Natural Science Report, Ochanomizu University, Vol. 29, No. 2, 1978 
お茶の水女子大学 自然科学報告 第29巻 第2号

# On Conformally Flat Spaces with Warped Product Riemannian Metric

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(Received, September 6, 1978)

## Introduction

Conformally flat hypersurfaces of the Euclidean space are studied by E. Cartan, J. Schouten, Nishikawa-Maeda and others. S. Nishikawa has determined such hypersurfaces under the assumption of analyticity. They are the examples of conformally flat spaces which are not constant curvature.

Let $M^n(c)$ be an $n$-dimensional Riemannian space of constant sectional curvature $c$. It is well known that $M^n(c)\times M^1$ and $M^n(c)\times M^m(-c)$ are conformally flat and clearly not of constant curvature. In this paper, we shall give other examples of conformally flat spaces making the product metric twisted.

## 1. Warped product spaces.

Let $M^n$ and $M'^m$ be the Riemannian spaces with dimension $n$ and $m$. We take a positive $C^\infty$-function $f$ on $M^m$. Then a warped product (Riemannian) space $\tilde{M}^{n+m}=M^n\times_f M'^m$ is defined by the Riemannian metric $d\tilde{\sigma}^2=d\sigma^2+f^2d\sigma'^2$, where $d\sigma$ and $d\sigma'$ are line elements of $M^n$ and $M'^m$ respectively. More precisely, taking the natural projections $p$: $\tilde{M}^{n+m}\to M^n$ and $p'$: $\tilde{M}^{n+m}\to M'^m$, the warped product metric is given by

$$d\tilde{\sigma}^2=p_*(d\sigma^2)+(p_*f)^2p'^*(d\sigma'^2)$$

where $p_*$ and $p_*'$ are the pull back operators induced by $p$ and $p'$.

In the following we make the consensus that the indices $i, j, k, \cdots$ run over the range $1,\cdots, n$, $\alpha, \beta, \gamma,\cdots$ the range $n+1,\cdots, n+m$ and $A, B, C,\cdots$ the range $1,\cdots, n+m$.

We take the orthonormal vector fields $e_i$, $e_\alpha$ and the dual basis $\omega_i$, $\omega_\alpha$ on $M^n$ and $M'^m$. The Riemannian connection forms $\omega_{ij}$ and $\omega_{\alpha\beta}$ satisfy

$$d\omega_i=\sum\omega_{ij}\wedge\omega_j\,,\qquad \omega_{ij}+\omega_{ji}=0\,,$$

$$d\omega_\alpha=\sum\omega_{\alpha\beta}\wedge\omega_\beta\,,\qquad \omega_{\alpha\beta}+\omega_{\beta\alpha}=0\,.$$

The orthonormal vector fields on $M^{n+m}$ are given by

$$\tilde{e}_i=e_i\,,\qquad \tilde{e}_\alpha=(p_*f)^{-1}e_\alpha$$

where $e_i$ and $e_\alpha$ are identified with the vector fields on $\tilde{M}^{n+m}$ by the natural injections. Then the dual basis $\varpi_A$ of 1-forms on $\tilde{M}^{n+m}$ are given by

$$\varpi_i = p_* \omega_i , \qquad \varpi_\alpha = (p_* f)\, p'_* \omega_\alpha .$$

Notations: We write down only quantities on $\tilde{M}^{n+m}$. Curvature form and cruvature tensor: $\tilde{\Omega}_{AB}=(1/2)\sum \tilde{R}_{ABCD}\, \varpi^C{}_\wedge \varpi^D$, coefficients of Riemannian connection: $\varpi_{AB}=\sum \tilde{\gamma}_{ABC}\, \varpi_C$ , Ricci tensor; $\tilde{R}_{AB}=\sum \tilde{R}_{CABC}=\sum \tilde{\Omega}_{CA}(\tilde{e}_B, \tilde{e}_C)$, scalar curvature; $\tilde{R}=\sum \tilde{R}_{AA}$. Then we have

$$d\varpi_A = \sum \varpi_{AB\wedge}\varpi_B ,$$

$$\tilde{\Omega}_{AB} = d\varpi_{AB} - \sum \varpi_{AC\wedge}\varpi_{CB} .$$

LEMMA 1. *The connection forms* $\varpi_{AB}$ *satisfy*

$$\varpi_{ij} = p^* \omega_{ij} , \qquad \varpi_{i\alpha} = \tilde{a}_i \varpi_\alpha ,$$

$$\varpi_{\alpha\beta} = p'_* \omega_{\alpha\beta} ,$$

*where we put* $p_* d(\log f) = \sum \tilde{a}_i \varpi_i$ .

PROOF. Making use of the structure equations on $M^n$, $M'^m$ and $\tilde{M}^{n+m}$, we have

$$\tilde{\gamma}_{ijk} = p_* \gamma_{ijk}, \qquad \tilde{\gamma}_{\alpha\beta\gamma} = p'_* \gamma'_{\alpha\beta\gamma}$$

$$\tilde{\gamma}_{ij\alpha} = -\tilde{\gamma}_{i\alpha j} = -\tilde{\gamma}_{\alpha ij} = 0 , \qquad \tilde{\gamma}_{\alpha\beta i} = 0 ,$$

$$\tilde{\gamma}_{i\alpha\beta} = \tilde{a}_i\, \delta_{\alpha\beta} = -\tilde{\gamma}_{\alpha i\beta} ,$$

from which the lemma follows easily.

LEMMA 2. *The curvature forms* $\tilde{\Omega}_{AB}$ *satisfy*

$$\tilde{\Omega}_{ij} = p_* \Omega_{ij} , \qquad \tilde{\Omega}_{i\alpha} = \sum p_* (\nabla_k f_i / f) \varpi_{k\wedge} \varpi_\alpha ,$$

$$\tilde{\Omega}_{\alpha\beta} = p'_* \Omega'_{\alpha\beta} + p_* \left(\sum f_k^2 / f^2\right) \varpi_{\alpha\wedge} \varpi_\beta ,$$

*where* $df = \sum f_k \omega_k$ *and* $\nabla_k f_i$ *is the covariant derivative of* $f_i$ *by the connection* $\omega_{ij}$.

PROOF. We only show the second equation.

$$\begin{aligned}
\tilde{\Omega}_{i\alpha} &= d\varpi_{i\alpha} - \sum \varpi_{iA\wedge} \varpi_{A\alpha} \\
&= d\tilde{a}_{i\wedge}\varpi_\alpha + \tilde{a}_i \left(\sum \varpi_{\alpha j\wedge}\varpi_j + \sum \varpi_{\alpha\beta\wedge}\varpi_\beta\right) \\
&\qquad - \sum (p_*\omega_{ik})_\wedge (\tilde{a}_k \varpi_\alpha) - \sum (\tilde{a}_i \varpi_\beta)_\wedge (p'_* \omega_{\beta\alpha}) \\
&= \left(d\tilde{a}_i + \tilde{a}_i \sum \tilde{a}_j \varpi_j - \sum (p_* \omega_{ik})\, \tilde{a}_k\right)_\wedge \varpi_\alpha .
\end{aligned}$$

If we put $d(\log f) = \sum a_i \omega_i$, then $p_* a_i = \tilde{a}_i$ holds good. Since $p_*$ and $d$ commute each other, we have

$$\begin{aligned}
\tilde{\Omega}_{i\alpha} &= p_* \left(da_i + \sum a_k \omega_{ki}\right)_\wedge \varpi_\alpha + (p_* d(\log f)\, a_i)_\wedge \varpi_\alpha \\
&= \sum p_* \left(\nabla_k (f_i / f) + (f_k / f)(f_i / f)\right) \varpi_{k\wedge} \varpi_\alpha \\
&= \sum p_* (\nabla_k f_i / f) \varpi_{k\wedge} \varpi_\alpha .
\end{aligned}$$

LEMMA 3. *The Ricci tensors* $\tilde{R}_{AB}$ *satisfy*

$$\tilde{R}_{ij} = p_* R_{ij} - m p_* (\nabla_j f_i / f),$$
$$\tilde{R}_{i\alpha} = 0,$$
$$\tilde{R}_{\alpha\beta} = (p_* f^2)^{-1} p'_* R'_{\alpha\beta} + p_* (\triangle f/f - (m-1) \sum f_k^2/f^2)\, \delta_{\alpha\beta}$$

*where* $\triangle f = -\sum \nabla_k f_k$ *is the Laplacian of* $f$ *on* $M^n$.

PROOF. By the definition of Ricci tensor and making use of Lemma 2, we have

$$\begin{aligned}
\tilde{R}_{ij} &= \sum \tilde{\Omega}_{ki}(\tilde{e}_j, \tilde{e}_k) + \sum \tilde{\Omega}_{\alpha i}(\tilde{e}_j, \tilde{e}_\alpha) \\
&= p_* R_{ij} - \sum p_* (\nabla_k f_i / f)\, \delta_{kj} \sum \varpi_\alpha(\tilde{e}_\alpha) \\
&= p_* R_{ij} - m p_* (\nabla_j f_i / f), \\
R_{i\alpha} &= \sum p_* \Omega_{ki}(\tilde{e}_\alpha, \tilde{e}_k) + \sum \tilde{\Omega}_{\beta i}(\tilde{e}_\alpha, \tilde{e}_\beta) = 0, \\
\tilde{R}_{\alpha\beta} &= \sum \tilde{\Omega}_{i\alpha}(\tilde{e}_\beta, \tilde{e}_i) + \sum \tilde{\Omega}_{\gamma\alpha}(\tilde{e}_\beta, \tilde{e}_\gamma) \\
&= \sum p_* (\nabla_k f_i)(-\delta_{ik}\delta_{\alpha\beta}) + (p_* f^2)^{-1} p'_* R'_{\alpha\beta} + p_* (\sum f_k^2/f^2) \\
&\qquad \times \sum (\delta_{\gamma\beta}\delta_{\alpha\gamma} - \delta_{\gamma\gamma}\delta_{\alpha\beta}) \\
&= (p_* f^2)^{-1} p'_* R'_{\alpha\beta} + p_* (\triangle f/f) - (m-1) \sum f_k^2/f^2)\, \delta_{\alpha\beta}.
\end{aligned}$$

LEMMA 4. *The scalar curvature of the warped product metric structure is*

$$\tilde{R} = p_* R + (p_* f^{-1})^2 p'_* R' + p_* (2 m f^{-1} \triangle f - m(m-1) f^{-2} \sum f_k^2).$$

PROOF. It is only direct calculation from Lemma 3.

## 2. Conformally flat spaces.

We study the conditions under which a warped product space $\tilde{M}^{n+m}$ be conformally flat ($n+m \geqq 4$). For this purpose, we put

$$\Psi_{AB} = -\frac{1}{n+m-2} (\varpi_A \wedge (\sum \tilde{R}_{CB} \varpi_C) - \varpi_B \wedge (\sum \tilde{R}_{CA} \varpi_C))$$
$$+ \frac{\tilde{R}}{(n+m-1)(n+m-2)} \varpi_A \wedge \varpi_B.$$

Then $\tilde{M}^{n+m}$ is conformally flat if and only if

$$\tilde{\Omega}_{AB} = \Psi_{AB}. \tag{2.1}$$

We assume that $M^n$ and $M'^m$ are conformally flat. Then we have

$$\Omega_{ij} = \frac{-1}{n-2} (\omega_i \wedge (\sum R_{kj} \omega_k) - \omega_j \wedge (\sum R_{ki} \omega_k))$$
$$+ \frac{R}{(n-1)(n-2)} \omega_i \wedge \omega_j, \tag{2.2}$$

$$(2.3)\qquad \Omega'_{\alpha\beta} = \frac{-1}{m-2}\left(\omega_{\alpha\wedge}\left(\sum R'_{\gamma\beta}\omega_\gamma\right) - \omega_{\beta\wedge}\left(\sum R'_{\gamma\alpha}\omega_\gamma\right)\right) + \frac{R'}{(m-1)(m-2)}\,\omega_{\alpha\wedge}\omega_\beta\,.$$

Since $\tilde{\Omega}_{AB}$ on $\tilde{M}^{n+m}$ is given by Lemma 2, (2.1) means that the three equations

$$(2.4)\qquad \begin{aligned} &p_*\Omega_{ij} = \Psi_{ij}\,,\\ &\sum p_*\left(\nabla_k f_i/f\right)\varpi_{k\wedge}\varpi_\alpha = \Psi_{i\alpha}\,,\\ &p_*'\Omega'_{\alpha\beta} + p_*\left(\sum f_k^2/f^2\right)\varpi_{\alpha\wedge}\varpi_\beta = \Psi_{\alpha\beta}\end{aligned}$$

are valid. In the following, we will omit the pull back operators $p_*$ and $p_*'$ to simplify the expressions. Then the first equation becomes

$$\begin{aligned}\Omega_{ij} &= \frac{-1}{n+m-2}\left(\omega_{i\wedge}\left(\sum R_{kj}\omega_k\right) - \omega_{j\wedge}\left(\sum R_{ki}\omega_k\right)\right)\\ &\quad + \frac{1}{(n+m-1)(n+m-2)}\left(R + f^{-2}R' + 2mf^{-1}\triangle f - m(m-1)f^{-2}\sum f_k^2\right)\omega_{i\wedge}\omega_j\\ &= \frac{n-2}{n+m-2}\,\Omega_{ij} + \frac{1}{(n+m-1)(n+m-2)}\Big(-\frac{m}{n-1}R + f^{-2}R' + m(2f^{-1}\triangle f\\ &\quad - (m-1)f^{-2}\sum f_k^2)\Big)\omega_{i\wedge}\omega_j + \frac{mf^{-1}}{n+m-2}\left(\omega_{i\wedge}\left(\sum \nabla_k f_j\omega_k\right) - \omega_j\left(\sum \nabla_k f_i\omega_k\right)\right)\end{aligned}$$

from which we obtain

$$\begin{aligned}\Omega_{ij} &= \frac{1}{n+m-1}\left(-\frac{1}{n-1}R + \frac{1}{m}f^{-2}R' + 2f^{-1}\triangle f - (m-1)f^{-2}\sum f_k^2\right)\omega_{i\wedge}\omega_j\\ &\quad + f^{-1}\left(\omega_{i\wedge}\sum \nabla_k f_j\omega_k - \omega_{j\wedge}\sum \nabla_k f_i\omega_k\right).\end{aligned}$$

Hence we have

$$\begin{aligned}R_{ij} &= \sum \Omega_{ik}(e_k, e_j)\\ &= \frac{1}{n+m-1}\Big(R - \frac{n-1}{m}f^{-2}R' + (m-1)(n-1)f^{-2}\sum f_k^2\\ &\quad + (m-n+1)f^{-1}\triangle f\Big)\delta_{ij} - (n-2)f^{-1}\nabla_j f_i\end{aligned}$$

and contracting $i$ and $j$, we get

$$(2.5)\qquad \frac{m-1}{n}R + \frac{n-1}{m}f^{-2}R' = (n-1)(m-1)\left(f^{-2}\sum f_k^2 + \frac{2}{n}f^{-1}\triangle f\right).$$

Using (2.5), we get

$$\Omega_{ij}=\Big(-\frac{R}{n(n-1)}+\frac{2}{n}f^{-1}\triangle f\Big)\omega_{i\wedge}\omega_j+f^{-1}(\omega_{i\wedge}\sum\nabla_k f_j\omega_k \tag{2.6}$$
$$-\omega_{j\wedge}\sum\nabla_k f_i\omega_k)$$

and

$$R_{ij}=\Big(\frac{R}{n}-\frac{n-2}{n}f^{-1}\triangle f\Big)\delta_{ij}+(n-2)f^{-1}\nabla_j f_i\,. \tag{2.7}$$

From the second equation of (2.4), we have

$$\sum\nabla_k f_i\,\omega_{k\wedge}\omega_\alpha=-\frac{1}{n+m-2}\Big(f^{-2}\omega_{i\wedge}\sum R'_{\beta\alpha}\varpi_\beta-\varpi_{\alpha\wedge}\sum R_{ki}\omega_k$$
$$+((\triangle f/f)-(m-1)\sum f_k^2/f^2)\,\omega_{i\wedge}\varpi_\alpha+\frac{1}{m}f\nabla_k f_i\varpi_{\alpha\wedge}\omega_k\Big)$$
$$+\frac{\tilde R}{(n+m-1)(n+m-2)}\,\omega_{i\wedge}\varpi_\alpha$$
$$=\frac{1}{(n+m-1)(n+m-2)}\Big(-\frac{n-1}{m}f^{-1}R'-\frac{m-1}{n}fR+(n-1)(m-1)$$
$$\times\frac{2}{n}\triangle f+f^{-1}\sum f_k^2\Big)+\sum\nabla_k f_i\omega_{k\wedge}\omega_\alpha\,,$$

hence we get the equation (2.5) again. Lastly, the third equation of (2.4) leads to the form

$$\Omega'_{\alpha\beta}+f\sum f_k^2\omega_{\alpha\wedge}\omega_\beta=-\frac{1}{n+m-2}\,(f^{-2}(\varpi_{\alpha\wedge}\sum R'_{\gamma\beta}\varpi_\gamma$$
$$-\varpi_{\beta\wedge}\sum R'_{\gamma\alpha}\varpi_\alpha)-2(f^{-1}\triangle f-(m-1)\sum f_k^2/f^2)\,\varpi_{\alpha\wedge}\varpi_\beta)$$
$$+\frac{\tilde R}{(n+m-1)(n+m-2)}\,\varpi_{\alpha\wedge}\varpi_\beta$$
$$=\frac{m-2}{n+m-2}\,\Omega'_{\alpha\beta}+\frac{1}{(n+m-1)(n+m-2)}\Big(f^2R-\frac{n}{m-1}R'-2(n-1)f\triangle f$$
$$+(m-1)(2n+m-2)\sum f_k^2\Big)\,\omega_{\alpha\wedge}\omega_\beta\,,$$

and hence we have

$$n\Omega'_{\alpha\beta}=\frac{1}{n+m-1}\Big(f^2R-\frac{n}{m-1}R'-2(n-1)f\triangle f-n(n-1)\sum f_k^2)\,\omega_{\alpha\wedge}\omega_\beta\,.$$

This shows that $M'^m$ is of constant sectional curvature. Moreover the Ricci tensor of $M'^m$ is calculated as

$$n(n+m-1)\,R'_{\alpha\beta}=-(m-1)\Big(f^2R-\frac{n}{m-1}R'-(n-1)(2f\triangle f-n\sum f_k^2)\Big)\,\delta_{\alpha\beta}\,,$$

and therefore we have

$$n(n+m-1)\,R' = -m(m-1)\,f^2R + mnR' + nm(n-1)(m-1)\left(\frac{2}{n} f\,\triangle f - \sum f_k{}^2\right),$$

from which the equation (2.4) is obtained. Substituting (2.4) into the equation of curvature form, we have

$$\Omega'_{\alpha\beta} = \frac{-R'}{m(m-1)}\,\omega_\alpha{}_\wedge\omega_\beta \tag{2.8}$$

which is a trivial result. Conversely it is easy to see that (2.5), (2.6) and (2.8) are sufficient for the equation (2.1) to be valid. Thus concluding these results we have proved

THEOREM 1. *Let $M^n$ and $M'^m$ be conformally flat spaces. Then the warped product space $M^n \times_f M'^m$ for a certain positive function $f$ is also conformally flat if and only if the following three conditions hold good: (1) the curvature form of $M^n$ satisfies (2.6), (2) $M'^m$ is of constant sectional curvature, (3) the scalar curvatures of $M^n$ and $M'^m$ satisfy (2.5).*

## 3. The special case of $M(c) \times_f M^m(c')$.

In this section we want to determine the positive function $f$ on $M^n$ by which the warped product space $\tilde{M}^{n+m} = M^n \times_f M'^m$ is conformally flat. By virtue of Theorem 1 $M'^m$ is necessarily constant curvature $c'$. Now we suppose that $M^n$ is of constant sectional curvature $c$, too. Then the condition (2.6) becomes

$$\frac{2}{n}\triangle f\,\omega_i{}_\wedge\omega_j + \omega_i{}_\wedge \sum \omega_k \nabla_k f_j - \omega_j{}_\wedge \sum \omega_k \nabla_k f_i = 0\,.$$

When $n=1$, this trivially holds, and for $n=2$, it is also true since we have $\triangle f = -\nabla_1 f_1 - \nabla_2 f_2$. When $n=3$, taking any fixed $i \neq j$, we have

$$\left(\frac{2}{n}\triangle f + \nabla_i f_i + \nabla_j f_j\right)\omega_j + \sum_{k\neq i,j} \omega_k \nabla_k f_i = 0\,,$$

hence

$$\frac{2}{n}\triangle f + \nabla_i f_i + \nabla_j f_j = 0\,, \qquad i \neq j\,,$$

$$\nabla_k f_i = 0 \qquad k \neq i, j$$

hold. Since there exists another index $h \neq i$, $j$, we easily see that

$$\nabla_i f_i = \nabla_j f_j \left(= -\frac{1}{n}\triangle f\right) \tag{3.1}$$

$$\nabla_i f_j = 0 \qquad \text{for} \quad i \neq j$$

are true.

As (2.5) is satisfied when $n=1$, we have

THEOREM 2. *For $m>2$ and any real number $c'$, the warped product space $M^1\times_f M^m(c')$ is conformally flat for any positive function $f$ on $M^1$.*

We consider the special case in which the conformally flat warped product space $\tilde{M}^{m+1}=M^1\times_f M^m(c')$ is constant curvature. Since the conformally flat space is constant curvature if and only if it is the Einstein space, the condition is given by

$$\tilde{R}_{AB} = mk\delta_{AB} \tag{3.2}$$

where $k$ is constant, and hence (3.2) is equivalent to

$$\begin{aligned} \nabla_1 f_1 &= -kf\,, \\ c'-f_1^2 &= kf^2 \end{aligned} \tag{3.3}$$

by virtue of Lemma 3. Taking the local cordinate function $x$ of $M^1$, (3.3) is written as

$$\begin{aligned} f'' &= -kf\,, \\ (f')^2 + kf^2 - c' &= 0\,. \end{aligned}$$

The first equation implies $((f')^2+kf^2)'=0$, which means that the second one is an initial condition for $f$. Solving this differential equation in each case of $k>0$, $k=0$ and $k<0$, we have

(1) $k>0$:
$$\begin{aligned} &f = A\sin\sqrt{k}\,x + B\sin\sqrt{k}\,x\,, \\ &A^2+B^2 = c'/k\,. \end{aligned}$$

Hence $c'$ is necessarily positive, and $f>0$ in an open domain in $M^1$.

(2) $k=0$:
$$f=\pm\sqrt{c'}\,x+B\,,$$

the same remark as above is true in this case.

(3) $k<0$:
$$\begin{aligned} &f = A\sinh\sqrt{-k}\,x + B\cosh\sqrt{-k}\,x\,, \\ &B^2-A^2 = c'/k\,. \end{aligned}$$

Here $B>A>0$, $c'<0$ and $f$ is positive on $M^1$.

We consider the case $n=2$ and $m\geqq 2$. Then $M^2(c)\times_f M^m(c')$ is conformally flat if and only if the equation (2.5) holds. Since $R=2c$ and $R'=m(m-1)c'$, (2.5) can be written as

$$c + f^{-2}(c'-\textstyle\sum f_k^2) - f^{-1}\triangle f = 0\,. \tag{3.4}$$

We put $F=\log f$ on $M^2(c)$. Then (3.4) is equivalent to the differential equation

$$\triangle F = c + c'e^{-2F}\,. \tag{3.5}$$

THEOREM 3. *Let $M^2(c)=S^2(c)$ be a space form with $c>0$. Shen $S^2(c)\times_f M^m(c')$*

*can not be conformally flat for* $c' \geqq 0$, $m \geqq 2$.

PROOF. If $S^2(c) \times_f M^m(c')$ is conformally flat, the function $F$ satisfies

$$\triangle F > 0$$

by virtue of (3.5). This is impossible on $S^2(c)$.

REMARK. We denote by $R^n$ (resp. $H^n(c')$) the space form $M^m(c')$ with $c'=0$ (resp. $c'<0$). Then the special solutions of (3.5) are obtained: If $c+c'=0$, then $F=0$, and if $c=c'=0$, then $F$ is a harmonic polynomial on $R^2$. Hence the Riemannian product spaces $S^2(c) \times H^m(-c)$, $H^2(-c) \times S^m(c)$ and $R^2 \times_f R^m$ are conformally flat.

Next we consider the case $n \geqq 3$, $m \geqq 2$. Owing to the equations (3.1), we can put for any indices $i$, $j$

$$\nabla_i f_j = \varphi \delta_{ij}, \tag{3.6}$$

where $\varphi$ is a scalar function on $M^n(c)$.

LEMMA 5. *There exists a constant* $k$ *on* $M^n(c)$ *such that* $\varphi$ *is written as*

$$\varphi = k - cf.$$

Proof. From (3.6) and the equation

$$df_j + \sum f_k \omega_{kj} = \sum \omega_k \nabla_k f_j,$$

we have

$$df_j + \sum f_k \omega_{kj} = \varphi \omega_j.$$

Differentiating it, we get

$$\sum df_k{}_\wedge \omega_{kj} + \sum f_k(-c\omega_k{}_\wedge \omega_j + \sum \omega_{kh}{}_\wedge \omega_{hj}) = d\varphi_\wedge \omega_j + \varphi \sum \omega_{jk}{}_\wedge \omega_k$$

and hence

$$d(cf + \varphi)_\wedge \omega_j = 0$$

is obtained. Since $n \geqq 3$, we conclude that

$$d(cf + \varphi) = 0$$

which proves the lemma.

We now assume $c>0$. According to Lemma 5, we can define a function

$$\tilde{f} = f - k/c = -\varphi/c.$$

Then $\tilde{f}$ satisfies $\tilde{f}_j = f_j$ and

$$\nabla_i \tilde{f}_j = -c\tilde{f}\delta_{ij}. \tag{3.7}$$

Moreover the equation (2.5) is written as

$$\sum \tilde{f}_k{}^2 = -c\tilde{f}^2 + K/c, \qquad K = k^2 + cc'. \tag{3.8}$$

Hence $K$ is a non-negative constant on $M^n(c)$.

Let $M^n(c)$ be the space form $S^n(c)$. Then $S^n(c)$ is isometrically imbedded in

$R^{n+1}$ with coordinate functions $(x_1, \cdots, x_n, x_\Delta)$: $\sum x_i^2 + x_\Delta^2 = 1/c$. (3.7) shows that the function $f$ is the first eigen-function of the Laplacian on $S^n(c)$, which is given by restricting the harmonic polynomial of degree one on $R^{n+1}$ to the sphere $S^n(c)$. Thus we have

$$\tilde{f} = \sum a_i x_i + a_\Delta x_\Delta\,, \qquad \sum x_i^2 + x_\Delta^2 = 1/c\,.$$

LEMMA 6. *The constant functions $a_i$, $a_\Delta$ on $S^n(c)$ satisfy*

$$\sum a_i^2 + a_\Delta^2 = K/c\,.$$

PROOF. Since $\tilde{f}$ attains to a critical point on $S^n(c)$, we take one of such points $p \in S^n(c)$. Then $d\tilde{f}(p)=0$, and hence

$$\tilde{f}(p) = \pm \sqrt{K}/c$$

by virtue of (3.8). On the other hand $\nabla_j \tilde{f}(p)=0$ leads to

$$x_j(p)/a_j = x_\Delta(p)/a_\Delta = b$$

for some $b$, hence we have

$$b^2 (\sum a_j^2 + a_\Delta^2) = 1/c\,.$$

Therefore

$$\begin{aligned}\tilde{f}(p) &= \sum a_j x_j(p) + a_\Delta x_\Delta(p)\\ &= b(\sum a_j^2 + a_\Delta^2) = \pm \sqrt{K}/c\,,\end{aligned}$$

from which

$$b^2 (\sum a_j^2 + a_\Delta^2)^2 = K/c^2$$

is obtained. Comparing these equations, it is easy to see that

$$\sum a_j^2 + a_\Delta^2 = K/c$$

holds good.

Conversely, we have

LEMMA 7. *Let $\tilde{f} = \sum a_i x_i + a_\Delta x_\Delta$ be a function on $S^n(c)$ in $R^{n+1}$ whose coefficients satisfy $\sum a_i^2 + a_\Delta^2 = K/c$ for some non-negative constant $K$. Then $\tilde{f}$ is the solution of the differential equations*

$$\nabla_i \nabla_j \tilde{f} = -c\tilde{f} g_{ij}$$

*with*

$$\|d\tilde{f}\|^2 = -c\tilde{f}^2 + K/c$$

*where $g_{ij}$ is the canonical metric tensor on $S^n(c)$.*

PROOF. Since the first equation is well known, we show the second one. As we have

$$\nabla_j \tilde{f} = a_i + a_\Delta (-x_j)/x_\Delta\,,$$

$\|d\tilde{f}\|^2 = \sum g^{ij} \nabla_i \tilde{f} \nabla_j \tilde{f}$ becomes

$$\|d\tilde{f}\|^2 = \sum (\delta_{ij} - cx_ix_j)(a_i - a_\Delta x_i/x_\Delta)(a_j - a_\Delta x_j/x_\Delta)$$
$$= \sum a_j^2 - 2(a_\Delta/x_\Delta) \sum a_jx_j + (a_\Delta^2/x_\Delta^2)\left(\frac{1}{c} - x_\Delta^2\right)$$
$$- c\,(\sum a_jx_j)^2 + 2c(a_\Delta/x_\Delta) \sum a_jx_j \left(\frac{1}{c} - x_\Delta^2\right)$$
$$- c\,(a_\Delta^2/x_\Delta^2)\left(\frac{1}{c} - x_\Delta^2\right)^2 .$$

Making use of the condition $\sum a_j^2 + a_\Delta^2 = K/c$, we see that

$$\|d\tilde{f}\|^2 = -c\tilde{f}^2 + K/c$$

after some calculations.

THEOREM 4. *Let $n \geqq 3$, $m \geqq 2$. If $c' \leqq 0$, then there exists a positive function $f$ on $S^n(c)$ such that the warped product space $M^{n+m} = S^n(c) \times_f M^m(c')$ is conformally flat. The function $f$ is given by (3.9).*

PROOF. According to Lemma 7, for some non-negative constant $K$ the function $\tilde{f}$ satisfies (3.7) and (3.8). We define $f$ on $S^n(c)$ by

$$f = \tilde{f} + k/c \tag{3.9}$$

where $k = \sqrt{K - cc'}$. From (3.8), we see that

$$-c\tilde{f}^2 + K/c = -cf^2 + 2kf + c' = \|df\|^2 \geqq 0\,,$$

and hence $f > 0$ on $S^n(c)$. Since $f$ satisfies (3.1) and (2.5), $\tilde{M}^{n+m}$ is conformally flat.

REMARK. It is easy to see that $K = 0$ if and only if $\tilde{f} = 0$ and hence $f = \sqrt{-c'/c}$ is a unique constant function such that $\tilde{M}^{n+m}$ is conformally flat.

Next we assume $c = 0$. Then taking orthogonal coordinate functions $(x_1, \cdots, x^n)$ on $R^n$, the equation (3.6) becomes

$$\partial_i \partial_j f = k\delta_{ij}$$

by virtue of Lemma 5. Thus $f$ is of the form

$$f = (k/2) \sum x_j^2 + \sum a_jx_j + b \tag{3.10}$$

for some constants $a_j$, $b$. From (2.5) we have

$$\sum a_j^2 = c' + 2kb\,. \tag{3.11}$$

If $k > 0$ and $c' < 0$ then $f$ is positive all over $R^n$. But if $k \leqq 0$ or $k > 0$ and $c' \geqq 0$, then $f$ is possibly positive on some open domain $D^n$ of $R^n$. Consequently we have

THEOREM 5. *Let $n \geqq 3$ and $m \geqq 2$. If $c' < 0$, the warped product space $R^n \times_f M^m(c')$ is conformally flat, where $f$ is given by (3.10) with the condition (3.11). If $c' \geqq 0$, then $D^n \times_f M^m(c')$ is conformally flat, where $D^n$ is an open domain in $R^n$.*

REMARK. The former space $R^n \times_f M^m(c')$ is not of flat curvature. However the second one $D^n \times_f M^m(c')$ is flat if and only if $k=0$.

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