Natural Science Report, Ochanomizu University, Vol. 29, No. 1, 1978
お茶の水女子大学 自然科学報告 第29巻 第1号

On the variety of Borel subgroups containing a given diagonalizable subset

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(Received, April 3, 1978)

§ 0. Introduction.

Let G be a connected affine algebraic group defined over an algebraically closed field K of any characteristic. Then the set $\mathscr{B}$ of all Borel subgroups of G can be regarded as a projective variety. Given a subset S of G, we want to know about the subvariety $\mathscr{B}^S$ which consists of Borel subgroups containing S. In this paper we consider the dimension of $\mathscr{B}^S$.

In connection with this problem, R. Steinberg conjectured that for any reductive algebraic group G and any element x of G, the following equality holds.

$$\dim Z_G(x) = \mathrm{rank}\ G + 2 \dim \mathscr{B}^x,$$

where $Z_G(x)$ denotes the centralizer of x in G. And he proved that if x is a unipotent element, then the left-hand side is not less than the right-hand side ([3]). The main result of the present paper is Corollary 9, which says that if G is a reductive group and S is a diagonalizable subset of G satisfying a certain condition, we can compute the dimension of $\mathscr{B}^S$, using the root system of G. It is derived from a relation between dimensions of $Z_G(S)$ and $\mathscr{B}^S$. As an application of our result, we note that the above conjecture is true for semi-simple elements.

Throughout this paper, G denotes a connected affine algebraic group over an algebraically closed field K. The terminology and notations we employ are mostly those of Borel ([1]).

§ 1. The moduli of Cartan subgroups.

Given an algebraic group H, we write H^0 for its identity component. Given an element or subset S of H then ${}^xS$ stands for xSx^{-1}.

We recall that the set $\mathscr{B} = \mathscr{B}(G)$ of all Borel subgroups of G is naturally endowed with an algebraic structure. That is for a fixed $B \in \mathscr{B}$, we have a bijection $G/B \to \mathscr{B}$, mapping a coset gB to a point ${}^gB$ of $\mathscr{B}$. Through this bijection, we can regard $\mathscr{B}$ as a flag variety. Furthermore if

S is a subset of G, then the fixed point set $(G/B)_S$ of S in G/B corresponds to the set $\mathscr{B}^S$ of Borel subgroups containing S.

Suppose that G is defined over a subfield k of K and has a k-split Borel subgroup and that S consists of k-rational points. Then $\mathscr{B}^S$ is a k-closed subvariety of $\mathscr{B}$. Moreover if k is perfect, then k-group $Z_G(S)$ acts on k-variety $\mathscr{B}^S$, k-morphically.

Now we shall apply a similar method to Cartan subgroups of G to construct an algebraic variety whose dimension is $\dim G - \operatorname{rank} G$. Let $\mathscr{C} = \mathscr{C}(G)$ denote the set of all Cartan subgroups of G. For a fixed $C \in \mathscr{C}$, define a mapping

$$\varphi:\ G/N_G(C) \to \mathscr{C},\quad \varphi(g\cdot N_G(C)) = {}^gC.$$

By the way, it is well-known that $G/N_G(C)$ is an irreducible smooth quasi-projective variety (see e.g. [1], § 6). Through this G-equivariant bijection φ, one can give $\mathscr{C}$ the variety structure of $G/N_G(C)$, not depending on the choice of C. This is an analogue of $\mathscr{B}$, however, it is not so natural as in the case of $\mathscr{B}$, for $N_G(C)$ is not equal to C in general. But we have a canonical morphism ψ of G/C onto $G/N_G(C)$. ψ is surjective and every fibre of ψ is finite, because $N_G(C)^\circ = C$. Hence we have

$$\dim \mathscr{C} = \dim G - \operatorname{rank} G.$$

Furthermore for any subset S of G, we write $\mathscr{C}^S$ for the set of Cartan subgroups containing S. Then we have

PROPOSITION 1. For any subset S of G, $\mathscr{C}^S$ is a closed subvariety of $\mathscr{C}$.

PROOF. G acts on G/C in the natural way, and the set $(G/C)_S$ of fixed points of S in G/C is closed. But since the condition $S\cdot gC = gC$ is equivalent to $S^g = g^{-1}Sg \subset C$, it follows that $\psi((G/C)_S) = \mathscr{C}^S$. On the other hand, if $g^{-1}g' \in N_G(C)$ and $S^g \subset C$, then $S^{g'} \subset C$. Since ψ is a quotient morphism, and hence open, we conclude that $\mathscr{C}^S$ is closed.

PROPOSITION 2. If S is a subgroup of G consisting of semi-simple elements, then $\dim \mathscr{C}^S \geqq \dim \mathscr{B}^S$.

PROOF. If $\mathscr{B}^S$ is empty then so is $\mathscr{C}^S$. We assume $\mathscr{B}^S$ is not empty. We define

$$Y = \{(C, B) \in \mathscr{C}^S \times \mathscr{B}^S | S \subset C \subset B\}.$$

Since the projection of Y to $\mathscr{B}^S$ is surjective, $\dim Y \geqq \dim \mathscr{B}^S$. On the other hand, the projection of Y to $\mathscr{C}^S$ is surjective, and every fibre is finite. Hence $\dim Y = \dim \mathscr{C}^S$, so the proposition follows.

§ 2. The centralizer of a diagonalizable subset.

We shall assume from now on that G is reductive. Let T be a maximal torus in G and $\Phi=\Phi(T, G)$ the root system of G relative to T. Let B and B^- be opposite Borel subgroups relative to T, and put $\Phi(B)=\Phi^+$, $\Phi(B^-)=\Phi^-$. We write U (resp. U^-) for the unipotent part of B (resp. B^-). In the following lemma, $\boldsymbol{G}_a$ denotes the additive group of K.

Lemma (cf. [1]-§ 14). For each $\alpha \in \Phi$, there exists a connected unipotent subgroup U_α of G, normalized by T, and an isomorphism $\theta_\alpha : \boldsymbol{G}_a \to U_\alpha$ such that $t\,\theta_\alpha(u)t^{-1}=\theta_\alpha(\alpha(t)u)$ ($u \in \boldsymbol{G}_a$, $t \in T$). Furthermore the morphism σ of $\boldsymbol{G}_a^n \times T \times \boldsymbol{G}_a^n$ onto U^-TU defined by $\sigma((\prod_{\alpha\in\Phi^-} u_\alpha, t, \prod_{\beta\in\Phi^+} u_\beta)) = (\prod_{\alpha\in\Phi^-} \theta_\alpha(u_\alpha))\, t\, (\prod_{\beta\in\Phi^+} \theta_\beta(u_\beta))$ (arbitrary but fixed order of the factors) is an isomorphism, where $n={}^{\#}\Phi^+$, the number of the set Φ^+.

Now let S be a diagonalizable subset of G with $\mathscr{B}^S \neq \phi$. We shall clarify the structure of $Z_G(S)$ in the following

Proposition 3. Let G be a reductive group with a maximal torus T, and S a subset of T. Put $\Phi=\Phi(T, G)$ and

$$\Phi_S = \{\alpha \in \Phi \mid \alpha(S) = 1\}.$$

(i) Φ_S is a closed subsystem of Φ.
(ii) $Z_G(S)^0$ is generated by T and $U_\alpha(\alpha \in \Phi_S)$.
(iii) $Z_G(S)^0$ is a reductive group with the maximal torus T, and $\Phi(T, Z_G(S)^0)=\Phi_S$.

Proof. For any element x of U^-TU of the form $(\prod_{\alpha<0} \theta_\alpha(u_\alpha))\, t\, (\prod_{\beta>0} \theta_\beta(u_\beta))$ and any element s of S, we have

$$sxs^{-1}=(\prod_{\alpha<0} \theta_\alpha(\alpha(s)u_\alpha))\, t\, (\prod_{\beta>0} \theta_\beta(\beta(s)u_\beta))$$

It then follows that $sxs^{-1}=x$ if and only if $\gamma(s)=1$ for any root γ with $u_\gamma\neq 0$. Hence $Z_G(S) \ni x$ if and only if $\gamma \in \Phi_S$ for any root with $u_\gamma\neq 0$. Thus $Z_G(S) \cap U^-TU$ equals $(\prod_{\substack{\alpha<0\\ \alpha\in\Phi_S}} U_\alpha)\, T\, (\prod_{\substack{\beta>0\\ \beta\in\Phi_S}} U_\beta)$. But U^-TU is open in G, and since the subgroup generated by T and $U_\alpha(\alpha \in \Phi_S)$ is connected, it coincides with $Z_G(S)^0$, as claimed.

The first and last assertions are obvious.

Thus we obtain

Theorem 4. Let G be reductive and S a diagonalizable subset of G with $\mathscr{B}^S \neq \phi$. Then

$$\dim Z_G(S) = \operatorname{rank} G + 2\ {}^{\#}\{\alpha \text{ a positive root} \mid \alpha(S) = 1\},$$

where the root system is taken relative to any maximal torus of G containing S.

REMARK. We can obtain another proof of Theorem 4, by investigating the Lie algebra $\mathfrak{g}$ of G. The sketch of the proof is as follows. Since S is diagonalizable, the Lie algebra $L(Z_G(S))$ of $Z_G(S)$ is equal to $\mathfrak{g}^S$ (where $\mathfrak{g}^S$ is defined by : $\mathfrak{g}^S = \{X \in \mathfrak{g} \mid (\mathrm{Ad}\, s)X = X \text{ for all } s \in S\}$). On the other hand,

$$\begin{aligned} \mathfrak{g}^S &= L(T)^S \oplus \coprod_{\alpha \in \Phi} \mathfrak{g}_\alpha^S \\ &= L(T) \oplus \coprod_{\substack{\alpha \in \Phi \\ \alpha(S)=1}} \mathfrak{g}_\alpha \end{aligned}$$

is the root space decomposition of $\mathfrak{g}^S$ relative to a maximal torus T containing S. Hence

$$\dim \mathfrak{g}^S = \operatorname{rank} G + {}^{\#}\Phi_S.$$

Thus we have the desired conclusion.

Theorem 4 also implies that the number of positive roots whose kernels contain S does not depend on the choice of a root system, i.e. the choice of $T \in \mathscr{C}^S$.

By the way, $\dim Z_G(x) \geqq \operatorname{rank} G$ for any element x of a reductive group G; and x is called *regular* if the equality holds. It should be remarked that our definition is different from that of [1]. It is known that if x is regular, then $Z_G(x)^0$ is abelian ([5]). But since a maximal torus is a maximal abelian subgroup, the converse is true for semi-simple elements.

PROPOSITION 5. Let x be a semi-simple element of a reductive group G. Then x is regular if and only if $Z_G(x)^0$ is abelian.

As an immediate consequence of Theorem 4, we have

COROLLARY 6. The dimension of any semi-simple or regular conjugacy class in a reductive group is even.

§ 3. Dimension of $\mathscr{B}^S$.

THEOREM 7. Let G be a reductive group and S a diagonalizable subset of G with $\mathscr{B}^S \neq \phi$. Then there exist (not necessarily distinct) irreducible components F_1 and F_2 of $\mathscr{B}^S$ whose dimensions satisfy the following equality.

$$\dim Z_G(S) = \operatorname{rank} G + \dim F_1 + \dim F_2$$

PROOF. The proof consists of four parts.

(a) Fix a maximal torus T and opposite Borel subgroups B and B^- relative to T. Set $\mathscr{B}=G/B$, $\mathscr{B}^-=G/B^-$ and $\mathscr{C}=G/N_G(T)$. Define X by: $X=\{({}^gB, {}^gB^-)\in\mathscr{B}\times\mathscr{B}^-|g\in G\}$. Consider a morphism of G into $\mathscr{B}\times\mathscr{B}^-$ given by $g\mapsto({}^gB, {}^gB^-)$. It is a quotient onto its image X and the fibre over (B, B^-) is $N_G(B)\cap N_G(B^-)$, which is $B\cap B^-=T$. X is therefore isomorphic to G/T.

Moreover we claim that X is an open subvariety of $\mathscr{B}\times\mathscr{B}^-$. To verify this, first of all we note that the inclusion $X\subset\{({}^gB, {}^{g'}B^-)|g^{-1}g'\in BU^-\}$ is an equality. Indeed, if $g^{-1}g'=bu$ with $b\in B$, $u\in U^-$, then $g'u^{-1}=gb$. Put $h=gb$. Then ${}^hB^-={}^{g'}B^-$, ${}^hB={}^gB$. Define a morphism ρ ; $G\times G\to G$, by $\rho((g,g'))=g^{-1}g'$. Then above fact shows that X is the image of $\rho^{-1}(BU^-)$ under a quotient morphism $G\times G\to(G/B)\times(G/B^-)$. Since the big cell BU^- is open in G, and a quotient morphism is open, it follows that X is open dense in $\mathscr{B}\times\mathscr{B}^-$.

(b) Now to prove the theorem, we may assume that S is finite. In fact, we can choose a finite subset S_n of S such that both $\mathscr{B}^S=\mathscr{B}^{S_n}$ and $Z_G(S)=Z_G(S_n)$ holds. (This is possible since $Z_G(S)$ and $\mathscr{B}^S$ are noetherian.) If we put $S_n=\{s_1, \cdots\cdots, s_n\}$, then we have only to prove the theorem for S_n.

(c) We write $C(s_i)$ for the conjugacy class of s_i. They are irreducible closed subvarieties of G. Let $C(S_n)$ be a subset of $C(s_1)\times\cdots\cdots\times C(s_n)$ defined by: $C(S_n)=\{({}^gs_1, \cdots\cdots, {}^gs_n)|g\in G\}$. Through a morphism of G to $C(s_1)\times\cdots\cdots\times C(s_n)$, given by $g\mapsto({}^gs_1,\cdots\cdots, {}^gs_n)$, we have $\dim G=\dim Z_G(S_n)+\dim C(S_n)$. Put $s=(s_1, \cdots\cdots, s_n)$. For any subset A of G, we write ${}^gs\in A$ when each ${}^gs_i (i=1, \cdots\cdots, n)$ belongs to A. Define

$$U=\{({}^gB, {}^{g'}B^-, y)\in\mathscr{B}\times\mathscr{B}^-\times C(S_n)|{}^gB_\cap{}^{g'}B^-\ni y\}$$

and put $U_X=U_\cap(X\times C(S_n))$. Since ${}^gB_\cap{}^gB^-={}^gT$, U_X is equal to the set $\{({}^gB, {}^gB^-, y)|{}^gT\ni y\}$ and it is not empty by the semi-simplicity of s_i. Let p (resp. p_X) be a projection of U (resp. U_X) to $C(S_n)$. They are surjective. Furthermore since the fibres over all points of $C(S_n)$ are isomorphic to each other, we have

$$\begin{aligned}\dim U_X &= \dim C(S_n)+\dim p_X^{-1}(s)\\ &= \dim G-\dim Z_G(S_n)+\dim p_X^{-1}(s) \quad\cdots\cdots(1)\end{aligned}$$

On the other hand, there exists a surjective morphism $X\times C(S_n)\to\mathscr{C}\times C(S_n)$ with finite fibre. Let V_X be the image of U_X under this morphism. Then $\dim V_X=\dim U_X$. Projecting V_X onto the first factor, we have

$$\dim V_X=\dim\mathscr{C}+\dim(C(S_n)_\cap(\overbrace{T\times\cdots\cdots\times T}^{n\text{-times}}))$$

But $C(S_n)_\cap(T\times\cdots\cdots\times T)\subset(C(s_1)_\cap T)\times\cdots\cdots\times(C(s_n)_\cap T)$ and the dimension of the latter is 0, because in general the intersection of a torus with every conjugacy class is always finite ; it is easily verified by embedding G into $\boldsymbol{GL}_m$. Thus we have

$$\dim V_X = \dim G - \operatorname{rank} G. \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (2)$$

Hence by (1) and (2)

$$\dim Z_G(S_n) = \operatorname{rank} G + \dim p_X^{-1}(s) \quad \cdots\cdots\cdots\cdots\cdots\cdots (3)$$

(d) Let $\mathscr{B}^{S_n} = \bigcup_i F_i$ be a decomposition into irreducible components. Then $\mathscr{B}^{S_n} \times \mathscr{B}^{-S_n} = \bigcup_{ij} F_i \times F_j$. Identifying $p^{-1}(s)$ and $\mathscr{B}^{S_n} \times \mathscr{B}^{-S_n}$ by the isomorphism, we write $p_X^{-1}(s) = \bigcup_{ij} (F_i \times F_j) \cap X$. Hence there exist F_i and F_j such that $\dim p_X^{-1}(s) = \dim (F_i \times F_j) \cap X$. But since X is open, we have $\dim p_X^{-1}(s) = \dim F_i + \dim F_j$. Owing to (3), these F_i and F_j are the components which satisfy the requirement. q.e.d.

Combining the Theorem 4 and Theorem 7, we obtain the following results.

THEOREM 8. Let G and S be as in the Theorem 7. Then there exist irreducible components F_1 and F_2 of $\mathscr{B}^S$ such that

$$\dim F_1 + \dim F_2 = 2^{\#}\{\alpha \text{ a positive root} \mid \alpha(S) = 1\},$$

where the root system of the right-hand side is taken relative to any maximal torus of G containing S.

COROLLARY 9. Let G and S be as above. Then

$$\dim Z_G(S) \leqq \operatorname{rank} G + 2 \dim \mathscr{B}^S.$$

In particular, if the dimensions of all irreducible components of $\mathscr{B}^S$ are equal, then equality holds above. And

$$\dim \mathscr{B}(G)^S = \dim \mathscr{B}(Z_G(S)^0)$$

It is known that for any $x \in G$, all components of $\mathscr{B}^x$ have the same dimension ([6]).

Finally we remark that if G is not reductive, the conjecture stated in § 0 does not make sense. Indeed in case $G = B$, the left-hand side can be strictly larger than the right-hand side. On the other hand, the left-hand side can be strictly less than rank G as the following example shows; for any integer $n \geq 3$, define

$$U_n = \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \in \boldsymbol{GL}(n, K) \right\}$$

Let G be the direct product of a torus T and U_n, and take $x=(t, \begin{pmatrix} 1 & c & & 0 \\ & \ddots & \ddots & \\ & & \ddots & c \\ 0 & & & 1 \end{pmatrix})$ where $t \in T$ and c is a non zero element of K. Then rank $G = \dim G = \dim T + n(n-1)/2$, and $\dim Z_G(x) = \dim T + n - 1$.

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