2(t)+iH_1^2(t))=\left(\frac{2}{\pi t}\right)^{1/2} e^{-i(t-\pi/4)} \sum_{m=0}^{\infty} \frac{b_m}{(2it)^m}$$

$$k_-(t)=\frac{1}{2}(H_0^2(t)-iH_1^2(t))=\left(\frac{2}{\pi t}\right)^{1/2} e^{-i(t-\pi/4)} \sum_{m=0}^{\infty} \frac{a_m}{(2it)^m}$$

One sees that as $t\to\infty$,

$$h_+\to\left(\frac{2}{\pi t}\right)^{1/2} e^{i(t-\pi/4)} , \qquad h_-\to\left(\frac{2}{\pi t}\right)^{1/2} e^{i(t-\pi/4)} \frac{-b_1}{2it}$$

$$k_+\to\left(\frac{2}{\pi t}\right)^{1/2} e^{-i(t-\pi/4)} \frac{b_1}{2it} , \qquad k_-\to\left(\frac{2}{\pi t}\right)^{1/2} e^{-i(t-\pi/4)} .$$

§ 4. The singular part of an integral

If the range of integration for the Laplacian integrals is divided into two parts at $t=1$, the integrals from 0 to 1 of the Laplacian integrals are regular in z. The integrals from 1 to ∞, however, may be singular. An integral

$$\int_1^\infty e^{-\lambda t} \frac{dt}{t^{\nu+1}}$$

converges at $\lambda=0$ if $\nu>0$, but it is not always regular in λ. Making use of the Mellin integral representation of e^{-x}

$$e^{-x}=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\Gamma(s)x^{-s}\,ds\,,\quad \sigma>0$$

one gets

$$\int_1^\infty e^{-\lambda t}\frac{dt}{t^{\nu+1}}=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\Gamma(s)\lambda^{-s}\frac{1}{s+\nu}\,ds$$
$$=\Gamma(-\nu)\lambda^{\nu}+\sum_{n=0}^{\infty}\frac{(-)^n\lambda^n}{n!(\nu-n)}\,.$$

If ν is a half-integer, the integral turns out to be the sum of a regular function in λ and a singlar (two-valued) function in λ. The singular part is given by $\Gamma(-\nu)\lambda^{\nu}$. So one sees that

$$\text{Singular part of }\int_1^\infty e^{-\lambda t}\frac{dt}{t^{\nu+1}}=\Gamma(-\nu)\lambda^{\nu}\,.$$

The integral

$$\int_1^\infty e^{-izt}(J_0(t)+iJ_1(t))^3\,dt=\int_1^\infty e^{-izt}(h_+(t)+k_+(t))^3\,dt$$

may be split into the sum of four integrals,

$$\int_1^\infty e^{-izt}h_+^3(t)\,dt+3\int_1^\infty e^{-izt}h_+^2(t)k_+(t)\,dt+3\int_1^\infty e^{-izt}h_+(t)k_+^2(t)\,dt+\int_1^\infty e^{-izt}k_+^3(t)\,dt\,.$$

The first integral is singular at $z=3$, since

$$\int_1^\infty e^{-izt}h_+^3(t)\,dt=\int_1^\infty e^{-izt}\left(\frac{2}{\pi t}\right)^{3/2}e^{i3(t-\pi/4)}\left\{1+O\left(\frac{1}{t}\right)\right\}dt$$
$$=\left(\frac{2}{\pi}\right)^{3/2}e^{-(3/4)\pi i}\Gamma\left(-\frac{1}{2}\right)[i(z-3)]^{1/2}\{1+O(z-3)\}\,.$$

Similarly the second interal is singular at $z=1$, the third integral at $z=-1$ and the fourth integral at $z=-3$, these three integrals being all regular at $z=3$. So one gets

$$\text{Singular part of }\ v_0=-\frac{2}{\pi}\sqrt{z-3}\{1+O(z-3)\}\qquad\text{at}\quad z=3\,.$$

In the same way one gets

$$\text{Singular part of }\ v_1=-i\frac{2}{\pi}\sqrt{z-1}\{1+O(z-1)\}\qquad\text{at}\quad z=1\,,$$

$$\text{Singular part of }\ v_2=\frac{2}{\pi}\sqrt{z+1}\{1+O(z+1)\}\qquad\text{at}\quad z=-1\,,$$

Singular part of $v_3 = i\frac{2}{\pi}\sqrt{z+3}\{1+O(z+3)\}$ at $z=-3$.

§ 5. The expansion at a singular point

One denotes four eigenvalues of the matrix A by α_k, $k=0, 1, 2, 3$. They are singular points of the differential equation (13). Substitution of a series expansion of $\boldsymbol{v}$

$$\boldsymbol{v}=(z-\alpha_k)^{\lambda}(\boldsymbol{b}_0+(z-\alpha_k)\boldsymbol{b}_1+(z-\alpha_k)^2\boldsymbol{b}_2+\cdots)$$

into (13) gives the conditions to be satisfied by $\boldsymbol{b}_k$

$$\lambda(\alpha_k-A)\boldsymbol{b}_0=0 \tag{15}$$

$$(\lambda+1)(\alpha_k-A)\boldsymbol{b}_1=(B-\lambda)\boldsymbol{b}_0 \tag{16}$$

$$(\lambda+2)(\alpha_k-A)\boldsymbol{b}_2=(B-\lambda-1)\boldsymbol{b}_1 \tag{17}$$

The first condition requires that either $\lambda=0$ or $(\alpha_k-A)\boldsymbol{b}_0=0$. While the first case gives a regular solution, the second case may lead to a singular solution.

Case 1. $\lambda=0$. Regular solution

$$\boldsymbol{v}=\boldsymbol{b}_0+(z-\alpha_k)\boldsymbol{b}_1+(z-\alpha_k)^2\boldsymbol{b}_2+\cdots$$

$$(\alpha_k-A)\boldsymbol{b}_1=B\boldsymbol{b}_0 \tag{18}$$

$$2(\alpha_k-A)\boldsymbol{b}_2=(B-1)\boldsymbol{b}_1\,. \tag{19}$$

This case must lead to a solution with indices 0, 0, 0. So three components of $\boldsymbol{b}_0$ are adjustable. Since A is a diagonal matrix and α_k is an eigenvalue of A, the k-th component of $B\boldsymbol{b}_0$ must be 0. This condition allows to determine the remaining component of $\boldsymbol{b}_0$ by other components. The condition (18) gives $\boldsymbol{b}_1$ except for its k-th component, which is to be determined by the condition that the k-th component of $(B-1)\boldsymbol{b}_1$ must vanish. In this manner, successive coefficients $\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3, \cdots$ will be determined in terms of $\boldsymbol{b}_0$.

Case 2. $\lambda\neq 0$, $A\boldsymbol{b}_0=\alpha_k\boldsymbol{b}_0$. Singular solution.

This condition shows that $\boldsymbol{b}_0$ is an eigenvector of the matrix A corresponding to its eigenvalue α_k. Since the matrix A is diagonal, the vector $\boldsymbol{b}_0$ is a constant multiple of unit vector consisting of the k-th component 1 alone. The condition (16) imposes that the k-th component of the vector $(B-\lambda)\boldsymbol{b}_0$ vanish. The equation (13) gives therefore $\lambda=1/2$, for every value of k. Singular parts of v_k computed at the end of § 4 serve to determine $\boldsymbol{b}_0$

completely for all values of k. The coefficient vector $\boldsymbol{b}_1$ is determined by (16) except for its k-th component, which will be given by the condition the k-th component of $(B-3/2)\boldsymbol{b}_1=0$. Successive coefficients $\boldsymbol{b}_2, \boldsymbol{b}_3, \cdots$ will be determined similary. So the singular solution for every k will uniquely be determined. The singular solution, however, vanishes at $z=\alpha_k$, since its lowest term is a multiple of $(z-\alpha_k)^{1/2}$. Therefore one sees

$$\boldsymbol{v}(\alpha_k)=\boldsymbol{b}_0\,.$$

The value of $\boldsymbol{v}(\alpha_k)$, however, must be determined by a different approach,

§ 6. Anisotropic lattice Green functions

If the interaction of neighbouring spins differs according to the direction, one has eight lattice integrals

$$u_0(z)=\frac{1}{\pi^3}\iiint_0^\pi \frac{d\theta_1 d\theta_2 d\theta_3}{z-a\cos\theta_1-b\cos\theta_2-c\cos\theta_3}=\int_0^\infty e^{-zt}I_0(at)I_0(bt)I_0(ct)\,dt$$

$$u_1(z)=\frac{1}{\pi^3}\iiint_0^\pi \frac{\cos\theta_1\, d\theta_1 d\theta_2 d\theta_3}{z-a\cos\theta_1-b\cos\theta_2-c\cos\theta_3}=\int_0^\infty e^{-zt}I_1(at)I_0(bt)I_0(ct)\,dt$$

$$u_2(z)=\frac{1}{\pi^3}\iiint_0^\pi \frac{\cos\theta_2\, d\theta_1 d\theta_2 d\theta_3}{z-a\cos\theta_1-b\cos\theta_2-c\cos\theta_3}=\int_0^\infty e^{-zt}I_0(at)I_1(bt)I_0(ct)\,dt$$

$$u_3(z)=\frac{1}{\pi^3}\iiint_0^\pi \frac{\cos\theta_1\cos\theta_2\, d\theta_1 d\theta_2 d\theta_3}{z-a\cos\theta_1-b\cos\theta_2-c\cos\theta_3}=\int_0^\infty e^{-zt}I_1(at)I_1(bt)I_0(ct)\,dt$$

$$u_4(z)=\frac{1}{\pi^3}\iiint_0^\pi \frac{\cos\theta_3\, d\theta_1 d\theta_2 d\theta_3}{z-a\cos\theta_1-b\cos\theta_2-c\cos\theta_3}=\int_0^\infty e^{-zt}I_0(at)I_0(bt)I_1(ct)\,dt$$

$$u_5(z)=\frac{1}{\pi^3}\iiint_0^\pi \frac{\cos\theta_1\cos\theta_3\, d\theta_1 d\theta_2 d\theta_3}{z-a\cos\theta_1-b\cos\theta_2-c\cos\theta_3}=\int_0^\infty e^{-zt}I_1(at)I_0(bt)I_1(ct)\,dt$$

$$u_6(z)=\frac{1}{\pi^3}\iiint_0^\pi \frac{\cos\theta_2\cos\theta_3\, d\theta_1 d\theta_2 d\theta_3}{z-a\cos\theta_1-b\cos\theta_2-c\cos\theta_3}=\int_0^\infty e^{-zt}I_0(at)I_1(bt)I_1(ct)\,dt$$

$$u_7(z)=\frac{1}{\pi^3}\iiint_0^\pi \frac{\cos\theta_1\cos\theta_2\cos\theta_3\, d\theta_1 d\theta_2 d\theta_3}{z-a\cos\theta_1-b\cos\theta_2-c\cos\theta_3}=\int_0^\infty e^{-zt}I_1(at)I_1(bt)I_1(ct)\,dt\,.$$

Partial integration and use of the formula (6) on modified Bessel functions give a set of differential equations to be satisfied by $u_k(z)$, $k=0, 1, 2, \cdots, 7$ as follows

$$(z-P)\frac{d\boldsymbol{u}}{dz}=Q\boldsymbol{u} \tag{20}$$

$$P=\begin{bmatrix} 0 & a & b & 0 & c & 0 & 0 & 0 \\ a & 0 & 0 & b & 0 & c & 0 & 0 \\ b & 0 & 0 & a & 0 & 0 & c & 0 \\ 0 & b & a & 0 & 0 & 0 & 0 & c \\ c & 0 & 0 & 0 & 0 & a & b & 0 \\ 0 & c & 0 & 0 & a & 0 & 0 & b \\ 0 & 0 & c & 0 & b & 0 & 0 & a \\ 0 & 0 & 0 & c & 0 & b & a & 0 \end{bmatrix}, \quad Q=\frac{1}{2}\begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}, \quad \boldsymbol{u}=\begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{bmatrix}$$

Use of Pauli matrices σ_1, σ_3 and the direct product of matrices allows one to write

$$P=a\cdot\sigma_1\times1\times1+b\cdot1\times\sigma_1\times1+c\cdot1\times1\times\sigma_1$$

$$Q=\frac{1}{2}(1\times1\times1-\sigma_3\times1\times1-1\times\sigma_3\times1-1\times1\times\sigma_3)$$

where are used the abbreviations

$$\sigma_1=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \sigma_3=\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad 1=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

It is to be noted that elimination of $u_1, u_2, u_3, \cdots, u_7$ from the equation (20) is very difficult even though the elimination leads to a differential equation of the fifth order for u_0, the coefficient of the fifth derivative of of u_0 being a polynomial of the eighth degree in z. So a direct approach to (20) may be preferable.

If one introduces an orthogonal matrix T defined by

$$T=\tau\times\tau\times\tau=\frac{1}{2\sqrt{2}}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{bmatrix}$$

$$\tau=\frac{\sigma_1+\sigma_3}{\sqrt{2}}=\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},$$

and puts

$$\boldsymbol{u}=T\boldsymbol{v}$$

one gets then

$$\boldsymbol{v}=T\boldsymbol{u}$$

and the following integral representations of v_k, $k=0, 1, \cdots, 7$,

$$v_0(z)=\frac{1}{2\sqrt{2}}\int_0^\infty e^{-zt}(I_0(at)+I_1(at))(I_0(bt)+I_1(bt))(I_0(ct)+I_1(ct))\,dt$$
$$v_1(z)=\frac{1}{2\sqrt{2}}\int_0^\infty e^{-zt}(I_0(at)-I_1(at))(I_0(bt)+I_1(bt))(I_0(ct)+I_1(ct))\,dt$$
$$v_2(z)=\frac{1}{2\sqrt{2}}\int_0^\infty e^{-zt}(I_0(at)+I_1(at))(I_0(bt)-I_1(bt))(I_0(ct)+I_1(ct))\,dt$$
$$v_3(z)=\frac{1}{2\sqrt{2}}\int_0^\infty e^{-zt}(I_0(at)-I_1(at))(I_0(bt)-I_1(bt))(I_0(ct)+I_1(ct))\,dt$$
$$v_4(z)=\frac{1}{2\sqrt{2}}\int_0^\infty e^{-zt}(I_0(at)+I_1(at))(I_0(bt)+I_1(bt))(I_0(ct)-I_1(ct))\,dt$$
$$v_5(z)=\frac{1}{2\sqrt{2}}\int_0^\infty e^{-zt}(I_0(at)-I_1(at))(I_0(bt)+I_1(bt))(I_0(ct)-I_1(ct))\,dt$$
$$v_6(z)=\frac{1}{2\sqrt{2}}\int_0^\infty e^{-zt}(I_0(at)+I_1(at))(I_0(bt)-I_1(bt))(I_0(ct)-I_1(ct))\,dt$$
$$v_7(z)=\frac{1}{2\sqrt{2}}\int_0^\infty e^{-zt}(I_0(at)-I_1(at))(I_0(bt)-I_1(bt))(I_0(ct)-I_1(ct))\,dt\,.$$

The differential equation (20) for $\boldsymbol{u}$ is transformed into the differential equation for $\boldsymbol{v}$

$$(z-A)\frac{d\boldsymbol{v}}{dz}=B\boldsymbol{v} \tag{21}$$

$$A=T^{-1}PT=a\cdot\sigma_3\times1\times1+b\cdot1\times\sigma_3\times1+c\cdot1\times1\times\sigma_3$$

$$B=T^{-1}QT=\frac{1}{2}(1\times1\times1-\sigma_1\times1\times1-1\times\sigma_1\times1-1\times1\times\sigma_1)$$

$$A=\begin{bmatrix} a+b+c & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & -a+b+c & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & a-b+c & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -a-b+c & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & a+b-c & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & -a+b-c & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & a-b-c & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -a-b-c \end{bmatrix}$$

$$B=\frac{1}{2}\begin{bmatrix} 1 & -1 & -1 & 0 & -1 & 0 & 0 & 0\\ -1 & 1 & 0 & -1 & 0 & -1 & 0 & 0\\ -1 & 0 & 1 & -1 & 0 & 0 & -1 & 0\\ 0 & -1 & -1 & 1 & 0 & 0 & 0 & -1\\ -1 & 0 & 0 & 0 & 1 & -1 & -1 & 0\\ 0 & -1 & 0 & 0 & -1 & 1 & 0 & -1\\ 0 & 0 & -1 & 0 & -1 & 0 & 1 & -1\\ 0 & 0 & 0 & 1 & 0 & -1 & -1 & 1 \end{bmatrix}.$$

The differential equation (21) has eight regular singular points $a+b+c\equiv\alpha_0$, $-a+b+c\equiv\alpha_1$, $a-b+c\equiv\alpha_2$, $-a-b+c\equiv\alpha_3$, $a+b-c\equiv\alpha_4$, $-a+b-c\equiv\alpha_5$, $a-b-c\equiv\alpha_6$, $-a-b-c\equiv\alpha_7$ and a regular singular point $z=\infty$.

A similar reasoning to that of § 2 leads to the expansion at $z=\infty$,

$$u=\frac{c_0}{z}+\frac{c_1}{z^2}+\frac{c_2}{z^3}+\frac{c_3}{z^4}+\cdots$$

$$c_0=\begin{bmatrix}1\\0\\0\\0\\0\\0\\0\\0\end{bmatrix},\quad c_1=\begin{bmatrix}0\\a/2\\b/2\\0\\c/2\\0\\0\\0\end{bmatrix},\quad c_2=\begin{bmatrix}(a^2+b^2+c^2)/2\\0\\0\\2ab/3\\0\\2ca/3\\2bc/3\\0\end{bmatrix},\quad c_3=\begin{bmatrix}0\\a(3a^2+7b^2+7c^2)/8\\b(7a^2+3b^2+7c^2)/8\\0\\c(7a^2+7b^2+3c^2)/8\\0\\0\\abc\end{bmatrix},$$

$$c_4=\begin{bmatrix}3(a^2+b^2+c^2)^2/8+b^2c^2+c^2a^2+a^2b^2\\0\\0\\ab(5a^2+5b^2+11c^2)/6\\0\\ca(5a^2+11b^2+5c^2)/6\\bc(11a^2+5b^2+5c^2)/6\\0\end{bmatrix},\quad\ldots.$$

A similar reasoning to that of § 4 leads to the computation of singular parts of $v_k(z)$ at $z=\alpha_k$, α_k denoting one of eight singular points,

$$[\text{Singular part of } v_k(z) \text{ at } z=\alpha_k]=\varepsilon_k\frac{2}{\pi}\left(\frac{z-\alpha_k}{abc}\right)^{1/2}\{1+O(z-\alpha_k)\}$$

where $\varepsilon_0=-1$, $\varepsilon_1=\varepsilon_2=\varepsilon_4=-i$, $\varepsilon_3=\varepsilon_5=\varepsilon_6=1$, $\varepsilon_7=i$.

In the same way as in § 5, the expansion at a singular point α_k needs the value of $v(\alpha_k)$, which must be provided by a different approach.

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