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On the Peripheral Spectrum of Operators

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§ 1. Introduction.

The peripheral spectrum of positive operators in Banach lattices has been studied by many mathematicians. Among them, it is well known that if a positive operator T is irreducible and $r(T)=1$ is a pole of $R(\lambda, T)$, then the peripheral spectrum of T is a group of roots of unity consisting entirely of simple poles of $R(\lambda, T)$ [4, 5]. Moreover, if either an operator T is quasi-compact [1, 8], or $r(T)=1$ is a pole of $R(\lambda, T)$ of a positive operator T and the residuum P of R (λ, T) at $\lambda=1$ is of finite rank [3, 5], then the peripheral spectrum of T consists entirely of poles of $R(\lambda, T)$. M.A. Kaashoek and T.T. West showed that the peripheral spectrum of an element T in a Banach algebra is a set of simple poles, including $r(T)=1$ if and only if $A(T)$ is locally compact, semi-simple and strict (Th. III. 2.1 of [2]), by examing the connection between the structure of locally compact semi-algebra and spectral theory.

Recently, author has extended the relationship between uniform ergodicity for a positive operator T and a simple pole of $R(\lambda, T)$ to that between convergence of a certain sequence of polynomials $\{f_{k,n}(T)\}$ and poles of order k [7]. In this paper, by examing the structure of the cone to which the polynomial $f_{k,n}(T)$ belongs, we have obtained a necessary and sufficient condition for the peripheral spectrum to consist entirely of poles which are not necessarily simple poles.

§ 2. Preliminary and definitions.

Let E be a Banach space. The peripheral spectrum of $T \in \mathscr{L}(E)$ is the set

$$\text{Per } \sigma(T) = \{\lambda \in \sigma(T) : |\lambda| = r(T)\} .$$

Let $S(T)$ be the norm closure in $\mathscr{L}(E)$ of the set

$$\{T, T^2, \cdots\} ,$$

$A(T)$ be the norm closure of the set

$$\left\{ \sum_{i=1}^{m} a_i T^i : a_i \geqq 0 , \quad m \in Z^+ \right\}$$

and $C_k(T)$ be the set

$$\left\{\sum_{i=1}^{m} a_i T_{k,i}:\quad a_i \geqq 0\,,\quad m \in Z^+\right\},$$

where

$$T_{k,i} = \binom{k+i-1}{k} k \sum_{s_{k-1}=0}^{1} \cdots \sum_{s_1=0}^{1} (-1)^{\sum_{r=1}^{k-1} s_r} \frac{T^{i+\sum_{r=1}^{k-1} s_r}}{i+\sum_{r=1}^{k-1} s_r}$$

$$= \binom{k+i-1}{k} k \sum_{s=0}^{k-1} (-1)^s \binom{k-1}{s} \frac{T^{i+s}}{i+s}$$

for $k \geqq 1$.
In case of $k=1$, $T_{1,i}=T^i$ and the norm closure of $C_1(T)$ is $A(T)$. In case of $k \neq 1$, $C_k(T)$ is not semi-algebra, but a cone, although $A(T)$ is semi-algebra. A semi-algebra $A(T)$ is called *strict* if $A(T) \cap (-A(T)) = \{0\}$ and *locally compact* if $A(T)$ contains nonzero elements and the set

$$\{x \in A(T):\quad \|x\| \leqq 1\}$$

is a compact subset of $\mathscr{L}(E)$.

DEFINITION. *We call a cone $C_k(T)$ k-relatively locally compact if the set*

$$\{g(T) \in C_k(T):\ \sup_{|\beta|=1} \|g(\beta T)(I-\beta T)^{k-1}\| \leqq 1\}$$

is a relatively compact subset of $\mathscr{L}(E)$.

If $A(T)$ is locally compact, $C_1(T)$ is 1-relatively locally compact. The converse is not necessarily true. But we have

LEMMA 1. *Let T be an element in $\mathscr{L}(E)$ with $r(T)=1$. Then the following are equivalent.*

i) *$A(T)$ is locally compact, semi-simple*) and strict.*

ii) *$S(T)$ is compact and $1 \in \sigma(T)$.*

iii) *For any α, $|\alpha|=1$, $C_1(\alpha T)$ is 1-relatively locally compact, the set $\{\|T^n\|: n=1, 2, \cdots\}$ is bounded and $1 \in \sigma(T)$.*

iv) *$C_1(T)$ is 1-relatively locally compact, the set $\{\|T^n\|: n=1, 2, \cdots\}$ is bounded and $1 \in \sigma(T)$.*

PROOF. By III. 2.3 of [2], i) and ii) are equivalent. ii)$\Rightarrow$iii): Let $D=\{\lambda x: |\lambda| \leqq 1,\ x \in S(T)\}$. Then $\overline{\text{co}\ D}$ (convex closure of D) is compact by ii).
For α, $|\alpha|=1$, consider

$$F_\alpha = \{g(\alpha T) \in C_1(\alpha T):\ \sup_{|\beta|=1} \|g(\beta T)\| \leqq 1\}\,.$$

*) A semi-algebra A is said to be semi-simple if the zero ideal is the only closed two-sided ideal J in A with $J^2=(0)$.

For $g(\alpha T)\in F_\alpha$, $g(T)$ is written as

$$g(T)=\sum_{n=1}^{\infty}a_nT^n\,,\qquad a_n\geqq 0$$

and we have

$$\sum_{n=1}^{\infty}a_n\leqq r(g(T))\leqq\|g(T)\|\leqq\sup_{|\beta|=1}\|g(\beta T)\|\leqq 1\,,$$

by using $1\in\sigma(T)$ and the spectral mapping theorem. Therefore

$$g(\alpha T)=\sum_{n=1}^{\infty}a_n(\alpha T)^n\in\overline{\mathrm{co}\,D}\,.$$

Hence $F_\alpha\subset\overline{\mathrm{co}\,D}$, that is, F_α is relatively compact. So $C_1(\alpha T)$ is 1-relatively locally compact.
iii)$\Rightarrow$iv) is obvious.
iv)$\Rightarrow$ii): By iv), there exists $M\geqq 0$ such that $\|T^n\|\leqq M$ for $n=1, 2,\cdots$. Since $\sup_{|\beta|=1}\|(\beta T)^n\|=\|T^n\|\leqq M$, we have $S(T)\subset$ closure of $\{g(T)\in C_1(T)\colon \sup_{|\beta|=1}\|g(\beta T)\|\leqq M\}$ Therefore $S(T)$ is compact.

§ 3. Main result.

In the definition of k-relatively locally compact, we consider the norm $\|g(T)(I-T)^{k-1}\|$. As for the relation between this norm and the spectral radius, we have the following lemma.

LEMMA 2. *Let T be an operator with the spectral radius $r(T)=1$ in a Banach space and* 1 *be a pole of $R(\lambda, T)$ of order k. Then there exists $L\geqq 0$ such that*

$$r(h(T))\leqq L\,\|h(T)(I-T)^{k-1}\|$$

for every polynomial $h(\lambda)$.

PROOF. If $k=1$, $L=1$ satisfies the lemma. We suppose $k\geqq 2$. Let P_1 be the spectral idempotent of T corresponding to the spectral set $\{1\}$. Put $Q_{k-1}=P_1(I-T)^{k-1}$. Then $Q_{k-1}\neq 0$ and $TQ_{k-1}=Q_{k-1}$, since 1 is a pole of order $k\geqq 2$. We have $r(h(T)P_1)=|h(1)|$, by using the relation

$$\sigma(h(T)P_1)=\sigma(h(TP_1))=h(\sigma(TP_1))=\{h(1)\}\,.$$

On the other hand, we have

$$\begin{aligned}\|h(T)(I-T)^{k-1}P_1\|&=\|h(T)Q_{k-1}\|=\|h(1)Q_{k-1}\|\\&=|h(1)|\,\|Q_{k-1}\|\,.\end{aligned}$$

Therefore we have

$$\begin{aligned}r(h(T)P_1)&\leqq\|h(T)(I-T)^{k-1}P_1\|/\|Q_{k-1}\|\\&\leqq\|h(T)(I-T)^{k-1}\|\,\|P_1\|/\|Q_{k-1}\|\,.\end{aligned}$$

Since $(I-T)^{k-1}$ is invertible on $(I-P_1)\,E$ by the definition of P_1, there exists $S_1 \in \mathscr{L}(E)$ such that

$$(I-P_1)(I-T)^{k-1}S_1 = I-P_1\,.$$

So we have

$$\|(h(T)(I-P_1))^m\| = \|(h(T)(I-T)^{k-1}(I-P_1)\,S_1)^m\|$$
$$\leqq \|h(T)(I-T)^{k-1}\|^m\,\|(I-P_1)\,S_1\|^m$$

and

$$r(h(T)(I-P_1)) \leqq \|(I-P_1)\,S_1\|\,\|h(T)(I-T)^{k-1}\|\,.$$

Therefore we have

$$r(h(T)) = r(h(T)P_1 + h(T)(I-P_1))$$
$$\leqq r(h(T)P_1) + r(h(T)(I-P_1)) \leqq L\|h(T)(I-T)^{k-1}\|\,,$$

where

$$L = \|P_1\|/\|Q_{k-1}\| + \|(I-P_1)\,S_1\|\,.$$

As a relation between polynomials and poles, author has obtained Theorem 2 of [7]. The polynomial $f_{k,n}(T)$ in Th. 2 of [7] can be rewritten as follows:

$$f_{k,n}(T) = \sum_{i=0}^{n-k} b_{k,i}\,T_{k,i} \tag{1}$$

where

$$T_{k,0} = I\,,$$

$T_{k,i}$ is defined in §2 for $i \geqq 1$,

$$b_{k,0} = 1 - \binom{n+k-2}{2k-1}\Big/\binom{n+2k-2}{2k-1}$$

and

$$b_{k,i} = \binom{n+k-i-2}{2k-2}\Big/\binom{n+2k-2}{2k-1} \qquad \text{for } i \geqq 1\,.$$

Here $\sum_{i=0}^{n-k} b_{k,i}=1$ and $b_{k,i}\geqq 0$. Therefore $f_{k,n}(T)$ is the convex combination of $T_{k,i}$ and $(f_{k,n}(T)-b_{k,0}I)$ belongs to $C_k(T)$. By using Th. 2 of [7] and the above lemma 2, we obtain the following theorem.

THEOREM. *Let E be a Banach space and T be an element in $\mathscr{L}(E)$ with $r(T)=1$. Then the following statements are equivalent for $k\geqq 1$.*

i) *Per $\sigma(T)$ is a set of poles of $R(\lambda, T)$ of order at most k and the maximal order of the poles in Per $\sigma(T)$ is k.*

ii) *For each α, $|\alpha|=1$, $C_k(\alpha T)$ is k-relatively locally compact and the order $\|T^n\|$ is n^{k-1}**)*

PROOF. i)$\Rightarrow$ii); In order to show that $C_k(T)$ is k-relatively locally compact,

**) This means that the set $\{\|T^n/n^{k-1}\| : n=1,2,\cdots\}$ is bounded, but the set $\{\|T^n/n^{k-2}\| : n=1,2,\cdots\}$ is not bounded.

take $g_n(T)\in C_k(T)$ and assume $\sup_{|\beta|=1}\|g_n(\beta T)(I-\beta T)^{k-1}\|\leqq 1$ for $n=1, 2,\cdots$. We will prove that the sequence $\{g_n(T)\}$ has a convergent subsequence. Here $g_n(T)$ is written as

$$g_n(T)=\sum_{i=1}^{\infty} a_i(n)\, T_{k,i}$$

where $a_i(n)\geqq 0$ and $a_i(n)=0$ for i sufficiently large. Let α_0 be the pole of $R(\lambda, T)$ of order k. Then 1 is a pole of $R(\lambda, \alpha_0^{-1}T)$ of order k and we have

$$\sum_{i=1}^{\infty} a_i(n)\, 1_{k,i}=\sum_{i=1}^{\infty} a_i(n)\in\sigma(g_n(\alpha_0^{-1}T))$$

by the spectral mapping theorem and the relation

$$\begin{aligned}1_{k,i}&=\binom{k+i-1}{k}k\sum_{s=0}^{k-1}(-1)^s\binom{k-1}{s}\frac{1}{i+s}\\&=1\,.\end{aligned}$$

Therefore we have

$$\begin{aligned}\sum_{i=1}^{\infty} a_i(n)&=\left|\sum_{i=1}^{\infty} a_i(n)\right|\leqq r(g_n(\alpha_0^{-1}T))\\&\leqq\|g_n(\alpha_0^{-1}T)(I-\alpha_0^{-1}T)^{k-1}\|\,L\leqq L\end{aligned}\tag{2}$$

by lemma 2 and assumption. By using compactness arguments and the diagonal process, we obtain an increasing sequence $\{n_j\}$ in Z^+, such that for each i, the sequence $\{a_i(n_j)\}$ converges. By passing to this subsequence, we may suppose that $b_i=\lim_n a_i(n)$ exists. Observe that

$$b_i\geqq 0\qquad\text{and}\qquad\sum_{i=1}^{\infty} b_i\leqq L\,.\tag{3}$$

Let P_0 be the spectral idempotent associated with the spectral set $\{\lambda\in\sigma(T)\colon |\lambda|=1\}$ and put $S=T(I-P_0)$. Since $r(S)<1$, we see that the set $\{\|n^{k-1}S^n\|\colon n=1, 2,\cdots\}$ is bounded and therefore the set

$$\{\|S_{k,i}\|\colon\ i=1,2,\cdots\}\tag{4}$$

is bounded.

This and (3) imply that the sequence $\{\sum_{i=1}^{n} b_iS_{k,i}\}$ converges in the norm of $\mathscr{L}(E)$. Let D be its sum, i.e.

$$D=\sum_{i=1}^{\infty} b_i S_{k,i}\,.$$

Then we have $D=\lim_n g_n(T)(I-P_0)$ by using (2), (3) and (4).

Now $g_n(T)=g_n(T)(I-P_0)+g_n(T)P_0$ for each $n\in Z^+$. Hence in order to prove that $\{g_n(T)\}$ has a convergent subsequence, it suffices to show that $\{g_n(T)P_0\}$ has a convergent subsequence in the closed subalgebra of $\mathscr{L}(E)$ generated by TP_0. If $1\in\rho(T)$, $I-T$ is invertible and

$$\begin{aligned}\|g_n(T)\| &\leqq \|g_n(T)(I-T)^{k-1}\|\ \|R(1,T)\|^{k-1}\\ &\leqq \|R(1,T)\|^{k-1}.\end{aligned}$$

If $1\in\sigma(T)$, 1 is a pole of $R(\lambda, T)$ of order at most k. Let P' be the spectral idempotent associated with the spectral set $\{1\}$. Then we have $g_n(T)P'=\sum_{i=1}^{\infty}a_i(n)T_{k,i}P'=\sum_{i=1}^{\infty}a_i(n)P'$ and the set $\{\|g_n(T)P'\|;\ n=1,2,\cdots\}$ is bounded. Since $I-T$ is invertibe on $(P_0-P')E$, the set $\{\|g_n(T)(P_0-P')\|:\ n=1,2,\cdots\}$ is bounded. Therefore $\{g_n(T)P_0\}$ is a bounded sequence in the closed subalgebra of $\mathscr{L}(E)$ generated by TP_0. From the spectral properties of T it follows that this algebra is finite dimensional. So $\{g_n(T)P_0\}$ has a convergent subsequence.

For any α, $|\alpha|=1$, we can replace T by αT in condition i). So we can prove $C_k(\alpha T)$ is k-relatively locally compact in the same way.

Next, let Per $\sigma(T)=\{\lambda_1,\cdots,\lambda_s\}$ and $P_1,\cdots, P_s$ be the associated spectral idempotents. Then $P_0=P_1+\cdots+P_s$. Since $r(T(I-P_0))<1$, we have

$$T^n(I-P_0)\to 0 \qquad (n\to\infty).$$

Since each λ_j is a pole of order at most k, we have

$$T^nP_j=\lambda_j{}^nP_j+\sum_{m=1}^{k-1}\binom{n}{m}\lambda_j{}^{n-m}Q_{j,m} \qquad \text{for } n\geqq k$$

where $$Q_{j,m}=P_j(\lambda_jI-T)^m \qquad 1\leqq m\leqq k-1$$

and therefore

$$\begin{aligned}\frac{T^n}{n^{k-1}}&=\sum_{j=1}^{s}\frac{T^nP_j}{n^{k-1}}+\frac{1}{n^{k-1}}T^n(I-P_0)\\ &=\sum_{j=1}^{s}\left\{\frac{\lambda_j{}^n}{n^{k-1}}P_j+\sum_{m=1}^{k-1}\frac{1}{n^{k-1}}\binom{n}{m}\lambda_j{}^{n-m}Q_{j,m}\right\}+\frac{1}{n^{k-1}}T^n(I-P_0).\end{aligned}$$

So the set $\{\|T^n/n^{k-1}\|:\ n=1,2,\cdots\}$ is bounded. If the set $\{\|T^n/n^{k-2}\|:\ n=1,2,\cdots\}$ is bounded, the maximal order is at most k-1 by Th. 2 of [7], which is a contradiction. So the order of $\|T^n\|$ is n^{k-1}.

ii)$\Rightarrow$i): Since the order of $\|T^n\|$ is n^{k-1}, we have the set $\{\|f_{k,n}(\beta T)(I-\beta T)^{k-1}\|:\ |\beta|=1,\ n=1,2,\cdots\}$ is bounded (see p. 57 of [7]), where $f_{k,n}(T)$ is defined in (1) above. Since $C_k(\alpha T)$ is k-relatively locally compact for any α, $|\alpha|=1$, $\{f_{k,n}(\alpha T)\}$ has a convergent subsequence. Therefore, by Th. 2 of [7], this means 1 is either in $\rho(\alpha T)$ or else a pole of $R(\lambda, \alpha T)$ of order at most k, that is, α^{-1} is either in $\rho(T)$ or else a pole of $R(\alpha,T)$ of order at most k. So Per $\sigma(T)$ is a set of poles of $R(\lambda, T)$.

If the maximal order is at most $k-1$, we get the set $\{\|T^n/n^{k-2}\|: n=1, 2,\cdots\}$ is bounded, which is a contradiction. So the maximal order is k.

It is well known that if an operator T in an ordered Banach space is positive and $r(T)$ is a pole of $R(\lambda, T)$, it is of maximal order in Per $\sigma(T)$ [6]. Taking account of this fact, we have the following Corollary.

COROLLARY. *Let E be a Banach space and T be an element in $\mathscr{L}(E)$ with $r(T)=1$. Then the following statements are equivalent for $k\geqq 1$.*

i) Per $\sigma(T)$ is a set of poles of $R(\lambda, T)$ of order at most k and 1 is a pole of order k.

ii) For any α, $|\alpha|=1$, $C_k(\alpha T)$ is k-relatively locally compact, the order of $\|T^n\|$ is n^{k-1} and $A(T)$ is strict.

PROOF. By Th. II. 1.2 and Th. IV. 1.7 of [2], we can prove easily.

REMARK. In case of $k=1$, the above Corollary is the same with Kaashoek and West's theorem (Th. III.2.1 of [2]) by lemma 1.

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