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On Type(τ, 1) Cylindrical Measures on Banach Spaces with Schauder Bases

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1. Introduction

In [2] and [3], L. Schwartz has discussed the connection between cylindrical measures on a locally convex Hausdorff space and random functions on its topological dual space. A special case of this result is that cylindrical measures on l^p are representable as sequences of real random variables. As further generalization of these discussions, the author has gotten the similar result for every Banach space with a shrinking basis ([5]). If we restrict cylindrical measures to be of type(τ, 1), which will be defined later, the same result for more general Banach spaces will be able to be verified. In this note we deal with the relation between cylindrical measures of type(τ, 1) on Banach spaces with Schauder bases and sequences of real random variables.

2. Preliminaries

First we present the following theorem, which is due to L. Schwartz.

THEOREM ([2]). *Let E be a locally convex Hausdorff space and E' be a topological dual space of E. There exists a bijective correspondence between the cylindrical measures on E and the isonomy classes of linear random functions on E'. Moreover, the following two statements are equivalent. Let $\mathfrak{S}$ be a saturated collection of subsets of E.*

(i) *The cylindrical measure μ on E is scalarly $\mathfrak{S}$-concentrated.*

(ii) *The associated random function $f : E_{\mathfrak{S}}' \to L^{\circ}(\Omega, m)$ is continuous if L° is equipped with the topology of convergence in probability, where $E_{\mathfrak{S}}'$ means that E' is endowed with $\mathfrak{S}$-topology and $L^{\circ}(\Omega, m)$ is the space of all real random variables defined on the probability space (Ω, m).*

Now we explain the type(τ, 1) cylindrical measures which have been discussed in [3] and [4].

DEFINITION. Let X be a Banach space. A cylindrical measure μ on X is called to be of *type*(τ, 1) if the associated linear random function f is a continuous map from X' equipped with the Mackey topology $\tau(X', X)$ into $L^1(\Omega, m)$ (not only into L°). Of course, in this case the space L^1 is equipped with the usual norm topology.

The above theorem shows that every type(τ, 1) cylindrical measure is scalarly concentrated on the saturated collection of weakly compact subsets of X.

3. Cylindrical measures on $\sigma(X, M')$

Let X be a Banach space with a Schauder basis $\{x_n\}_{n=1}^\infty$, $\{x_n'\}_{n=1}^\infty$ its biorthogonal functionals and M' the closed linear span of the set $\{x_n'\}_{n=1}^\infty$. It is checked that M' is a closed subspace of X' in the sense of its norm topology and also dense in X' for the weak* topology. Clearly, the pair (X, M') is a dual system. We denote by Y the space X equipped with the weak topology $\sigma(X, M')$. In this section we shall investigate the characterization of the cylindrical measures on Y.

To begin with, we consider some sorts of topology on M' which is the dual of Y. The first one is the relative topology of the Mackey topology $\tau(X', X)$ on X' and we denote it by t_1. The next one is the Mackey topology $\tau(M', X)$ denoted by t_2, and the last one is the relative topology of the norm topology on X', denoted by t_3. Obviously, t_2 is finer than or equal to t_1 and t_3 is finer than or equal to t_2. We denote by M_i' the space M' equipped with the topology t_i and by $\mathfrak{F}^i$ the class of all continuous linear random functions from M_i' into L° for $i=1, 2, 3$.

The next result is easily verified from the result of [2] (p. 265, Th. 2); its proof will be omitted.

PROPOSITION 1. *Let E be a locally convex Hausdorff space, μ a cylindrical measure on E, $f : E' \to L^\circ$ a linear random function associated with μ, and $\mathfrak{M}$ a set of all Radon probabilities on* **R**. *The following conditions are equivalent*:

(i) *the map $\xi \mapsto \xi(\mu)$ from E' into $\mathfrak{M}$ equipped with the narrow topology is continuous* ;

(ii) *the random function $f : E' \to L^\circ$ is continuous if L° is equipped with the topology of convergence in probability.*

Here we can consider arbitrary compatible topology on E'. Then we have the next result.

PROPOSITION 2. *If a linear random function f belongs to $\mathfrak{F}^i$, then every linear random function which is isonomous to f, is contained in $\mathfrak{F}^i$ for $i=1, 2, 3$.*

This proposition shows that the condition that f belongs to $\mathfrak{F}^i$, depends only on the isonomy class. Therefore, by the theorem in §2 we can classify the cylindrical measures on Y in the following way. Let $\mathfrak{C}^i$ be the class of all cylindrical measures on Y such that the associated random functions are contained in $\mathfrak{F}^i$ for $i=1, 2, 3$. Using the result that t_i is finer than or equal to t_{i-1} for $i=2, 3$, we have

PROPOSITION 3. $\mathfrak{C}^1 \subset \mathfrak{C}^2 \subset \mathfrak{C}^3$.

THEOREM 1. *There exists a bijective correspondence between the cylindrical measures on* Y *belonging to* $\mathfrak{C}^3$, *and the isonomy classes of sequences* $\{\varphi_n\}$ *of real random variables satisfying the condition that for every* $x' \in M'$, *the series* $\sum \varphi_n < x_n, x' >$ *converges in probability.*

The proof will be followed by the same method of [5] (p. 3, Th. 2). Avoiding duplication, we omit to state this proof.

REMARK 1. We assume X to be a dual of some Banach space, i.e. $X = A'$ where A is a Banach space. It follows immediately from [1] and the hypothesis that X has a Schauder basis, that A has a shrinking basis, and we denote it by $\{a_n\}$. We have the biorthogonal functionals $\{a_n'\}$ of $\{a_n\}$. Since the sequence $\{a_n'\}$ is a Schauder basis of X (not necessarily the same basis as original one), we can define several terms, e.g. M_*', Y_*, $\mathfrak{F}_*^i$ and $\mathfrak{C}_*^i$ respectively corresponding to M', Y, $\mathfrak{F}^i$ and $\mathfrak{C}^i$, using the new basis $\{a_n'\}$. In this case, we have a bijective correspondence between the cylindrical measures on Y_* belonging to $\mathfrak{C}_*^2$, and the isonomy classes of sequences $\{\psi_n\}$ of real random variables satisfying the condition that for every $y' \in M_*'$, the series $\sum \psi_n \langle a'_n, y' \rangle$ converges in probability, because M_*' coincides with A and so $\mathfrak{C}_*^2$ coincides with $\mathfrak{C}_*^3$.

REMARK 2. Through this section, we can replace the space L° by L^1 and the topology of convergence in probability by the usual norm topology, and in this case we say C^i instead of $\mathfrak{C}^i$.

4. Cylindrical measures on X

We adopt the same notation employed in the preceding section.

The next result has been shown by L. Schwartz.

THEOREM ([2]). *Let* E *and* F *be locally convex Hausdorff spaces and* $j : E \to F$ *a continuous, injective, linear map. If* μ *and* ν *are two cylindrical measures on* E, *both scalarly concentrated on a collection of balanced, convex, weakly compact subsets of* E, *and if* $j(\mu) = j(\nu)$, *then* $\mu = \nu$.

PROPOSITION 4. *Let* i *be an identity map from* X *onto* Y. *If* μ *and* ν *are two cylindrical measures of type*(τ, 1) *on* X, *and if* $i(\mu) = i(\nu)$ *then* $\mu = \nu$.

Proposition 4 is an easy consequence of the above theorem, and also it is clearly seen that $i(\mu)$ belongs to C^1 if μ is of type(τ, 1).

We now pass to the following theorem.

THEOREM 2. *There exists a bijective correspondence between the cylindrical measures of type*(τ, 1) *on* X *and the cylindrical measures on* Y *belonging to* C^1.

PROOF. We have already checked that the identity map i induces an injective map from the collection of all cylindrical measures of type(τ, 1) on X into the collection C^1. It is remained to show that the map is surjective. Given any cylindrical measure μ of C^1, we have the associated random function f which is continuous from M' into L^1 if M' is equipped with the topology t_1. We recall the next result that M' is the closed subspace of X' in the sense of its norm topology and dense in X' for the weak* topology, and therefore also for the Mackey topology. Then there exists the extension $\tilde{f}$ of f such that $\tilde{f}$ is continuous from X' into L^1 if X' is equipped with the Mackey topology $\tau(X', X)$. We have the cylindrical measure of type(τ, 1) associated with $\tilde{f}$, say λ. It is obvious that $i(\lambda)=\mu$. Thus the proof is complete.

It follows from Theorems 1 and 2 that every cylindrical measure of type(τ, 1) on X is representable as a sequence of real random variables, which is uniquely determined, in disregard of isonomy. However, at present we do not get the sufficient condition for the sequence of real random variables to define the cylindrical measure of type(τ, 1).

The restriction of cylindrical measures to type(τ, 1) is not essential damage for the investigation of cylindrical measure theory. This is supported by the fact that every cylindrical Gauss measure on an arbitrary Banach space is of type(τ, 1).

Finally, we add some argument about bases of Banach spaces. A sequence $\{x_n\}_{n=1}^{\infty}$ in a Banach space X is called a Schauder basis of X if for every $x \in X$ there is a unique sequence of scalars $\{\alpha_n\}_{n=1}^{\infty}$ so that $x=\sum_{n=1}^{\infty} \alpha_n x_n$. The important examples of Schauder bases are the Haar system in $L^p(0, 1)(1 \leq p < \infty)$, the Schauder system in $C(0, 1)$ and the sequence of unit vectors in each of the spaces c_0 and $l^p(1 \leq p < \infty)$. In particular, each of the spaces $L^p(0, 1)$, $l^p(1 < p < \infty)$ and c_0 has a shrinking basis, i.e. its dual has a Schauder basis. The object of this note is a generalization of the theory for the spaces which have Schauder bases but not shrinking bases.

References

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