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The Existence of Positive Harmonic Functions and Green Operators

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§ 0. Introduction.

Characterization of spaces on which Green function associated with Laplacian Δ exists, has been treated as the classification theory of Riemann surfaces. In particular, it is well known that the existence of a nonconstant positive harmonic function implies the existence of a Green function [3]. Similar result has been obtained for the partial differential operator of the form $\Delta+\sum_{i=1}^{n} a_i(x)\partial/\partial x_i$ in the n-dimensional space by the computational method in the theory of partial differential equations.

Such problem is investigated as that of potential theory. On the other hand, G. Hunt [2], K. Yosida [5], A. Yamada [4] and others have treated Green function as an operator in certain function space and characterized the corresponding operator, such as Laplacian, as the generator of a suitable semigroup of operators.

In the present paper, we consider an operator L satisfying certain axioms and, by means of abstract method, characterize the space where a Green operator associated with L exists. Main difference of our results from those of Hunt and Yosida is as follows; we first give an operator L and discuss the existence of the corresponding Green operator; we choose axioms for L in such a way that our result contains the case of differential operators of the form Δ+*first order terms*, and we do not restrict the function space to $\overline{C_0(X)}$(=the completion of the space of all continuous functions with compact support with respect to the supremum norm).

§ 1. Preliminary notions and the main result.

Throughout this paper, all functions are assumed to be real valued.

Let X be a locally compact and σ-compact Hausdorff space, $C(X)$ be the set of all continuous functions on X and $C_0(X)$ be the set of all functions in $C(X)$ with compact support. For any subdomain D of X, $C(D)$, $C_0(D)$ and $C_0(\overline{D})$ are defined analogously. We define the norm $\|f\|$ of any bounded function f on X (or D, $\overline{D}$) by $\|f\|=\sup_x |f(x)|$, and denote by $\overline{C_0(X)}$, $\overline{C_0(D)}$ respectively the completion of

$C_0(X)$ and $C_0(D)$ with respect to the norm.

Let L be the linear operator defined as follows and satisfying the axioms (α), (β), (γ) and (δ) mentioned later. The domain $\mathscr{D}(L)$ is a linear subspace of $C(X)$ such that $\mathscr{D}(L)\cap C_0^+(D)$ is dense in $C_0^+(D)$ for any subdomain D of X. L is a linear operator (generally unbounded) of $\mathscr{D}(L)$ into $C(X)$; in particular, any constant c belongs to $\mathscr{D}(L)$ and $Lc=0$. Furthermore L is assumed to be a local operator, that is, if $f\in\mathscr{D}(L)$ and $f(x)\equiv 0$ in a neighborhood of a point $x_0\in X$, then $(Lf)(x_0)=0$.

For any subdomain D of X, let

$$\mathscr{D}(L_D)=\{g\in C(D) \mid g=f|_D \quad \text{for some} \quad f\in\mathscr{D}(L)\}$$

and define $L_D g=(Lf)|_D$. Then, since L is a local oprator, L_D is defined independently of the chocie of f, and accordingly L_D is a linear operator of $\mathscr{D}(L_D)$ into $C(D)$. Hereafter we shall denote L_D by L briefly.

DEFINITION 1. A subdomain D of X is called a *regular domain* if the closure $\bar{D}$ is compact and, for any $\varphi\in C(\partial D)$, there exists a solution $u\in\mathscr{D}(L)\cap C(\bar{D})$ of the boundary value problem: $Lu=0$ in D and $u=\varphi$ on ∂D.

DEFINITION 2. A function u on a domain $D\subset X$ is said to be *L-harmonic* if $u\in\mathscr{D}(L)$ and satisfies $Lu=0$ in D.

DEFINITION 3. A linear operator G of $C_0(X)$ into $C(X)$ is called a *Green operator* associated with L if, for any $f\in\mathscr{D}(L)\cap C_0(X)$, $u=Gf$ belongs to $\mathscr{D}(L)$ and satisfies $Lu=-f$ on X.

The main purpose of this paper is to prove the following:

THEOREM 1. *If the space X admits a positive nonconstant L-harmonic function, then exists a Green operator associated with L.*

We assume that the operator L satisfies the following axioms; these are justified not only in the case of usual Laplacian, but also in the case of differential operators of the form $\Delta+\sum_{i=1}^{n}a_i(x)\partial/\partial x_i$ mentioned in §0 ([1], [7]).

(α) If $Lu\geqq 0$ and u is nonconstant in D, then u does not take its maximum in the interior of D (maximum principle).

(β) If $\{u_n\}$ and $\{Lu_n\}$ are uniformly bounded on D, then a subsequence $\{u_{n_\nu}\}$ of $\{u_n\}$ converges uniformly on every compact subset of D (cf. Harnack theorem).

(γ) For any regular domain D, any $\lambda>0$ and any $f\in\mathscr{D}(L)\cap C(\bar{D})$, there exists $u\in\mathscr{D}(L)\cap\overline{C_0(D)}$ satisfying $(\lambda-L)u=f$.

The dual space of $C(\bar{D})$ is the set of all signed measures on $\bar{D}$, which is denoted by $\mathfrak{M}(\bar{D})$. We denote by $\mathfrak{M}_0(D)$ the set of $\rho\in\mathfrak{M}(\bar{D})$ whose support is a compact set in the interior of D, and by L^* the dual operator of the restriction of L on $C(\bar{D})$ $\cap\mathscr{D}(L)$. Here we add the following axiom which corresponds to Weyl's lemma in the case of Laplacian.

(δ) If $u\in\mathscr{D}(\bar{D})$ satisfies $\langle u, L^*\rho\rangle=0$ for any $\rho\in\mathscr{D}(L^*)$, then $u\in\mathscr{D}(L)$ and

$Lu=0$.

We assume that there exist sufficiently many regular domains, that is, for any domains D_1 and D_2 satisfying $\bar{D}_1 \subset D_2$, there exists a regular domain D such that $\bar{D}_1 \subset D \subset D_2$.

§ 2. Preliminary lemmas.

LEMMA 1. *If $u \in \mathscr{D}(L_D)$ takes its maximum at $x_0 \in D$, then $Lu(x_0) \leqq 0$.*

LEMMA 2. *Suppose $\lambda > 0$. If $(\lambda - L)u \leqq 0$ in D, then u does not take its maximum in the interior of D. If $(\lambda - L)u \geqq 0$ in D, then u does not take its minimum in the interior of D.*

LEMMA 3. *Suppose $\lambda > 0$, $f \in \mathscr{D}(L) \cap C(\bar{D})$ and $u \in \mathscr{D}(L) \cap \overline{C_0(D)}$. If $(\lambda - L)u = f$, then $\|u\| = \|f\|/\lambda$. Accordingly the function u in (γ) is uniquely determined by f.*

LEMMA 4. *If $f \geqq 0$ in (γ), then the corresponding u is $\geqq 0$.*

LEMMA 5. *The function u in Definition 1 is uniquely determined by φ.*

Lemma 1 is proved from the axiom (α). Lemmas 2 and 3 easily follow from Lemma 1. Lemmas 4 and 5 may be proved by using Lemma 2.

LEMMA 6. *Suppose that $\{u_n\}$ and $\{Lu_n\}$ are uniformly bounded in D, and that $f_n = -Lu_n \in \mathscr{D}(L)$. If $\lim_{n\to\infty} f_n = 0$ uniformly on every compact subset of D, then a subsequence $\{u_{n_\nu}\}$ of $\{u_n\}$ converges to a function $u \in \mathscr{D}(L)$ and $Lu=0$ holds in D.*

PROOF. The existence of convergent subsequence $\{u_{n_\nu}\}$ of $\{u_n\}$ is assured by (β); we shall denote the subsequence simply by $\{u_n\}$ again. For any $\rho \in \mathscr{D}(L^*) \cap \mathfrak{M}_0(D)$ we have $\langle u_n, L^*\rho \rangle = \langle Lu_n, \rho \rangle = \langle -f_n, \rho \rangle$. Accordingly

$$\langle u, L^*\rho \rangle = \lim_{n\to\infty} \langle u_n, L^*\rho \rangle = \lim_{n\to\infty} \langle -f_n, \rho \rangle = 0 .$$

Hence $u \in \mathscr{D}(L)$ and $Lu=0$ by (δ).

§ 3. The existence of Green operator in the case of compact domain.

We fix a regular domain D. Suppose $\lambda > 0$ and $f \in \mathscr{D}(L) \cap C(\bar{D})$. Then by (γ) there exists $u \in \mathscr{D}(L) \cap \overline{C_0(D)}$ such that $(\lambda - L)u = f$. Since $\|u\| \leqq \|f\|/\lambda$ by Lemma 3, $(\lambda - L)^{-1}$ is defined and bounded on $\mathscr{D}(L) \cap C(\bar{D})$. We put $J_\lambda = (\lambda - L)^{-1}$. Then we have $\|J_\lambda\| \leqq 1/\lambda$. J_λ is defined as an operator of $\mathscr{D}(L) \cap C(\bar{D})$ into $\mathscr{D}(L) \cap \overline{C_0(D)}$, but we may consider J_λ as an operator of $\mathscr{D}(L) \cap \overline{C_0(D)}$ into $\overline{C_0(D)}$. Since $\overline{C_0(D)}$ is a Banach space and $\mathscr{D}(L) \cap \overline{C_0(D)}$ is dense in $\overline{C_0(D)}$, J_λ is extended to a bounded operator in $\overline{C_0(D)}$; we denote the extended operator by J_λ again.

By the Hille-Yosida theorem, there exists a unique semigroup $\{T_t\}$ of bounded operators in $\overline{C_0(D)}$ whose generator is a closed extention of the restriction of L to $\mathscr{D}(L) \cap \overline{C_0(D)}$, and we have

$$J_\lambda f = \int_0^\infty e^{-\lambda t} T_t f \, dt \quad (f \in \overline{C_0(D)}) . \tag{3.1}$$

The following proposition may be proved easily.

PROPOSITION 1. *J_λ satisfies the resolvent equation*:

$$J_\lambda - J_\mu = (\mu - \lambda) J_\lambda J_\mu .$$

Hence, by the result of Yosida [6], there exists a Green operator G defined by $Gf = \text{s-}\lim\limits_{\lambda \downarrow 0} J_\lambda f$ for any $f \in \overline{C_0(D)}$ if and only if the following (3.2) and (3.3) hold:

$$\text{s-}\lim_{\lambda \uparrow \infty} \lambda J_\lambda f = f \tag{3.2}$$

$$\text{s-}\lim_{\lambda \downarrow 0} \lambda J_\lambda f = 0 . \tag{3.3}$$

We shall prove (3.2) and (3.3).

PROOF OF (3.2). It follows from (3.1) that

$$\lambda J_\lambda f - f = \int_0^\infty \lambda e^{-\lambda t} (T_t f - f) \, dt .$$

Hence for any $\delta > 0$,

$$\begin{aligned} \| J_\lambda f - f \| &\leqq \int_0^\delta \lambda e^{-\lambda t} \| T_t f - f \| \, dt + \int_\delta^\infty \lambda e^{-\lambda t} (\| T_t f \| + \| f \|) \, dt \\ &\leqq \max_{0 < t < \delta} \| T_t f - f \| + 2 \| f \| \, e^{-\lambda \delta} . \end{aligned}$$

Accordingly

$$\overline{\lim_{\lambda \uparrow \infty}} \| \lambda J_\lambda f - f \| \leqq \max_{0 < t < \delta} \| T_t f - f \| .$$

Since δ is arbitrary and the right side tends to 0 as $\delta \to 0$, we have

$$\lim_{\lambda \uparrow \infty} \| \lambda J_\lambda f - f \| = 0 .$$

PROOF OF (3.3). For any $\lambda > 0$ and any $f \in \mathscr{D}(L) \cap \overline{C_0(D)}$, we put $v_\lambda = \lambda J_\lambda f$. Then

$$\| v_\lambda \| \leqq \lambda \| J_\lambda \| \, \| f \| \leqq \| f \| \qquad \text{and} \qquad (\lambda - L) \, v_\lambda = \lambda f .$$

Accordingly

$$\| L v_\lambda \| \leqq \lambda (\| v_\lambda \| + \| f \|) \leqq 2 \lambda \| f \| .$$

Thus we see that $\{v_\lambda\}_{\lambda > 0}$ and $\{L_\lambda\}_{\lambda > 0}$ are uniformly bounded. Hence by the axiom (β),

(3.4) $\{v_\lambda\}$ has a subsequence which converges uniformly on every compact subset of D_0 as $\lambda \downarrow 0$.

Let $\mu < \lambda$. Then

(3.5)
$$\begin{aligned}\lambda J_\lambda f-\mu J_\mu f&=\{\lambda(J_\lambda-J_\mu)+(\lambda-\mu)J_\mu\}f\\&=\{\lambda(\mu-\lambda)J_\lambda J_\mu+(\lambda-\mu)J_\mu\}f\\&\qquad\text{(by Proposition 1)}\\&=(\mu-\lambda)J_\mu(\lambda J_\lambda-I)f.\end{aligned}$$

Let $f_1(x)\equiv c>0$ and put $w_\lambda=\lambda J_\lambda f_1$. Then

(3.6)
$$\lambda J_\lambda f_1-f_1=c(\lambda J_\lambda 1-1)\leqq c(\|\lambda J_\lambda\|-1)\leqq 0.$$

If f in the above argument is replaced by f_1, then v_λ is replaced by w_λ, and accordingly $\{w_\lambda\}$ is uniformly bounded and $\|Lw_\lambda\|\leqq 2\lambda\|f_1\|\to 0$ as $\lambda\downarrow 0$. Since J_μ is a positive operator by Lemma 4, $\{w_\lambda\}$ decreases monotonously as $\lambda\downarrow 0$ by (3.5) and (3.6). From this fact and (3.4), it follows that w_λ converges to a function w uniformly on every compact subset of D. Since every w_λ is in $\overline{C_0(D)}$, we may prove

PROPOSITION 2. $\lim\limits_{\lambda\downarrow 0} w_\lambda=w$ *holds uniformly on* $\overline{D}$.

Thus $\{w_\lambda\}$ and $\{Lw_\lambda\}$ respectively converge to w and 0 uniformly on D. Hence we have $w\in\mathscr{D}(L)$ and $Lw=0$ in D by Lemma 6. Accordingly, from (α), w does not take its maximum in the interior of D. Since w is continuous on $\overline{D}$ and $w\equiv 0$ on ∂D from Proposition 2, we have $w\equiv 0$ on $\overline{D}$.

Thus we obtain that $\operatorname{s-lim}\limits_{\lambda\downarrow 0}\lambda J_\lambda f_1=0$ for any positive constant function f_1. Similary we may get the same result for any negative constant function. For general $f\in\mathscr{D}(L)\cap\overline{C_0(D)}$ we take constant functions f_1 and f_2 such that $f_1\leqq f\leqq f_2$. Since J_λ is a positive operator, we have

$$\lambda J_\lambda f_1\leqq\lambda J_\lambda f=\lambda J_\lambda f_2.$$

Since the extreme left side and the extreme right side converge to 0 uniformly on $\overline{D}$ as $\lambda\downarrow 0$, we get $\operatorname{s-lim}\limits_{\lambda\downarrow 0}\lambda J_\lambda f=0$.

From the above results we may conclude the existence of Green operator in the case of a compact domain.

§ 4. The existence of Green operator in the case of the whole space X.

Let $\{D_n\}_{n=0,1,2,\cdots}$ be a sequence of subdomains of X satisfying that $\overline{D}_n$ is compact and $\overline{D}_n\subset D_{n+1}$ for each n and that $\bigcup\limits_{n=0}^{\infty}D_n=X$; such sequence $\{D_n\}$ is called an *exhaustion of* X. Since X is locally compact and σ-compact, such an exhaustion always exists. Here we may assume every D_n to be a regular domain.

Let ω_n be the function L-harmonic in $D_n-\overline{D}_0$, continuous on $\overline{D}_n-D_0$, and satisfying that $\omega_n=0$ on ∂D_0 and $=1$ on ∂D_n. Then the following Propositions 3 and 4 may be proved by using (α) (maximum principle) and Lemma 6.

PROPOSITION 3. *$\omega_n(x)$ decreases montonously at every point x, as $n\to\infty$, and converges to an L-harmonic function ω uniformly on every compact subset of X.*

PROPOSITION 4. *The function ω in Proposition 3 does not depend on the choice of the sequence $\{D_n\}_{n\geqq 1}$.*

The function ω defined by the above proposition satisfies either $\omega\equiv 0$ on $X-\bar{D}_0$ or $\omega>0$ everywhere on $X-\bar{D}_0$ by the axiom (α).

PROPOSITION 5. *The situation:* $\omega\equiv 0$ or $\omega>0$ *on $X-\bar{D}_0$, does not depend on the choise of D_0. Accordingly the situation depends only on topological properties of X and the operator L.*

This fact may be proved from Proposition 4 and by the maximum principle (α).

PROPOSITION 6. $\omega\equiv 0$ *if and only if the following fact holds: for any subdomain V of X and any function u bounded and continuous on $\bar{V}$, and L-harmonic in V, both*

$$\sup_{\partial V} u = \sup_{V} u \qquad \text{and} \qquad \inf_{\partial V} u = \inf_{V} u$$

are satisfied.

Proof of this proposition is achieved in the same way as that of Theorem 3 in [3].

The following proposition also is proved by the same argument as in the case of Riemann surfaces, namely, by applying Proposition 6 to a connected component V of the domain $\{x\,|\,u(x)<u(x_0)\}$ where u is any nonconstant positive L-harmonic function and x_0 is any fixed point in X.

PROPOSITION 7. *If there exists a nonconstant positive L-harmonic function on X, then* $\omega\not\equiv 0$.

Under the assumption of Theorem 1 in §1, we have $\omega\not\equiv 0$ by Proposition 7, and accordingly $\omega>0$ in $X-\bar{D}_0$. Henceforth we assume $\omega>0$.

Let G_n be the Green operator defined in §3 for each $D_n (n\geqq 1)$. Then the function $u_n=G_nf$ satisfies $Lu_n=-f$ in D_n. In each D_n, the operator J_λ considered in §3 is a positive operator and $G_n=\text{s-}\lim_{\lambda\downarrow 0} J_\lambda$. Hence G_n also is a positive operator.

Now we take a nonnegative function $f\in C_0(X)\cap\mathscr{D}(L)$. We may assume that $\operatorname{supp} f\subset D_0$. Then,

PROPOSITION 8. *$G_nf\leqq G_mf$ in D_n if $1\leqq n\leqq m$.*
This may be proved by using (α).

PROPOSITION 9. $\max_{\partial D_1} u_n\leqq(1-r)\max_{\partial D_0} u_n$ *for a sutitable $r>0$ independent of $n\geqq 1$.*

PROOF. Let r be a positive number such that $\max_{\partial D_1}(1-\omega)=1-r$ and put $c_n=\max_{\partial D_0} u_n$. Then

$$L(c_n(1-\omega)-u_n)=0 \quad \text{in} \quad D_n-\bar{D}_0,$$
$$c_n(1-\omega)=c_n\geqq u_n \quad \text{on} \quad \partial D_0,$$
$$c_n(1-\omega)\geqq 0=u_n \quad \text{on} \quad \partial D_n.$$

Hence $c_n(1-\omega)-u_n\geqq 0$ on $D_n-\bar{D}_0$ by (α). Accordingly

$$u_n\leqq c_n(1-\omega) \quad \text{on} \quad \partial D_1.$$

Therefore

$$\max_{\partial D_1} u_n \leqq \max_{\partial D_1} c_n(1-\omega)=(1-r)c_n=(1-r)\max_{\partial D_0} u_n.$$

Hereafter we fix r as defined above.

Since $L(u_n-u_1)=0$ in D_1, we have, by (α) and Proposition 9,

$$u_n-u_1\leqq \max_{\partial D_1} u_n\leqq (1-r)\max_{\partial D_0} u_n\leqq (1-r)\max_{\bar{D}_0} u_n \quad \text{on} \quad \bar{D}_1.$$

Hence $\max_{\bar{D}_0}(u_n-u_1)\leqq(1-r)\max_{\bar{D}_0} u_n$. Accordingly

$$\max_{\bar{D}_0} u_n-\max_{\bar{D}_0} u_1\leqq \max_{\bar{D}_0} u_n-r\max_{\bar{D}_0} u_n,$$

which implies

(4.2) $$\max_{\bar{D}_0} u_n\leqq \frac{1}{r}\max_{\bar{D}_0} u_1.$$

Since $Lu_n=0$ in $D_n-\bar{D}_0$ by supp $f\subset D_0$ and since $u_n=0$ on ∂D_n, we have $u_n\leqq \max_{\partial D_0} u_n$ in $D_n-\bar{D}_0$. This fact and (4.2) imply that

(4.3) $$u_n\leqq\frac{1}{r}\max_{\partial D_0} u_n \quad \text{in} \quad D_n-\bar{D}_0.$$

From (4.2) and (4.3) we obtain

(4.4) $$u_n\leqq\frac{1}{r}\max_{\bar{D}_0} u_1 \quad \text{in} \quad D_n.$$

The sequence $\{u_n\}$ is monotone increasing in n by Proposition 8 and uniformly bounded by (4.4). Hence $u=\lim_{n\to\infty} u_n$ exists, and

(4.5) $$0\leqq u\leqq\frac{1}{r}\max_{\bar{D}_0} u_1.$$

Since $\{u_n\}$ and $\{Lu_n\}$ are uniformly bounded, a subsequence of $\{u_n\}$ converges uniformly on every compact subset of X by (β). By virtue of the monotonicity in n, the original sequence $\{u_n\}$ converges uniformly on every compact subset of X.

We fix an arbitrary n_0. Then $L(u_n-u_{n_0})=0$ in D_{n_0} and $\{u_n-u_{n_0}\}$ is uniformly bounded in D_{n_0}. Hence, by Lemma 6, $u-u_{n_0}\in\mathscr{D}(L)$ and $L(u-u_0)=0$ in D_{n_0}. Accordingly $Lu=Lu_{n_0}=-f$ in D_{n_0}. Since n_0 is arbitrary, we have $Lu=-f$ in X.

The above result is obtained for nonnegative function $f\in C_0(X)\cap\mathscr{D}(L)$. For any $f\in C_0(X)\cap\mathscr{D}(L)$, we may find $g\in C_0(X)\cap\mathscr{D}(L)$ such that $g\geqq 0$ and $g\geqq f$ on X. Then there exist bounded functions u_1 and $u_2\in\mathscr{D}(L)$ such that $Lu_1=-g$ and $Lu_2=-(g-f)$. Hence $u=u_1-u_2$ satisfies $Lu=-f$.

Thus we have obtained a linear operator which maps $f\in C_0(X)\cap\mathscr{D}(L)$ to u defined above; we denote the linear operator by G. Then $u=Gf$ implies $Lu=-f$. Therefore G is a Green operator associated with L considered in the whole space X.

The proof of Theorem 1 is thus complete.

§ 5. Integral representation of the Green operator.

THEOREM 2. *For the green operator G defined in the preceding section, there exists a family of measures $\{\Phi(x, E)\,|\,x\in X\}$ such that*

$$(5.1)\qquad (Gf)(x)=\int_X \Phi(x, dy)f(y) \quad \text{for any} \quad f\in C_0(X)\,.$$

PROOF. We fix a sequence of domains $\{D_n\}$ such that $\bar{D}_1\subset D_2\subset\bar{D}_2\subset D_3\cdots$, each $\bar{D}_n$ is compact and $\bigcup_{n=1}^{\infty} D_n=X$. For each n, we fix a function $g_n\in\mathscr{D}(L)\cap C_0^+(X)$ such that $1\leqq g_n\leqq 2$ on $\bar{D}_n$ and $\operatorname{supp} g_n\subset D_{n+1}$. Then $M_n=\sup_{x\in X}(Gg_n)(x)$ is finite by (4.5). For every n we shall prove

$$(5.2)\qquad |Gf(x)|\leqq M_n\|f\| \quad \text{for any} \quad f\in C_0(D_n)\cap\mathscr{D}(L)\,.$$

We may consider $f=0$ in $X-D_n$ and we have $-g_n\leqq f/\|f\|\leqq g_n$ on X; here we may assume $\|f\|>0$, otherwise (5.2) is trivial. Since G is a positve operator, we get

$$-M_n\leqq -Gg_n\leqq \frac{1}{\|f\|}\,Gf\leqq Gg_n\leqq M_n \quad \text{on} \quad X\,.$$

this shows (5.2). Since $C_0(D_n)\cap\mathscr{D}(L)$ is dense in $C_0(D_n)$, for each $x\in X$, $(Gf)(x)$ is extended to a positive bounded linear functional $\Phi_x^{(n)}(f)$ defined on $C_0(D_n)$ which satisfies

$$|\Phi_x^{(n)}(f)|\leqq M_n\|f\|$$

by (5.2). Hence there exists a measure $\Phi^{(n)}(x, E)$ in D_n depending on x such that

$$(5.3)\qquad \Phi_x^{(n)}(f)=\int_{D_n}\Phi^{(n)}(x, dy)f(y)$$

for any $f\in C_0(D_n)$ and any $x\in X$.

Next we extend the measure to the whole space X. If $n<m$ and $f\in C_0(D_n)$ $\cap\mathscr{D}(L)$, then

$$\int_{D_n}\Phi^{(n)}(x,dy)f(y)=\Phi_x^{(n)}(f)=(Gf)(x)=\Phi_x^{(m)}(f)$$

$$=\int_{D_m}\Phi^{(m)}(x,dy)f(y)\,.$$

Since $C_0(D_n)\cap\mathscr{D}(L)$ is dense in $C_0(D_n)$, we have

$$\int_{D_n}\Phi^{(n)}(x,dy)f(y)=\int_{D_m}\Phi^{(m)}(x,dy)f(y)\qquad\text{for any }\ f\in C_0(D_n)\,,$$

which implies $\Phi^{(n)}(x,E)=\Phi^{(m)}(x,E)$ for any $x\in X$ and any Borel set $E\subset D_n$. Hence, for every x, there exists a measure $\Phi(x,E)$ on X such that

$$\Phi(x,E)=\Phi^{(n)}(x,E)\qquad\text{for}\quad E\subset D_n\,. \tag{5.4}$$

For any $f\in C_0(X)\cap\mathscr{D}(L)$ we consider $D_n\supset\operatorname{supp} f$ and we have

$$(Gf)(x)=\Phi_x^{(n)}(f)=\int_{D_n}\Phi^{(n)}(x,dy)f(y)=\int_X\Phi(x,dy)f(y)$$

by (5.3) and (5.4). Our assertion (5.1) is thus proved.

§ 6. An example of the case where the range of Green operator is not contained in $C_0(X)$.

G. Hunt [2], K. Yosida [5] and others have constructed abstract potential theory in the function space $\overline{C_0(X)}$. However, if we consider the differential operator of the form $L=\Delta+\sum_{i=1}^{n}a_i(x)\partial/\partial x_i$ in $X=R^n$, we may find a case where no solution of the equation $Lu=-f$ for given $f\in C_0(X)$ is contained in the space $\overline{C_0(X)}$. We shall give an exmaple of such case. (The result of the present paper covers that kind of case.)

EXAMPLE. The space $X=\{(x,y)\,|\,0<x<\infty,\ 0<y<\pi\}$ with usual Euclidean topology is locally compact and σ-compact. We consider the operator L:

$$Lu=u_{xx}+u_{yy}-(e^x-1)\,u_x\,.$$

Then,

[a] There exists a nonconstant positive L-harmonic function in X.

In fact the function $u(x,y)=\int_0^x e^{e^{\xi}-\xi}\,d\xi$ satisfies $Lu=0$ in X. On the other hand we may prove that

[b] If $f\in C_0^2(X)$, $f\geqq 0$ and $f\not\equiv 0$, then no solution of the equation $Lu=-f$ belongs to $\overline{C_0(X)}$.

References

1) R. Courant-D. Hilbert: Methods of Mathematical Physics, **II** (Partial Differential Equations), Interscience Publ., New York, 1962.
2) G. Hunt: Markov processes and potentials, I, II, III, Ill. J. Math. **1** (1957) 44–93, 316–369, **2** (1958), 151–213.
3) A. Mori: On the existence of harmonic functions on a Riemann surface, J. Fac. Sci. Univ. Tokyo, Sec. I, **6** (1951), 247–257.
4) A. Yamada: On the correspondence between potential operators and semigroups associated with Markov processes, Z. Wahrscheinlichkeitstheorie u. verw. Gebiete, **15** (1970), 230–238.
5) K. Yosida: Positive resolvents and potentials, ibid. **8** (1967), 210–218.
6) K. Yosida: On the existence and characterization of abstract potential operators, Proc. Colloq. Funct. Anal., Liège (1970), 129–136.
7) K. Yosida: Functional Analysis, 4th ed. Springer 1974.