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On the Proper Space of Δ for m-Forms in $2m$ Dimensional Conformally Flat Riemannian Manifolds

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Introduction

Let M^n be a compact orientable conformally flat Riemannian manifold with positive constant scalar curvature R. If the Ricci form is positive definite [8] or the curvature operator is positive [6], then M^n is a space of constant curvature. So, we are interested in M^n without these rather strong assumptions. Recently A. Avez and A. Heslot [1] showed that the proper value λ of the Laplacian operator Δ for m-forms in M^{2m} satisfies $\lambda \geqq m^2 k$, where $k=R/2m(2m-1)$. As they have written, this lower bound is not best and we shall show in this note that the best one is $m(m+1)k$ by making use of Gallot-Meyer's method, [2], [3].

On the other hand, in a compact orientable Riemannian manifold of positive curvature operator, the proper form of Δ corresponding to the possible minimal proper value is Killing or closed conformal Killing, [7], [9]. We shall see a similar fact for m-forms in M^{2m} and show that the proper space corresponding to the proper value $m(m+1)k$ is just the vector space of conformal Killing m-forms.

Throughout the paper, manifolds are assumed to be connected and the differentiability be C^∞.

§ 1. Preliminaries.

Let M^n be a compact orientable Riemannian manifold. Let us denote by g_{ji}, $R_{kji}{}^r$, $R_{ji}=R_{rji}{}^r$ and $R=g^{ji}R_{ji}$ the metric, the curvature, the Ricci tensors and the scalar curvature respectively. We shall represent tensors by their components with respect to the natural base, and the summation convention is assumed. ∇ means the operator of covariant differentiation and $\Delta=d\delta+\delta d$ denotes the Laplacian operator.

For an m-form $u=(1/m!)\,u_{i_1\cdots i_m}\,dx^{i_1}\wedge\cdots\wedge dx^{i_m}$, Δu is written explicitely as

$$(\Delta u)_{i_1\cdots i_m} = -\nabla^r\nabla_r u_{i_1\cdots i_m} + H(u)_{i_1\cdots i_m}$$

for $m\geqq 2$, where

$$H(u)_{i_1\cdots i_m}=\sum_h R_{i_h}{}^r u_{i_1\cdots r\cdots i_m}+\sum_{h<l} R_{i_h i_l}{}^{rs} u_{i_1\cdots r\cdots s\cdots i_m}\,.$$

If a non-zero m-form u satisfies $\Delta u=\lambda u$ with a constant λ, it is called a proper form of Δ corresponding to the proper value λ.

The quadratic form $F_m(u)$ $(m\geqq 2)$ of u is defined by

$$F_m(u)=\frac{1}{m!}H(u)_{i_1\cdots i_m}u^{i_1\cdots i_m}$$

$$=\frac{1}{(m-1)!}\left(R_{rs}u^{ri_2\cdots i_m}u^s{}_{i_2\cdots i_m}+\frac{m-1}{2}R_{rsab}u^{rsi_3\cdots i_m}u^{ab}{}_{i_3\cdots i_m}\right).$$

We denote

$$\langle u, v\rangle=(1/m!)\,u_{i_1\cdots i_m}v^{i_1\cdots i_m}\,,\qquad |u|^2=\langle u, u\rangle\,,$$

$$|\nabla u|^2=(1/m!)\,\nabla_r u_{i_1\cdots i_m}\nabla^r u^{i_1\cdots i_m}$$

for m-forms u and v.

The following formula is well known for any m-form u:

$$\frac{1}{2}\Delta(|u|^2)=\langle u, \Delta u\rangle-|\nabla u|^2-F_m(u)\,. \tag{1.1}$$

An m-form u or a skew symmetric tensor $u_{i_1\cdots i_m}$ is called *Killing* if the tensor $\nabla_{i_0}u_{i_1\cdots i_m}$ is skew:

$$\nabla_{i_0}u_{i_1\cdots i_m}+\nabla_{i_1}u_{i_0 i_2\cdots i_m}=0\,.$$

A Killing m-form u is coclosed and satisfies

$$m\nabla_a\nabla_b u_{i_1\cdots i_m}+\sum_h R_{i_h ba}{}^c u_{i_1\cdots c\cdots i_m}-\sum_{h<l} R_{i_h i_l a}{}^c u_{i_1\cdots c\cdots b\cdots i_m}=0\,,$$

where c and b appear respectively at the h-th and l-th positions in lower indices of u. From this equation we know that

$$m\nabla^r\nabla_r u_{i_1\cdots i_m}+H(u)_{i_1\cdots i_m}=0 \tag{1.2}$$

holds good for any Killing m-form u. For further details, see [5].

An m-form u is called *conformal Killing* if it satisfies

$$\nabla_{i_0}u_{i_1\cdots i_m}+\nabla_{i_1}u_{i_0 i_2\cdots i_m}=2\theta_{i_2\cdots i_m}g_{i_0 i_1}$$
$$-\sum_{h>1}(-1)^h(\theta_{i_1\cdots\hat{i}_h\cdots i_m}g_{i_0 i_h}+\theta_{i_0 i_2\cdots\hat{i}_h\cdots i_m}g_{i_1 i_h})\,,$$

where θ is an $(m-1)$-form, and $\hat{i}_h$ means that i_h is deleted.

A conformal Killing m-form u satisfies

$$\nabla^r u_{ri_2\cdots i_m}=(n-m+1)\,\theta_{i_2\cdots i_m}\,, \tag{1.3}$$

$$m\nabla^r\nabla_r u_{i_1\cdots i_m}+H(u)_{i_1\cdots i_m}=(n-2m)\sum_h(-1)^{h-1}\nabla_{i_h}\theta_{i_1\cdots\hat{i}_h\cdots i_m}\,. \tag{1.4}$$

From (1.3) we know that a conformal Killing m-form is Killing if and only if it is coclosed. It should be noticed that (1.4) coincides with (1.2) when $n=2m$. For further details, see [4].

We need the following (Cf. [7])

Lemma 1. *For any m-form u in an n dimensional Riemannian manifold,*

$$|\nabla u|^2 \geqq \frac{1}{m+1}|du|^2+\frac{1}{n-m+1}|\delta u|^2$$

holds. If the equality holds for a coclosed (resp. closed) u, then u is Killing (resp. conformal Killing).

§2. Conformally flat manifold.

In this section, we assume that $n\geqq 4$ and M^n is conformally flat. If we define k and L_{ji} by

$$R=n(n-1)k\,,\qquad L_{ji}=R_{ji}-(n/2)kg_{ji}\,,$$

the curvature tensor satisfies

$$(n-2)R_{kjih}=L_{ji}g_{kh}-L_{ki}g_{jh}+g_{ji}L_{kh}-g_{ki}L_{jh}\,.$$

Hence we have the following formulas for any m-form u:

$$(n-2)R_{i_h i_l}{}^{rs}u_{i_1\cdots r\cdots s\cdots i_m}=-2(L_{i_l}{}^{r}u_{i_1\cdots i_h\cdots r\cdots i_m}+L_{i_h}{}^{r}u_{i_1\cdots r\cdots i_l\cdots i_m})\,,$$

$$(n-2)\sum_{h<l}R_{i_h i_l}{}^{rs}u_{i_1\cdots r\cdots s\cdots i_m}=-2(m-1)\left(\sum_h R_{i_h}{}^{r}u_{i_1\cdots r\cdots i_m}-\frac{nmk}{2}u_{i_1\cdots i_m}\right),$$

$$H(u)_{i_1\cdots i_m}=\frac{1}{n-2}\{(n-2m)\sum R_{i_h}{}^{r}u_{i_1\cdots r\cdots i_m}+nm(m-1)ku_{i_1\cdots i_m}\}\,,$$

$$F_m(u)=\frac{1}{(m-1)!\,(n-2)}\{(n-2m)R_{rs}u^{ri_2\cdots i_m}u^{s}{}_{i_2\cdots i_m}+n(m-1)ku_{i_1\cdots i_m}u^{i_1\cdots i_m}\}\,.$$

Especially $H(u)_{i_1\cdots i_m}$ and $F_m(u)$ are simplified in the case of $n=2m$ as follows:

(2.1) $$H(u)_{i_1\cdots i_m}=m^2ku_{i_1\cdots i_m}\,,\qquad F_m(u)=m^2k|u|^2\,.$$

A well known result deduced from (2.1) is that the m-th Betti number of M^{2m} vanishes if k is positive everywhere, [10], [1].

§3. Theorems.

In this section, we shall consider a conformally flat M^{2m} ($2m\geqq 4$) with positive constant scalar curvature.

By integration of (1.1) over M^{2m} with respect to the volume element we have

$$(u,\Delta u)=\|\nabla u\|^2+m^2k\|u\|$$

for any m-form u, where $(\,,\,)$ and $\|\ \|$ mean the global inner product and the global norm respectively. Hence, taking account of Lemma 1, we get

$$(u, \Delta u) \geqq \frac{1}{m+1} (\|du\|^2 + \|\delta u\|^2) + m^2 k \|u\|^2 = \frac{1}{m+1} (u, \Delta u) + m^2 k \|u\|^2 ,$$

from which it follows that

$$(u, \Delta u) \geqq m(m+1) k \|u\|^2 .$$

Thus we have

Theorem 1. *In a 2m dimensional compact orientable conformally flat Riemannian manifold with positive constant scalar curvature $R=2m(2m-1)k$, the proper value λ of Δ for m-forms satisfies*

$$\lambda \geqq m(m+1)\, k .$$

If there exists a coclosed (resp. closed) proper m-form corresponding to $\lambda=m(m+1)k$, then it is Killing (resp. conformal Killing).

Let C_p, $C_p{}^d$ and K_p be the vector spaces (over the real field) with natural structure defined by

C_p=the set of all conformal Killing p-forms,
$C_p{}^d$=the set of all closed conformal Killing p-forms,
K_p=the set of all Killing p-forms.

It is clear that $K_p \subset C_p$ and $C_p{}^d \subset C_p$.

Now, let $u \in C_m$ in our M^{2m}. Then we have from (1.4)

$$m \nabla^r \nabla_r u_{i_1 \cdots i_m} + H(u)_{i_1 \cdots i_m} = 0 .$$

Hence it follows that

$$(\Delta u)_{i_1 \cdots i_m} = -\nabla^r \nabla_r u_{i_1 \cdots i_m} + H(u)_{i_1 \cdots i_m} = \frac{m+1}{m} H(u)_{i_1 \cdots i_m} ,$$

and by virtue of (2.1) we have

$$\Delta u = m(m+1)\, ku .$$

Thus we know that any conformal Killing m-form is a proper form corresponding to $\lambda=m(m+1)k$. Therefore, combining with Theorem 1, we get a characterization of the Killing and the closed conformal Killing m-forms as follows:

Lemma 2. *Under the assumptions stated in Theorem 1, a coclosed (resp. closed) m-form u is Killing (resp. closed conformal Killing) if and only if it is a proper form of Δ corresponding to the possible minimal proper value $m(m+1)k$.*

It is known [4] that the direct sum

$$C_p = K_p \oplus C_p{}^d , \qquad 1 \leqq p \leqq n-1 ,$$

holds in manifolds of constant curvature $k \neq 0$ (without the assumption *compact orientable*).

Now we are in a position to prove the following

Theorem 2. *In a 2m dimensional compact orientable conformally flat Riemannian manifold with positive constant scalar curvature, the proper space of Δ for m-forms corresponding to the minimal proper value $m(m+1)k$ is C_m, and we have*

$$C_m = K_m \oplus C_m{}^d \qquad (\textit{direct sum}) .$$

Proof. Let P be the proper space of Δ corresponding to $m(m+1)k$. We have already shown that $C_m \subset P$. As there does not exist a non-trivial harmonic m-form in our M^{2m}, we have $K_m \cap C_m{}^d = \{0\}$ and $K_m \oplus C_m{}^d \subset C_m$.

Let $u \in P$, and u is decomposed uniquely into $u = v + w$, where v is coclosed and w is closed. As we have $\Delta v + \Delta w = m(m+1)k(v+w)$ and Δv(resp. Δw) is coclosed (resp. closed), it follows that

$$\Delta v = m(m+1)\,kv\,, \qquad \Delta w = m(m+1)\,kw$$

by the uniqueness of the decomposition. Thus $v \in K_m$ and $w \in C_m{}^d$ by virtue of Lemma 2, and hence $u \in K_m \oplus C_m{}^d$. Therefore $P = C_m = K_m \oplus C_m{}^d$ is proved. q.e.d.

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