Natural Science Report, Ochanomizu University, Vol. 29, No. 1, 1978
お茶の水女子大学 自然科学報告 第29巻 第1号

The cylindrical measures on some Banach space

Michie Maeda

Department of Mathematics, Faculty of Science
Ochanomizu University, Tokyo
(Received, March 28, 1978)

Introduction

This note deals with the characterization of cylindrical measures on a Banach space with a shrinking basis.

1. Preliminaries.

Let E and F be Banach spaces over the real field $\mathbf{R}$, E' be the topological dual space of E and $L(E, F)$ be the Banach space of continuous linear maps $u: E \to F$.

Our terminology shall primarily be in agreement with that of [3] and [4].

DEFINITION 1. *A cylindrical measure* (*or cylindrical probability*) μ on E is a correspondence which assigns to each $u \in L(E, H)$, where H is a finite dimensional space, a Radon probability μ_u on H such that the following coherent relation is satisfied: if $u \in L(E, H)$ and $v \in L(H, K)$, where K is a finite dimensional space, then $\mu_{v\circ u} = v(\mu_u)$.

This definition may be applicable to any other locally convex Hausdorff topological vector spaces. Moreover, we can restrict the H to spaces of the form $\mathbf{R}^n$ where n is a positive integer, since every n-dimensional Hausdorff topological linear space is topologically isomorphic to $\mathbf{R}^n$. Thus a cylindrical measure μ on E associates with each n-tuple $(\xi_1, \cdots\cdots, \xi_n)$, where each $\xi_i \in E'$, a Radon probability on $\mathbf{R}^n$.

DEFINITION 2. (i) Let μ be a cylindrical measure on E and $0 \leq \delta \leq 1$. Then μ is said to be *scalarly concentrated up to δ on a set $A \subset E$* if for every continuous linear form $\xi \in E'$, $(\mu_\xi)_*(\xi(A)) \geq 1 - \delta$ where $(\mu_\xi)_*$ is the inner measure associated with μ_ξ.

(ii) Let $\mathfrak{S}$ be a collection of subsets of E. A cylindrical measure μ is said to be *scalarly concentrated on* $\mathfrak{S}$ if for every $\varepsilon > 0$ there exists a set $A \in \mathfrak{S}$ such that μ is scalarly concentrated up to ε on A.

DEFINITION 3. A sequence $\{e_n\}_{n=1}^{\infty}$ in a Banach space E is called *a Schauder* *basis* of E if for every $x \in E$ there is a unique sequence of scalars $\{a_n\}_{n=1}^{\infty}$ such that $x = \sum_{n=1}^{\infty} a_n e_n$.

In this note we shall not consider any type of bases in infinite-dimensional Banach spaces besides Schauder bases. Therefore, from now on, we shall omit the word Schauder.

We assume the Banach space E to have a basis $\{e_n\}_{n=1}^{\infty}$. For every n, the linear functional e_n' on E defined by $e_n'\left(\sum_{i=1}^{\infty} a_i e_i\right) = a_n$ is continuous. These functionals are called the biorthogonal functionals associated to the basis $\{e_n\}_{n=1}^{\infty}$.

Definition 4. A basis $\{e_n\}_{n=1}^{\infty}$ of E is called *a shrinking basis* if the sequence $\{e_n'\}_{n=1}^{\infty}$ of the biorthogonal functionals associated to $\{e_n\}_{n=1}^{\infty}$ is a basis of E'.

The following theorem, due to W. B. Johnson, H. P. Rosenthal and M. Zippin, provides an interesting characterization of shrinking bases.

Theorem 1. ([2]) *Let E be a Banach space such that E' has a basis. Then E has a shrinking basis.*

Finally we shall make a brief mention of the relation between cylindrical measures and random functions. A sample space is a pair (Ω, m) consisting of a topological space Ω and a finite positive Radon measure m with total mass 1, defined on Ω. An equivalence class of m-measurable maps from Ω to $\mathbf{R}$ is called a real random variable, and we denote by $L^0(\Omega, m; \mathbf{R})$ the set of all real random variables. Let (Ω, m) and (Ω', m') be two sample spaces. We say that two families $\{x_i\}_{i \in I}$ and $\{x_i'\}_{i \in I}$ of real random variables such that $x_i : \Omega \to \mathbf{R}$ and $x_i' : \Omega' \to \mathbf{R}$ are isonomous if for every finite subset J of I they define the same probability distribution on $\mathbf{R}^{\|J\|}$, where $\|J\|$ means the cardinal number of J. Now we shall define a random function. Given an arbitrary set T, a random function f on T is a map from T to $L^0(\Omega, m; \mathbf{R})$. Unless T has some structure, we can regard a random function f as a family $\{f(t)\}_{t \in T}$ of random variables. Then it follows that the concept of isonomous random functions is well defined. In the case where T is a vector space, we define a random function f to be linear if for every $\lambda_1, \lambda_2 \in \mathbf{R}$ and $t_1, t_2 \in T$ it satisfies the relation

$$f(\lambda_1 t_1 + \lambda_2 t_2) = \lambda_1 f(t_1) + \lambda_2 f(t_2)$$

in the sense of equality of random variables.

The ensuing statements are well known results. (cf. [4])

Proposition 1. *There exists a bijective correspondence between the cylindrical measures on E and the isonomy classes of linear random functions on E'.*

PROPOSITION 2. *Let μ be a cylindrical measure on E, $f: E' \to L^0(\Omega, m; \mathbf{R})$ a linear random function associated with μ. The following are equivalent:*

(a) *μ is scalarly concentrated on the balls of E;*

(b) *f is continuous if $L^0(\Omega, m; \mathbf{R})$ is equipped with the topology of convergence in probability.*

PROPOSITION 3. *If μ is a Radon probability on E equipped with the weak topology $\sigma(E, E')$, then μ is also a Radon probability on E equipped with its norm topology, and vice versa.*

REMARK. In particular, when the topological dual E' of E is separable, we can replace E by E' and the weak topology $\sigma(E, E')$ by the weak* topology $\sigma(E', E)$ in the above statement.

2. Main results.

First, we refer to the following lemma without the proof, that is obtained from the Banach-Steinhaus theorem for spaces which are not necessarily locally convex spaces.

We keep the notation of the preceding section.

LEMMA 1. (cf. [4]) *Let $u_n: E \to L^0(\Omega, m; \mathbf{R})$ $(n=1, 2, \cdots\cdots)$ be a sequence of continuous linear maps which converges pointwise to a linear map $u: E \to L^0(\Omega, m; \mathbf{R})$. Then u is continuous.*

Here, we consider that L^0 is endowed with the topology of convergence in probability.

Now we shall prove the following

THEOREM 2. *Let E be a Banach space with a shrinking basis $\{e_n\}_{n=1}^{\infty}$. There exists a bijective correspondence between the cylindrical measures on E scalarly concentrated on the balls, and the strong isonomy classes of sequences $\{x_n(\omega)\}_{n=1}^{\infty}$ of real random variables with the following property:*

(*) *For every $x' \in E'$, the series $\sum_{n=1}^{\infty} x_n(\omega) \langle e_n, x' \rangle$ converges in probability.*

PROOF. Suppose that μ is a cylindrical measure on E scalarly concentrated on the balls. Then we have a continuous linear random function $f: E' \to L^0(\Omega, m; \mathbf{R})$ associated with μ, by Propositions 1 and 2. Denote by $\{e_n'\}_{n=1}^{\infty}$ the biorthogonal functionals associated to $\{e_n\}_{n=1}^{\infty}$. For each e_n', there corresponds a real random variable $f(e_n')$. Then clearly, letting $x_n = f(e_n')$, it can satisfy the condition (*). Indeed, each $x' \in E'$ can be represented as $x' = \sum_{n=1}^{\infty} a_n e_n'$, because the sequence $\{e_n'\}_{n=1}^{\infty}$ is a basis of E'. Therefore we only have to prove that the series $\sum_{n=1}^{\infty} a_n f(e_n')$ converges in probability for every scalar sequence

$\{a_n\}_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty} a_n e'_n \in E'$. But we can get easily this result using the continuity of the linear random function f.

Conversely, let (Ω, m) be a sample space and $\{x_n(\omega)\}_{n=1}^{\infty}$ a sequence of real random variables defined on Ω satisfying the condition (*). We obtain a linear random function f. It is as follows. For each $x' \in E'$ the series $\sum_{n=1}^{\infty} x_n(\omega) \langle e_n, x' \rangle$ converges in probability to some random variable, say $f(x')$. Obviously f is a linear random function on E'. Next we shall show the continuity of f. For any integer $N>0$, let f_N be the linear map such that

$$x' \longmapsto f_N(x') = \sum_{n=1}^{N} a_n x_n(\omega)$$

for every $x' = \sum_{n=1}^{\infty} a_n e'_n \in E'$. The map f_N is the composition of the projection

$$x' = \sum_{n=1}^{\infty} a_n e'_n \longmapsto (a_1, \cdots\cdots, a_N)$$

from E' into $\mathbf{R}^N$ and the linear map

$$(a_1, \cdots\cdots, a_N) \longmapsto \sum_{n=1}^{N} a_n x_n(\omega)$$

from $\mathbf{R}^N$ into $L^0(\Omega, m; \mathbf{R})$. The first one is obviously continuous and the second one is continuous, too, because $\mathbf{R}^N$ is a finite dimensional space. Thus f_N is continuous for every integer $N > 0$. Since the sequence $\{f_N\}_{N=1}^{\infty}$ tends pointwise to f, we can use Lemma 1 and so we can say that f is continuous. Propositions 1 and 2 show the existence of the cylindrical measure scalarly concentrated on the balls of E associated with the isonomy class of f.

Recapitulating the above discussion, we have the next result that every cylindrical measure scalarly concentrated on the balls of E defines the sequence of real random variables satisfying the condition (*) and vice versa. We want to show the existence of the bijective correspondence between these cylindrical measures and strong isonomy classes of sequences of real random variables. Given any cylindrical measure μ scalarly concentrated on the balls of E, there corresponds the sequence $\{x_n\}_{n=1}^{\infty}$ of real random variables. Assume that a sequence $\{x_n^*\}_{n=1}^{\infty}$ of real random variables is strongly isonomous to $\{x_n\}_{n=1}^{\infty}$ and satisfies the condition (*). If $\{x_n^*\}_{n=1}^{\infty}$ defines just the same cylindrical measure μ, then we complete the proof. Using the same way of the latter half of the above discussion, we get the random function f^* from $\{x_n^*\}_{n=1}^{\infty}$. It is clearly verified that f^* is isonomous to f which is associated with μ. Thus $\{x_n^*\}_{n=1}^{\infty}$ defines μ. □

We shall now characterize those sequences $\{x_n\}$ of real random variables for which the associated cylindrical measure μ is a Radon measure on E. For this purpose, we present the following proposition (cf. [4]).

We denote by $\check{\mathbf{R}}$ the Čech compactification of $\mathbf{R}$ and by $\check{\mathbf{R}}^{E'}$ the space of all maps from E' to $\check{\mathbf{R}}$ equipped with the product topology. Let μ be a cylindrical measure on E. To every n-tuple $u=(\xi_1, \cdots\cdots, \xi_n)$ of elements of E', we have a probability μ_u on $\check{\mathbf{R}}^n$ because it is the very definition of the cylindrical measure, and let π_u be the projection from $\check{\mathbf{R}}^{E'}$ onto the partial product $\check{\mathbf{R}}^n=\check{\mathbf{R}}^u$.

PROPOSITION 4. ([4]) *Given any cylindrical measure μ on E, there corresponds a Radon probability, say $\check{\mu}$, on the space $\check{\mathbf{R}}^{E'}$ such that $\pi_u(\check{\mu})=\mu_u$ for every u.*

REMARK. Thus we have the sample space $(\check{\mathbf{R}}^{E'}, \check{\mu})$. Let f be the function which is simply the ξ-th projection π_ξ: $\check{\mathbf{R}}^{E'} \to \check{\mathbf{R}}$, where ξ is an arbitrary element of E'. Then for almost every $\omega \in \check{\mathbf{R}}^{E'}$ the map $f(\xi)$ takes its value in $\mathbf{R}$ (not only in $\check{\mathbf{R}}$) and this function f is the random function from E' to $L_0(\check{\mathbf{R}}^{E'}, \check{\mu}; \mathbf{R})$ associated with the cylindrical measure μ originally given.

We can consider E as a subset of $\check{\mathbf{R}}^{E'}$ with the help of the canonical injection j: $E \to \check{\mathbf{R}}^{E'}$ defined by $j(x)=(\langle x, \xi\rangle)_{\xi \in E'}$. Then we have the following

LEMMA 2. ([4]) *If the Radon probability $\check{\mu}$ induced by μ is concentrated on $j(E)$, then the original cylindrical measure μ is the Radon probability on E equipped with the weak topology $\sigma(E, E')$.*

REMARK. Of course, these proposition and lemma are verified even if the Banach space E has no bases.

THEOREM 3. *Let E be a Banach space with a shrinking basis $\{e_n\}_{n=1}^{\infty}$. A cylindrical measure μ on E, scalarly concentrated on the balls, is a Radon measure if and only if the associated sequence $\{x_n(\omega)\}_{n=1}^{\infty}$ of real random variables satisfies the following condition:*

$\sum_{n=1}^{\infty} x_n(\omega)e_n$ converges in E for almost every $\omega \in \Omega$, where (Ω, m) is a sample space.

PROOF. First we observe that the condition that $\sum_{n=1}^{\infty} x_n(\omega)e_n$ converges in E for almost every $\omega \in \Omega$, depends only on the isonomy class of the random function f: $E' \to L^0(\Omega, m; \mathbf{R})$ associated with the cylindrical measure μ, or (it comes to the same thing) on the strong isonomy class of the sequence $\{x_n(\omega)\}_{n=1}^{\infty}$ associated with μ. The sequence $\{x_n(\omega)\}_{n=1}^{\infty}$ defines a map from Ω to $\mathbf{R}^{\mathbf{N}}$, where $\mathbf{N}$ is the set of all positive integers. Evidently, the space E with the basis $\{e_n\}_{n=1}^{\infty}$ can be considered as a sequence space by identifying each $x=\sum_{n=1}^{\infty} a_n e_n \in E$ with the unique sequence of coefficients $(a_1, a_2, a_3, \cdots\cdots)$. Therefore we can regard such a sequence space in the same light with E without any confusion. And so the condition that $\sum_{n=1}^{\infty} x_n(\omega)e_n$ converges in

E for almost every $\omega \in \Omega$, means that the image of the measure m with respect to the map $\{x_n\}$: $\Omega \to \mathbf{R}^{\mathbf{N}}$ is concentrated on the subspace E of $\mathbf{R}^{\mathbf{N}}$. By the definition of isonomy, the projections of $\{x_n\}(m)$ onto the products $\mathbf{R}^J$, where J is a finite subset of $\mathbf{N}$, depend only on the isonomy class of $\{x_n\}_{n=1}^{\infty}$, and therefore $\{x_n\}$ (m) depends only on the isonomy class of $\{x_n\}_{n=1}^{\infty}$.

Therefore we may consider the special random function f and sample space (Ω, m), i. e. $\Omega=\check{\mathbf{R}}^{E'}$, $m=\check{\mu}$ and $f(\xi)$ is the ξ-th projection π_ξ for every $\xi \in E'$. It follows from the consequence of Proposition 4 and its Remark that the above function f is the associated random function with respect to μ. There is a canonical injection $j: E \to \check{\mathbf{R}}^{E'}$ given by $j(x)=(\langle x, \xi\rangle)_{\xi \in E'}$ for $x=(a_1, a_2, \cdots\cdots) \in E$ (or $x=\sum_{n=1}^{\infty} a_n e_n \in E$), and the value of $x_n=f(e_n')$ at a point $\omega=j(x) \in j(E) \subset \check{\mathbf{R}}^{E'}$ is given by

$$x_n(\omega)=f(e_n')(\omega)=\pi_{e_n'}(j(x))=\langle x, e_n'\rangle=a_n.$$

Thus $\check{\mu}$ is concentrated on $j(E)$ if and only if the sequence $(x_1(\omega), x_2(\omega), \cdots\cdots)$ belongs to E for almost every $\omega \in \check{\mathbf{R}}^{E'}$, that is, $\sum_{n=1}^{\infty} x_n(\omega) e_n$ converges in E for almost every $\omega \in \check{\mathbf{R}}^{E'}$. According to Lemma 2 and Proposition 3, we complete the proof. □

3. Further discussions for cylindrical measures.

In the preceding section we studied the connection between cylindrical measures and sequences of real random variables. We shall now consider the class of the cylindrical measures such that the associated sequence $\{x_n(\omega)\}_{n=1}^{\infty}$ defined on Ω, where (Ω, m) is a sample space, satisfies the ensuing condition (**).

Throughout this section, let E be a Banach space with a shrinking basis $\{e_n\}_{n=1}^{\infty}$, and we always assume the cylindrical measure μ on E to be scalarly concentrated on the balls.

(**) *For almost every $\omega \in \Omega$, the sequence $\{x_n(\omega) e_n\}_{n=1}^{\infty}$ is scalarly in l^1, i.e. $\{x_n(\omega)\langle e_n, x'\rangle\}_{n=1}^{\infty} \in l^1$ for every $x' \in E'$.*

An interesting case is that of an E which is a C-space (cf. [1]), that is, every sequence which is scalarly in l^1 is summable. In this case the condition (**) implies that the associated cylindrical measure μ is a Radon measure.

Generally, we get the next result.

PROPOSITION 5. *Let μ be a cylindrical measure on E. If the associated sequence $\{x_n(\omega)\}_{n=1}^{\infty}$ defined on a sample space (Ω, m), satisfies the condition (**), then μ is a Radon probability on the second dual E'' of E, equipped with the weak* topology $\sigma(E'', E')$.*

PROOF. Let f be the random function $f: E' \to L^0(\Omega, m; \mathbf{R})$ associated with μ. Clearly, $f(\xi)(\omega) = \sum_{n=1}^{\infty} x_n(\omega) \langle e_n, \xi \rangle$ for every $\xi \in E'$. Now we define the random variable $\check{f}: \Omega \to \check{\mathbf{R}}^{E'}$ such that

$$\check{f}(\omega)(\xi) = f(\xi)(\omega) = \sum_{n=1}^{\infty} x_n(\omega) \langle e_n, \xi \rangle$$

for almost every $\omega \in \Omega$ and every $\xi \in E'$. For almost every $\omega \in \Omega$, $\check{f}(\omega)$ is a map from E' into $\mathbf{R}$ (not only into $\check{\mathbf{R}}$), and next we want to show that $\check{f}(\omega)$ is a continuous linear form on E'. This means that $\check{f}(\omega) \in j(E'')$ for almost every $\omega \in \Omega$, where j is the canonical injection $E'' \to \check{\mathbf{R}}^{E'}$ such that $j(x'') = (\langle \xi, x'' \rangle)_{\xi \in E'}$ for every $x'' \in E''$. As it is trivial that $\check{f}(\omega)$ is linear, we only have to prove the continuity. The linear map $\check{f}(\omega)$ is the composition of the linear map u from E' into l^1 such that

$$\xi \longmapsto \{x_n(\omega) \langle e_n, \xi \rangle\}_{n=1}^{\infty}$$

and the linear map v from l^1 into $\mathbf{R}$ such that

$$\{x_n(\omega) \langle e_n, \xi \rangle\}_{n=1}^{\infty} \longmapsto \sum_{n=1}^{\infty} x_n(\omega) \langle e_n, \xi \rangle.$$

Notice that the sequence $\{x_n(\omega) \langle e_n, \xi \rangle\}_{n=1}^{\infty}$ belongs to l^1 and the series $\sum_{n=1}^{\infty} x_n(\omega) \langle e_n, \xi \rangle$ converges for every $\xi \in E'$, because it is the very condition (**). It is easy to see the continuity of v. Then we shall check that of u. Denote by u_k $(k=1, 2, \cdots\cdots)$ the linear map from E' into l^1 such that

$$u_k(\xi) = \{x_1(\omega) \langle e_1, \xi \rangle, \cdots\cdots, x_k(\omega) \langle e_k, \xi \rangle, 0, 0, \cdots\cdots\}$$

for every $\xi \in E'$. Obviously each u_k is continuous and for every $\xi \in E'$, $\lim_{k \to \infty} u_k(\xi) = u(\xi) \in l^1$. Therefore it follows from the Banach-Steinhaus theorem that u is continuous.

On the other hand we may regard μ as a cylindrical measure on E'' equipped with $\sigma(E'', E')$. In order to assert that μ is a Radon probability on E'' endowed with $\sigma(E'', E')$, it is sufficient to see that $\check{\mu}$, which was introduced in the preceding section, is concentrated on $j(E'')$. But this means that $m(\check{f}^{-1}(j(E''))) = 1$, i.e. $\check{f}(\omega) \in j(E'')$ for almost every $\omega \in \Omega$. □

Finally we present some typical Banach spaces as examples that satisfy our requirement. It is trivial that l^p, $L^p(0, 1)$ $(1 < p < \infty)$ and c_0 have shrinking bases. In particular, since the spaces l^p and $L^p(0, 1)$ are separable and reflexive, the condition (**) implies that the associated cylindrical measure μ is a Radon probability on the original space equipped with its norm topology.

References

1) A. Badrikian, *Prolégomènes au calcul des probabilités dans les Banach*, Springer Lecture Notes in Math. **539** (1976), 1–166.
2) W. B. Johnson, H. P. Rosenthal and M. Zippin, *On bases, finite dimensional decompositions and weaker structures in Banach spaces*, Israel J. Math. **9** (1971), 488–506.
3) J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces I*, Springer-Verlag, Berlin Heidelberg New York, 1977.
4) L. Schwartz, *Radon measures on arbitrary topological spaces and cylindrical measures*, Oxford University Press, 1973.