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# On Idèle Class Groups of Imaginary Quadratic Fields

**Midori Onabe**

Department of Mathematics, Faculty of Science,
Ochanomizu University*

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## Intoroduction.

Let $\boldsymbol{Q}$ be the rational number field and $k$ be an algebraic number field of finite degree over $\boldsymbol{Q}$. We denote by $C_k$ the idèle class group of $k$. It is well-known that $C_k$ is a locally compact abelian group. In this paper, we shall consider the structure of $C_k$ as a topological group when $k$ is an imaginary quadratic field. Let $\boldsymbol{R}$ denote the additive group of the real numbers with usual topology, $\boldsymbol{T}$ the multiplicative group of all complex numbers of absolute value 1 with compact topology, and $Gal(A_k/k)$ the Galois grou of the maximal abelian extension $A_k$ over $k$ with the Krull topology. Then it will be shown that

$$C_k \cong \boldsymbol{T} \times \boldsymbol{R} \times Gal(A_k/k),$$

where $k$ is an imaginary quadratic field (Cor. 1 of Theorem 1).

Furthermore we shall see that even if the idèle class groups $C_k$ and $C_{k'}$ are isomorphic, the ideal class groups of $k$ and $k'$ are not necessarily isomorphic. In other words, the idèle class group of $k$ does not determine the structure of the ideal class group of $k$.

## § 1. Preliminaries.

Let $k$ be an algebraic number field which has $r_1$ real infinite primes and $r_2$ complex infinite primes. We shall denote by $I_k$ the idèle group of $k$, $k^\times$ the subgroup of principal idèles, and $C_k = I_k/k^\times$ the idèle class group of $k$. An idèle will be denoted by $a=(a_v)=(a_\mathfrak{p}, a_\lambda)$, where $v$ runs all primes of $k$, $\mathfrak{p}$ all finite primes and $\lambda$ all infinite primes of $k(\lambda=1, \cdots\cdots, r_1, r_1+1, \cdots\cdots, r_1+r_2)$.

As $C_k$ is a locally compact abelian group, its structure is determined by the character group by virtue of the duality theorem. We now consider the character group $C_k^*$ of $C_k$. If $\chi$ is a character of $C_k$, i.e. a continuous homomorphism of $C_k$ into $\boldsymbol{T}$, we can regard it as a character of $I_k$ such that $\chi(k^\times)=1$. Conversely a character $\chi$ of $I_k$ such that $\chi(k^\times)=1$ is regarded

* Present address, Tokyo Metropolitan University.

as a character of $C_k$. We consider the restriction of $\chi$ to the infinite part $\boldsymbol{R}^{\times r_1}\boldsymbol{C}^{\times r_2}$ of $I_k$. It is known that there exist $f_\lambda \in \boldsymbol{Z}$ and $\varphi_\lambda \in \boldsymbol{R}(\lambda=1, \cdots\cdots, r_1+r_2)$ such that

$$\chi((a_\lambda))=\prod_{\lambda=1}^{r_1+r_2}\left(\frac{a_\lambda}{|a_\lambda|}\right)^{f_\lambda}|a_\lambda|^{\sqrt{-1}\varphi_\lambda}, \qquad (a_\lambda)\in \boldsymbol{R}^{\times r_1}\boldsymbol{C}^{\times r_2},$$

and then we call $\chi$ a character of type $(f_\lambda, \varphi_\lambda)$.

Now we shall deal with Grössencharacters. For an integral idèal $\mathfrak{m}$ of $k$, we denote by $G(\mathfrak{m})$ the group of fractional ideals in $k$, prime to $\mathfrak{m}$, and $S(\mathfrak{m})$ the subgroup of $G(\mathfrak{m})$ which consists of principal ideals $(\xi)$ with $\xi\equiv 1 \pmod{\mathfrak{m}}$. A character $\psi$ of $G(\mathfrak{m})$ is called a Grössencharacter mod $\mathfrak{m}$ if there exist $f_\lambda \in \boldsymbol{Z}$ and $\varphi_\lambda \in \boldsymbol{R}$ $(\lambda=1, \cdots\cdots, r_1+r_2)$ such that for any $\xi \in k^\times$ with $\xi\equiv 1 \pmod{\mathfrak{m}}$,

$$\psi((\xi)) =\prod_{\lambda=1}^{r_1+r_2}\left(\frac{\xi^{\sigma_\lambda}}{|\xi^{\sigma_\lambda}|}\right)^{f_\lambda}|\xi^{\sigma_\lambda}|^{\sqrt{-1}\varphi_\lambda},$$

where $\sigma_\lambda$ are the embeddings of $k$ into $\boldsymbol{C}(\lambda=1, \cdots\cdots, r_1+r_2)$. $\psi$ is then called of type $(f_\lambda, \varphi_\lambda)$. We shall agree that two Gössencharakters are equivalent when they are identical on the common domain, i.e. $\psi_1\sim\psi_2$ if and only if $\psi_1=\psi_2$ on $G(\mathfrak{m}_1\mathfrak{m}_2)$, where $\psi_i$ is a Grössencharakter mod $\mathfrak{m}_i$ $(i=1, 2)$.

Here we recall the one-to-one correspondence between characters of $C_k$ and Grössencharakters of $k$. A character of $C_k$ of type $(f_\lambda, \varphi_\lambda)$ corresponds to a Grössencharakter of type $(-f_\lambda, -\varphi_\lambda)$. For an integral ideal $\mathfrak{m}=\prod \mathfrak{p}^{e_\mathfrak{p}}$ of $k(e_\mathfrak{p}\in \boldsymbol{Z},>0)$, we put

$I(\mathfrak{m})=\{a=(a_v)\in I_k:\ a_\mathfrak{p}=1 \text{ for } \mathfrak{p}|\mathfrak{m},\ a_\lambda=1\ (\lambda=1, \cdots\cdots, r_1+r_2)\}$
$I(\mathfrak{m})'=\{a=(a_v)\in I_k:\ a_\mathfrak{p}\equiv 1 \pmod{\mathfrak{p}^{e_\mathfrak{p}}} \text{ for } \mathfrak{p}|\mathfrak{m}\}$
$U=\{a=(a_v)\in I_k:\ a_\mathfrak{p} \text{ is an unit in } k_\mathfrak{p}\}$

Let $\chi$ be a character of $C_k$ with conductor $\mathfrak{m}$. Each idèle $a=(a_v)$ determines in an obvious manner an ideal of $k$; denote it by id $(a)$. Therefore we have the isomorphism $G(\mathfrak{m}) \cong I(\mathfrak{m})/(\mathfrak{m})\cap U$. Then we can define the Grössencharakter corresponding to $\chi$ through this isomorphism. Conversely assume that we have a Grössencharakter $\psi$ mod $\mathfrak{m}$ of type $(f_\lambda, \varphi_\lambda)$. We can define a character $\chi$ of $C_k$ as follows:

$$\chi(a)=\psi(\mathrm{id}\,(a))\prod_{\lambda=1}^{r_1+r_2}\left(\frac{a_\lambda}{|a_\lambda|}\right)^{-f_\lambda}|a_\lambda|^{-\sqrt{-1}\varphi_\lambda}, \text{ for any } a=(a_v)\in I(\mathfrak{m})'$$
$$\chi(a)=1, \qquad \text{for any } a\in k^\times.$$

Since $I(\mathfrak{m})'k^\times=I_k$, we obtain a character $\chi$ of $C_k$.

The following lemma is well-known ([2] p. 184).

LEMMA. Let $\chi$ be a character of $C_k$ of type $(f_\lambda, \varphi_\lambda)$.

The following three assertions are equivalent:

(i) $\chi$ is of finite order,
(ii) $\chi(D_k)=1$, where $D_k$ is the connected component of identity of $C_k$,
(iii) $\begin{cases} f_\lambda=0 & (\lambda: \text{ all complex primes}) \\ \varphi_\lambda=0 & (\lambda: \text{ all real and complex primes}). \end{cases}$

## § 2. The structure of $C_k$.

THEOREM 1. Let $k$ be an imaginary quadratic field and $C_k^*$ the character group of the idele class group $C_k$. Then

$$C_k^* \cong \boldsymbol{Z}\times\boldsymbol{R}\times T_k,$$

where $\boldsymbol{Z}$ is the additive group of rational integers with discrete topology, $\boldsymbol{R}$ the additive group of real numbers with usual topology, and $T_k$ the torsion subgroup of $C_k^*$ with discrete topology.

PROOF. Let $\mathfrak{p}$ be a prime ideal of $k$ such that $\varepsilon \not\equiv 1 \pmod{\mathfrak{p}}$ for any unit $\varepsilon$ $(\neq)$ of $k$. We obtain a character $\psi$ of $S(\mathfrak{p})$ by defining

$$\psi((\xi)) = \left(\frac{\xi}{|\xi|}\right)^{-1}$$

for any $\xi\equiv 1 \pmod{\mathfrak{p}}$. It is well-defined by our assumption on $\mathfrak{p}$. As the index $(G(\mathfrak{p}) : S(\mathfrak{p}))$ is finite, $\psi$ can be extended to a character of $G(\mathfrak{p})$. Take one of them and again denote it by $\psi$. This is a Grössencharacter mod $\mathfrak{p}$ of type $(-1, 0)$, then we define $\chi_{(1,0)}$ by the character of $C_k$ which corresponds to $\psi$; $\chi_{(1,0)}$ is of type $(1,0)$.

Next we will define

$$\psi(\mathfrak{a}) = \mathrm{N}\,\mathfrak{a}^{-\sqrt{-1}\frac{\varphi}{2}}$$

for any ideal $\mathfrak{a}$, where N is the norm in $k$ over $\boldsymbol{Q}$. Then $\psi$ is a Grössencharacter mod 1 of type $(0, -\varphi)$ and we put $\chi_{(0,\varphi)}$ the character of $C_k$ which corresponds to $\psi$. This is of type $(0, \varphi)$ and we have

$$\chi_{(0,\varphi)}(a)=\psi(\mathrm{id}(a))|a_\lambda|^{\sqrt{-1}\varphi},$$

for any idèle $a=(a_v)$.

Now for any $f\in\boldsymbol{Z}$ and $\varphi\in\boldsymbol{R}$, we define $\chi_{(f,\varphi)}$ as follows:

$$\chi_{(f,\varphi)}=\chi_{(1,0)}^{f}\chi_{(0,\varphi)}.$$

$\chi_{(f,\varphi)}$ is of type $(f, \varphi)$. Since

$$\chi_{(f_1+f_2,\varphi_1+\varphi_2)}=\chi_{(f_1,\varphi_1)}\cdot\chi_{(f_2,\varphi_2)}$$

for any $f_i\in\boldsymbol{Z}$ and $\varphi_i\in\boldsymbol{R}$ $(i=1, 2)$, the subgroup

$$\{\chi_{(f,\varphi)} \in C_k^* : f \in \boldsymbol{Z},\ \varphi \in \boldsymbol{R}\}$$

is isomorphic to the additive group $\boldsymbol{Z}\times\boldsymbol{R}$. Thus we can treat $\boldsymbol{Z}\times\boldsymbol{R}$ as the subgroup of $C_k^*$. Let $\chi$ be a character of $C_k$ of type $(f, \varphi)$. As $\chi\cdot\chi_{(f,\varphi)}^{-1}$ is of type (0, 0), it belongs to $T_k$ by lemma, that is $C_k=(\boldsymbol{Z}\times\boldsymbol{R})\cdot T_k$. By the same lemma, $\boldsymbol{Z}\times\boldsymbol{R}\cap T_k=1$. Therefore we have

$$C_k^*=\boldsymbol{Z}\times\boldsymbol{R}\times T_k \qquad \text{(as groups).}$$

Now it is well-known that $C_k \cong \boldsymbol{R}\times C_k^0$, where $C_k^0=I_k^0/k^\times$ and $I_k^0$ is the subgroup of all idèles of volume 1. Let $H$ be the annihilator of $C_k^0$, i.e.

$$H=\{\chi \in C_k^* \ :\ \chi(C_k^0)=1\}.$$

We will prove that $H=0\times\boldsymbol{R}\times 1$. In order to show it, it is sufficient to prove the following three statements :

(1) $H\cap\boldsymbol{Z}\times 0\times 1=1$,

(2) $H\supset 0\times\boldsymbol{R}\times 1$,

(3) $H\cap 0\times 0\times T_k=1$.

Indeed, let $(a, b, c)$ be any element of $H$. There exists a natural number $n$ such that $c^n=1$, then $(na, nb, 1)\in H$. By (2), $(0, -nb, 1)\in H$, and so $(na, 0, 1)\in H$; by (1), we have $na=0$, i.e. $a=0$. It is shown $c=1$ in the same way. Thus $H\subset 0\times\boldsymbol{R}\times 1$. Hence we have $H=0\times\boldsymbol{R}\times 1$.

PROOF of (1). We will show $\chi_{(f,0)}(C_k^0)\neq 1$ for any $f\in\boldsymbol{Z}(\neq 0)$. Let $a=(a_v)$ $(\neq 1)$ be an element of $I(\mathfrak{p})'\cap I_k^0$. If $\chi_{(f,0)}(a)\neq 1$, it is obvious that $\chi_{(f,0)}(C_k^0)\neq 1$. Then assume that $\chi_{(f,0)}(a)=1$. We put $a'=(a_\mathfrak{p}, za_\lambda)\in I(\mathfrak{p})'\cap I_k^0$, where $z\in\boldsymbol{T}$ such that $z^f\neq 1$. Then

$$\chi_{(f,0)}(a')=\psi(\mathrm{id}(a'))^f\left(\frac{za_\lambda}{|za_\lambda|}\right)^f=\psi\ (\mathrm{id}\ (a))^f\left(\frac{a_\lambda}{|a_\lambda|}\right)^f z^f=z^f\neq 1.$$

This shows that $\chi_{(f,0)}(C_k^0)\neq 1$.

PROOF of (2). We will show that $\chi_{(0,\varphi)}(C_k^0)=1$ for any $\varphi\in\boldsymbol{R}$. If $a=(a_v)\in I_k^0$, then $\mathrm{N}(\mathrm{id}\ (a))=|a_\lambda|^2$. Therefore

$$\chi_{(0,\varphi)}(a)=\mathrm{N}\ (\mathrm{id}\ (a))^{-\sqrt{-1}\frac{\varphi}{2}}|a_\lambda|^{\sqrt{-1}\varphi}=1.$$

PROOF of (3). We show that $\chi(C_k^0)\neq 1$ for any $\chi\in T_k$, $\neq 1$. As $\chi\neq 1$, there exists $a=(a_v)=(a_\mathfrak{p}, a_\lambda)\in I_k$ such that $\chi(a)\neq 1$. We can choose $a'_\lambda\in\boldsymbol{C}^\times$ such that $a'=(a_\mathfrak{p}, a'_\lambda)\in I_k^0$. Then we have $\chi(a')=\chi(a)\neq 1$ because $\chi(a_\lambda)=\chi(a'_\lambda)=1$.

Since $C_k\cong\boldsymbol{R}\times C_k^0$, then $C_k^*\cong\boldsymbol{R}\times(C_k^0)^*$. As we have seen,

$$C_k^*=\boldsymbol{Z}\times\boldsymbol{R}\times T_k$$

and

$$H=0\times\boldsymbol{R}\times 1.$$

Therefore

$$(C_k^0)^* \cong C_k^*/H \cong \boldsymbol{Z} \times \boldsymbol{R} \times T_k/0 \times \boldsymbol{R} \times 1 \cong \boldsymbol{Z} \times T_k.$$

Since $(C_k^0)^*$ is a discrete group, we have

$$C_k^* \cong \boldsymbol{R} \times \boldsymbol{Z} \times T_k \qquad \text{(as topological groups).}$$

This completes our proof.

Corollary 1. Let $k$ be an imaginary quadratic field and $C_k$ the idèle class group of $k$. Then

$$C_k \cong \boldsymbol{T} \times \boldsymbol{R} \times Gal\,(A_k/k).$$

Proof. We have $T_k \cong (C_k/D_k)^*$ by Lemma. From Theorem 1, duality theorem, and class field theory, we have

$$\begin{aligned} C_k &\cong \boldsymbol{Z}^* \times \boldsymbol{R}^* \times T_k^* = \boldsymbol{T} \times \boldsymbol{R} \times C_k/D_k \\ &= \boldsymbol{T} \times \boldsymbol{R} \times Gal(A_k/k). \end{aligned}$$

Corollary 2. For any imaginary quadratic fields $k$ and $k'$, the following two assertions are equivvlent :

(i) $C_k \cong C_{k'}$,
(ii) $Gal\,(A_k/k) \cong Gal\,(A_{k'}/k')$.

Consequently, for all imaginary quadratic fields $k$, except $\boldsymbol{Q}(\sqrt{-1})$, with class number 1, the idèle class groups $C_k$ are isomorphic to each other (cf. [1]).

Corollary 3. Let $\mathscr{C}_k$ be the idèal class group of $k$. Then there exist $k$ and $k'$ satisfying the following conditions :

(i) $C_k \cong C_{k'}$
(ii) $\mathscr{C}_k \ncong \mathscr{C}_{k'}$.

For example the idèle class groups $C_k$ are isomorphic to each other for $k = \boldsymbol{Q}(\sqrt{-2})$, $\boldsymbol{Q}(\sqrt{-5})$, $\boldsymbol{Q}(\sqrt{-23})$, $\boldsymbol{Q}(\sqrt{-47})$, and $\boldsymbol{Q}(\sqrt{-71})$, but their ideal class numbers are 1, 2, 3, 5 and 7, respectively (cf. [1]).

Theorem 2. Let $\boldsymbol{Q}$ be the rational number field, $C_{\boldsymbol{Q}}$ the idèle class group of $\boldsymbol{Q}$. Then

$$C_{\boldsymbol{Q}} \cong \boldsymbol{R} \times \mathrm{Gal}(A_{\boldsymbol{Q}}/\boldsymbol{Q}).$$

Proof. The proof is similar to Theorem 1.

Added in proof; Prof. Iwasawa has kindly written to me and remarked that theorem 1 can also be shown by considering the structure of $D_k$.

## References

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