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A potential operator with respect to a resolvent and ergodic properties

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§ 1. Introduction.

Let $(X, \mathfrak{A}, m)$ be a σ-finite measure space and F be a convex cone of all non-negative extended real-valued measurable functions. K. A. Astbury has defined in [2] a potential operator on F with respect to a positive linear operator on F which is not necessarily monotonically continuous and proved that the potential operator has some of the familiar properties enjoyed by potential operators obtained from kernels (e.g. Domination Principle, Riesz Decomposition, and Balayage). Further he has shown that a positive linear contraction of $L_\infty(X, \mathfrak{A}, m)$ determines a Hopf decomposition of the space X into the conservative and the dissipative regions.

In this paper we shall define a potential operator on F with respect to a resolvent of positive linear operators on F which are not necessarily monotonically continuous and show that the potential operator satisfies the domination principle. Further we shall give an example which shows that the potential operator fails to satisfy the balayage principle and the Riesz decomposition theorem fails to hold. Moreover we shall apply them to a decomposition of X into the conservative and the dissipative sets and show that, for a resolvent $(V_p)_{p>0}$ generated as usual by a strongly measurable semigroup of positive linear contractions on $L_1(X, \mathfrak{A}, m)$, our decomposition with respect to $(V_p)_{p>0}$ is identical with the well-known ergodic decomposition (cf. [4]).

Throughout this paper we shall consider that sets or functions are equal if they are equal almost everywhere, and consider that equalities or inequalities hold if they hold almost everywhere. The following lemma is fundamental.

LEMMA 1. 1. (cf. [5, Proposition II-4-1]) (I) *Let* $(f_\alpha)_{\alpha\in I}$ *be a subfamily of* F. *Then there exist two unique elements of* F *(up to equivalence class), denoted by* $\operatorname{ess\,sup}_{\alpha\in I} f_\alpha$ *and* $\operatorname{ess\,inf}_{\alpha\in I} f_\alpha$, *such that for* $f, g\in F$

(a) $f_\alpha \leqq f$ *for all* $\alpha\in I \iff \operatorname{ess\,sup}_{\alpha\in I} f_\alpha \leqq f$

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(b) $f_\alpha \geqq g$ *for all* $\alpha \in I \iff \operatorname*{ess\,inf}_{\alpha \in I} f_\alpha \geqq g$.

(II) *Let* $(A_\alpha)_{\alpha \in I}$ *be a subfamily of* $\mathfrak{A}$. *Then there exist two unique elements of* $\mathfrak{A}$ *(up to m-measure zero), denoted by* $\operatorname*{ess\,sup}_{\alpha \in I} A_\alpha$ *and* $\operatorname*{ess\,inf}_{\alpha \in I} A_\alpha$. *such that for* A, $B \in \mathfrak{A}$

(c) $A_\alpha \subset A$ *for all* $\alpha \in I \iff \operatorname*{ess\,sup}_{\alpha \in I} A_\alpha \subset A$,

(d) $A_\alpha \supset B$ *for all* $\alpha \in I \iff \operatorname*{ess\,inf}_{\alpha \in I} A_\alpha \supset B$.

§2. Potentials.

Let $(X, \mathfrak{A}, m)$ be a σ-finite measure and F the convex cone of all non-negative extended real-valued measurable functions. A monotone map T from F to F is called a positive linear operator on F if it satisfies

$$T(f+g) = Tf + Tg \text{ for every } f,\ g \in F,$$
$$T(af) = aTf \text{ for every } a \in \boldsymbol{R}^+ \text{ and every } f \in F.$$

Remark that $a \cdot \infty = \infty \cdot a = \infty (a > 0)$ and $0 \cdot \infty = \infty \cdot 0 = 0$.

A resolvent on F is a family $(V_p)_{p>0}$ of positive linear operators on F such that

$$(2.\ 1) \qquad V_p V_q = V_q V_p, \quad V_p = V_q + (q-p) V_q V_p$$

for every pair of real numbers p, q satisfying $q > p > 0$ and $(V_p f)(x) = \infty$ implies $(V_q f)(x) = \infty$ for every $q > 0$. By the definition we have immediatey the following proposition.

PROPOSITION 2. 1. *Let* f, $g \in F$ *and* p, q *be positive real numbers. Then* $pV_pV_qf + g \leqq V_qf$ *implies* $qV_qV_pf + g \leqq V_pf$.

For $f \in F$, put

$$\varphi_f = \{g \in F: \quad V_pf + pV_pg \leqq g \text{ for every } p > 0\}.$$

Since $\varphi_f \ni \infty$, φ_f is not empty. We define the potential of f by

$$Vf = \text{ess inf } \{g: \quad g \in \varphi_f\}.$$

We also write $V(A)$ for $V(1_A)$[1]. The potential operator $V: f \mapsto Vf$ has the following properties.

PROPOSITION 2. 2. (1) $Vf = V_p + pV_pVf$ $(p > 0)$,

(2) $f \leqq g$ *implies* $Vf \leqq Vg$,

(3) $V(f+g) = Vf + Vg$, $aVf = V(af)$ $(a \in \boldsymbol{R}^+)$,

(4) $g \geqq pV_pg + f$ *implies* $g \geqq pVf + f$.

PROOF. (1) Let $g \in \varphi_f$ and p be a positive real number. Then $V_pf + pV_pVf$

1) We denote by 1_A the characteristic function of a subset A of X.

$\leqq V_p f + pV_p g \leqq g$ implies $V_p f + pV_p Vf \leqq Vf$.
By (2. 1) we have

$$V_q f + qV_q(V_p f + pV_p Vf) = V_p f + pV_p V_q f + pqV_p V_q Vf$$
$$\leqq V_p f + pV_p Vf$$

for every $q>0$ and hence $Vf \leqq V_p f + pV_p Vf$. Thus $Vf = V_p f + pV_p Vf$.

(2) Let $f \leqq g$ and $h \in \varphi_g$. $V_p f + pV_p h \leqq V_p g + pV_p h \leqq h$ implies $Vf \leqq h$ and hence $Vf \leqq Vg$.

(3) From (1) it follows that $Vf + Vg = V_p(f+g) + pV_p(Vf + Vg)$ for every $p>0$ and hence $Vf + Vg = V_p(f+g) + pV_p(Vf + Vg)$ for every $p>0$ and hence $V(f+g) \leqq Vf + Vg$. On the other hand, since $Vg \leqq V(f+g)$, we define

$$h = \begin{cases} \infty & \text{where } V(f+g) = \infty \\ V(f+g) - Vg & \text{where } V(f+g) < \infty. \end{cases}$$

Then $h \in F$ is the largest solution of $h + Vg = V(f+g)$. The function $V_p f + pV_p h$, for each $p>0$, is also an solution of the equation. Consequently $V_p f + pV_p h \leqq h$ for every $p>0$ and $Vf \leqq h$. Thus we have $Vf + Vg \leqq V(f+g)$. Further, from (1) it follows that

$$V_p(af) + pV_p(aVf) = a(V_p f + pV_p Vf) = aVf$$

for every $a>0$ and $p>0$, whence $V(af) \leqq aVf$. We have also $V\left(\frac{1}{a} f\right) \leqq \frac{1}{a} Vf$. Consequently $V(af) = aVf$. It is trivial that the equality holds for $a=0$.

(4) Let $g \geqq pV_p g + f$. Since $V_q g \geqq pV_p V_q g + V_q f$ for every $q>0$, it follows from Proposition 2. 1 that $V_p g \geqq qV_q(V_p g) + V_q f$ for any $q>0$ and hence $V_p g \geqq Vf$. Consequently $g \geqq pV_p g + f \geqq pVf + f$.

A subset φ of F is a called a V-class for $f \in F$ if

(i) $V_p f \in \varphi$ for every $p>0$,

(ii) $g \in \varphi$ implies $pV_p g + V_p f \in \varphi$ for every $p>0$,

(iii) $(g_i)_{i \in I} \subset \varphi$ implies $\operatorname{ess\,sup}_{i \in I} g_i \in \varphi$.

Since the intersection of V-classes for f is again a V-class for f, there exists the smallest V-class ε_f for f. The following characterization of potential is useful.

Lemma 2. 1. $Vf = \operatorname{ess\,sup}\{g : g \in \varepsilon_f\}$.

Proof. Since ε_f is a V-class for f, we have $pV_p(\operatorname{ess\,sup}_{g \in \varepsilon_f} g) + V_p f \leqq \operatorname{ess\,sup}_{g \in \varepsilon_f} g$ for every $p>0$. Consequently $Vf \leqq \operatorname{ess\,sup}_{g \in \varepsilon_f} g$. On the other hand, put $\varphi = \{g \in \varepsilon_f : g \leqq Vf\}$. Then φ is a V-class for f contained in ε_f. Since ε_f is the smallest V-class for f, it follows that $\varepsilon_f = \varphi$, and hence $\operatorname{ess\,sup}_{g \in \varepsilon_f} g \leqq Vf$. Thus we have the conclusion.

A positive linear operator T on F is called monotonically continuous if it has the following property.

Let (f_n) be an increasing sequence of elements of F. Then $Tf_n \uparrow Tf$.

PROPOSITION 2. 3. *Let $(V_p)_{p>0}$ be a resolvent of monotonically continuous positive linear operators on F. Then $Vf=\lim_{p\to 0} V_p f$.*

PROOF. Let f be an element of F. According to (2. 1) the function $p \mapsto V_p f$ is decreasing. Put

$$V_0 f = \operatorname*{ess\,sup}_{p>0} V_p f = \lim_{p\to 0} V_p f.$$

Since V_p is monotonically continuous, it is easily verified that

$$V_0 f = V_p f + pV_p V_0 f \quad (p>0).$$

Hence $Vf \leqq V_0 f$. Further, $V_p f \in \varepsilon_f$ for every $p>0$ implies $V_0 f = \operatorname*{ess\,sup}_{p>0} V_p f \in \varepsilon_f$. From Lemma 2. 1 it follows that $V_0 f \leqq Vf$, and hence $V_0 f = Vf$.

PROPOSITION 2. 4. $\sum_{n=0}^{\infty}(pV_p)^n f \leqq pVf+f$. *The equality holds if V_p is monotonically continuous for every $p>0$.*

PROOF. By the definition of ε_f, we have $\sum_{n=0}^{\infty}(pV_p)^n V_p f \in \varepsilon_f$ and consequently $\sum_{n=0}^{\infty}(pV_p)^n V_p f \leqq Vf$. Hence $\sum_{n=0}^{\infty}(pV_p)^n f \leqq pVf+f$. If V_p is monotonically continuous, we have

$$pV_p\Big(\sum_{n=0}^{\infty}(pV_p)^n f\Big)+f = \sum_{n=0}^{\infty}(pV_p)^n f.$$

From Proposition 2. 2 it follows that $\sum_{n=0}^{\infty}(pV_p)^n f \geqq pVf+f$.

THEOREM 2. 1. *$VV_p f = V_p Vf$ for every $p>0$.*

PROOF. Let p be an arbitrary positive integer. From Proposition 2.2 it follows that $V_p f + pV_p Vf = Vf$ and hence $pV_p V_q Vf + V_p V_q f = V_q Vf$ for every $q>0$. According to Proposition 2. 1 the inequality $qV_q V_p Vf + V_q(V_p f) \leqq V_p Vf$ holds and therefore $VV_p f \leqq V_p Vf$. On the other hand, set

$$\varphi = \{g \in \varepsilon_f : \ pV_p g + V_p f \geqq g, \ V_p g \leqq VV_p f\}.$$

Then we shall show that φ is a V-class. (i) Using (2. 1) we have

$$V_p V_q f = V_q V_p f \leqq VV_p f \text{ and } pV_p V_q f + V_p f = V_q f + qV_p V_q f \leqq V_q f \ (q>0).$$

Hence $V_q f \in \varphi$. (ii) Let g be an element of φ. It follows that

$$V_p(qV_q g + V_q f) \leqq qV_q VV_p f + V_p V_q f = VV_p f$$

and

$$\begin{aligned} pV_p(qV_qg+V_qf)+V_pf &= pV_pqV_q\,g+(pV_pV_qf+V_pf) \\ &= pV_pqV_qg+(qV_qV_pf+V_qf) \\ &= qV_q(pV_pg+V_pf)+V_qf \\ &\geqq qV_qg+V_qf, \end{aligned}$$

whence $qV_qg+V_qf \in \varphi$. (iii) Let $(g_\alpha)\supset\varphi$. Then

$$\begin{aligned} pV_p(\operatorname{ess}\sup_\alpha g_\alpha)+V_pf &\geqq \operatorname{ess}\sup_\alpha\ (pV_pg_\alpha)+V_pf \\ &= \operatorname{ess}\sup_\alpha\ (pV_pg_\alpha+V_pf) \\ &\geqq \operatorname{ess}\sup_\alpha\ g_\alpha, \end{aligned}$$

and

$$\begin{aligned} V_p(\operatorname{ess}\sup\ g_\alpha) &\leqq V_p(\operatorname{ess}\sup_\alpha\ (pV_pg_\alpha+V_pf)) \\ &\leqq V_p(pVV_pf+V_pf) \\ &= (pV_pV+V_p)V_pf=VV_pf. \end{aligned}$$

Thus (i), (ii) and (iii) of the definition of a V-clasi hold.
By the minimality of ε_f, we obtain $\varphi=\varepsilon_f$. Since $Vf\in\varphi$, we have $V_pVf\leqq VV_pf$.

A function f in F is called supermedian (resp. invariant) if for every $p>0$ $pV_p\leqq f$ (resp. $pV_pf=f$).

THEOREM 2. 2. *Let g be an invariant function satisfying $g\leqq Vf$. Then $g+Vf=Vf$.*

PROOF. Put

$$h=\begin{cases}\infty \text{ where } Vf=\infty \\ Vf-g \text{ where } Vf<\infty.\end{cases}$$

Then $h\in F$ is the largest solution of $h+g=Vf$. Since pV_ph+V_pf is another solution for every $p>0$, we have $pV_ph+V_pf\leqq h$ and hence $Vf\leqq h$. From the definition of h it follows that $Vf+g\leqq Vf$ and consequently $Vf+g=Vf$.

The support of $f\in F$, by definition, is the set $\{f>0\}$ and denoted by supp f.

THEOREM 2. 3 (Domination principle) *Assume thvt $pV_p\,g\leqq g$ and $Vf+\frac{1}{p}f\leqq g$ on* supp f. *Then $Vf+\frac{1}{p}f\leqq g$ everywhere.*

PROOF. Put

$$h=\min\left(Vf+\frac{1}{p}f,\ g\right)=\min\left(Vf+\frac{1}{p}f,\ g+\frac{1}{p}f\right).$$

Then we have

$$pV_ph+\frac{1}{p}f\leqq pV_p\left(Vf+\frac{1}{p}f\right)+\frac{1}{p}f=Vf+\frac{1}{p}f$$

and also

$$pV_ph+\frac{1}{p}f \leqq pV_pg+\frac{1}{p}f \leqq g + \frac{1}{p}f.$$

Hence $pV_ph+\frac{1}{p}f \leqq \min\left(Vf+\frac{1}{p}f,\ g+\frac{1}{p}f\right)=h.$

From Proposition 2. 1 if follows that $h \geqq pV\left(\frac{1}{p}f\right)+\frac{1}{p}f= Vf+\frac{1}{p}f$ and hence $g \geqq Vf+\frac{1}{p}f.$

COROLLARY 2. 1. *Assume that f is dominated by a supermedian function h with $h<\infty$. Let g be a supermedian function satisfying*

$$Vf \leqq g \text{ on } \operatorname{supp} f.$$

Then $Vf \leqq g$ everywhere.

PROOF. Since $Vf+\frac{1}{p}f \leqq g+\frac{1}{p}h$ on supp f for every $p>0$ and $g+\frac{1}{p}h$ is supermedian, it follows from Theorem 2. 3 that

$$Vf+\frac{1}{p}f \leqq g+\frac{1}{h}h \text{ for every } p>0.$$

Converging p to ∞, we have $Vf \leqq g$.

COROLLARY 2. 2. *Assume that f is bounded and that 1 is supermedian. Then, for $a \in \boldsymbol{R}^+$,*

(i) *$Vf \leqq Vg+a$ on* supp *f implies $Vf \leqq Vg+a$ everywhere,*

(ii) *$V(A) \leqq a$ on A implies $V(A) \leqq a$ everywhere.*

PROOF. (i) Since Vg and a are supermedian, the conclusion follows immediately from Corollary 2. 1. (ii) is trivial.

COROLLARY 2. 3. *Suppose that there exists $v \in F$ satisfing $0<v<\infty$ and $pV_pv<v$ for a real number $p>0$. Let A be an element of $\mathfrak{A}$ with $V(A)<\infty$. Then there exists an increasing sequence (A_n) oj $\mathfrak{A}$ satisfying $\bigcap_{n=1}^{\infty} A_n=A$ and $V(A_n) \leqq \frac{nv}{p}$.*

PROOF. Put $A_n=A\cap\{pV(A)+1_A \leqq nv\}$.
Then

$$pV(A_n)+1_A \leqq pV(A)+1_A \leqq nv \text{ on } A_n.$$

From Theorem 2. 3 it follows that

$$pV(\mathrm{A}_n)+1_{A_n} \leqq nv \text{ everywhere}$$

and hence $V(A_n) \leqq \frac{nv}{p}$. Since $V(A)<\infty$, we have $\bigcup_{u=1}^{\infty} A_n=A$.

EXAMPLE. (cf. [2, Example 3. 4]) Let $X=\{0, 1, 2, \cdots\}$. Let $\mathfrak{A}$ be the family of all subsets of X and let m be a probability measure having positive mass at each point of X. Let μ be a positive linear functional on $L_\infty(X, \mathfrak{A}, m)$ satisfying $\mu(1)=1$ and $\mu(1_A)=0$ where A is a finite set.

For each $f\in L_\infty^+$, set

$$Tf(x) = \begin{cases} f(x) & \text{where } x\geqq 1 \\ \mu(f) & \text{where } x=0. \end{cases}$$

Further, define

$$V_p f = \begin{cases} \dfrac{1}{p+1} Tf & \text{where } f\in L_\infty^+ \\ \infty & \text{where } f\in F\backslash L_\infty^+. \end{cases}$$

Then $(V_p)_{p>0}$ is a resolvent of positive linear operators on F and has the following properties.

(1) V_p is not monotonically continuous. In fact, put $A_n=\{1, 2, \cdots, n\}$ and $A=\{1, 2, \cdots\}$. Then $1_{A_n}\uparrow 1_A$. But $V_p(1_{A_n})(0)=\mu(1_{A_n})=0$ and $V_p(1_A)(0)$ $=\dfrac{1}{p+1}\mu(1_A)=\dfrac{1}{p+1}>0$. Therefore $\lim\limits_{n\to 0} V_p(1_{A_n})\neq V_p(1_A)$.

(2) $Vf=Tf$ for all $f\in L_\infty^+$. In fact, since $pV_pTf+V_pf=Tf$ for each $p>0$, $Vf\leqq Tf$ holds. $\dfrac{1}{p+1}Tf=V_pf\in\varepsilon_f$ implies $Tf=\text{ess sup}\left\{\dfrac{1}{p+1}Tf\text{: } p>0\right\}$ $\in\varepsilon_f$. Hence $Tf\leqq Vf$.

(3) Let $f\in L_\infty^+$ be invariant. Then $f\equiv 0$. This is trivial.

(4) The Riesz decomposition theorem fails to hold. In fact, the supermedian function f defined

$$f(x) = \begin{cases} 0 & \text{where } x\geqq 1 \\ 1 & \text{where } x=0 \end{cases}$$

is not decomposed into the potential part and the invariant part.

(5) The potential operator V fails to satisfy the balayage principle. In fact, let $f\equiv 1$ and $A=\{0\}$. Then $Vf(0)=1$. But $Vh(0)=0$ for any $h\in L_\infty^+$ satisfving supp $h=\{0\}$. Therefore there is not $h\in F$ such that $Vh\leqq Vf$, Vh $=Vf$ on A and supp $h\supset A$.

§ 3. The ergodicde composition.

Let $(X, \mathfrak{A}, m)$ be a σ-finite measure space and $(V_p)_{p>0}$ be a resolvent of positive linear operators on F. Throughout this section we assume that there exists a supermedian function v with $0<v<\infty$. The set defined

$$D=\text{ess sup } \{A\in\mathfrak{A}\text{: } V(A)<\infty\}$$

is called the dissipative part of X and $C=X\backslash D$ is called the conservative part of X.

THEOREM 3. 1. *Suppose that* supp $f\subset C$. *Then for each* $p>0$ $pVf+f$ *takes only the value* 0 *or* ∞.

PROOF. Let $p>0$ and $m\in N$. By the assumption there exists a supermedian function v with $0<v<\infty$. Put, for each $n\in N$,

$$A_n=\{pVf+f\leqq mv\}\cap\left\{f>\frac{1}{n}\right\}.$$

Then $1_{A_n}\leqq nf$ and $V(A_n)\leqq nVf$. Hence $pV(A_n)+1_{A_n}\leqq pnVf+nf\leqq n(pVf+f)$ $\leqq mnv$ on A_n. From Theorem 2.3 it follows that $pV(A_n)+1_{A_n}\leqq mnv$ everywhere. Therefore $V(A_n)<\infty$. Since $A_n\subset C$, we obtain $A_n=\phi$ and also

$$\{pVf+f\leqq mv\}\cap\{f>0\}=\bigcup_{n\in N}A_n=\phi.$$

Since $0<v<\infty$, we have

$$\{pVf+f\}\cap\{f>0\}=\phi$$

and consequently $pVf+f=\infty$ on supp f. Further, define

$$g=\begin{cases}\infty & \text{where } pVf+f=\infty\\ 0 & \text{eleswhere.}\end{cases}$$

The inequality $g\leqq pVf+f$ implies $pV_pg\leqq p^2V_pVf+pV_pf=pVf$. Since g takes only the value 0 or ∞, so does pV_pg, Therefore we have $pV_pg\leqq g$. Since $pVf+f=\infty$ on supp f, it follows that $pV_qg+f\leqq g$. From Proposition 2.2 $g\geqq pVf+f$ holds. Thus we have $g=pVf+f$. Consequently the function $pVf+f$ takes only the value 0 or ∞.

COROLLARY 3.1. *Suppose that* supp $f\subset C$ and $f<\infty$. *Then* Vf *takes only the value* 0 or ∞.

This corollary is an immediate consequence of Theorem 3.1.

THEOREM 3.2. *Let g be a supermedian. Then $pV_pg=g$ on C for each $p>0$.*

PROOF. Let $p>0$. Set $A_n=C\cap\left\{pV_pg+\frac{1}{n}<g\right\}$ for each positive integer n. Then $pV_p(ng)+1_{A_n}\leqq ng$ and hence

$$pV_pV_q(ng)+V_q(1_{A_n})\leqq V_q(ng) \text{ for each } q>0.$$

From Proposition 2.1 it follows that

$$qV_qV_p(ng)+V_q(1_{A_n})\leqq V_p(ng) \text{ for each } q>0,$$

whence $V(A_n)\leqq V_p(ng)=nV_pg$. Since $A_n\subset C$, we have, by Theorem 3.1, $pV(A_n)+1_{A_n}=0$ or ∞. Since $V(A_n)\leqq nV_pg<\infty$ on A_n, we have $pV(A_n)+1_{A_n}=0$ on A_n and hence $A_n=\phi$. $C\cap\{pV_pg<g\}=\bigcup A_n=\phi$ implies $pV_pg=g$ on C.

THEOREM 3.3. *Let f be an element of F. Then Vf takes only the value 0 or ∞ on C.*

PROOF. Let $f\in F$. Set

$$\varphi=\{g\in F: g=Vf \text{ on } C,\ pV_pg\leqq g \text{ for each } p>0\}.$$

Then $Vf\in\varphi$. Put $h=\text{ess inf}\ \{g:\ g\in\varphi\}$.

Then $pV_ph\leqq h$ for each $p>0$. It follows from Theorem 3.2 that $pV_ph=h$ $=Vf$ on C. Also

$$qV_q(pV_ph)=pV_p(qV_qh)\leqq pV_ph \text{ for each } q>0.$$

Consequently $pV_ph\in\varphi$. The relation $pV_ph\leqq h$ and minimality of h imply $pV_ph=h$ for each $p>0$. By Theorem 2.2 we have $Vf+h=Vf$. Especially $2Vf=Vf$ on C whence $Vf=0$ or ∞ on C.

Let $(X, \mathfrak{A}, m)$ be a σ-finite measure space and $(T_t)_{t>0}$ be a strongly measurable semigroup of positive linear contractions on $L_1(X, \mathfrak{A}, m)=L_1$. It is known (cf. [3, p. 686]) that for every $f\in L_1$ there exists a function $g(t, x)$ measurable on $[0, \infty)\times X$ (which respect to the product of m and Lebesgue measure) such that for almost every fixed t the function $x\mapsto g(t, x)$ belongs to the equivalence class of T_tf. Moreover there exists a set $N(f)\subset X$ with $m(N(t))=0$, independent of t, such that if $x\notin N(f)$ then the function $t\mapsto g(t, x)$ is Lebesgue integrable ober every finite interval $(a, b)\subset(0, \infty)$ and the integral $\int_a^b g(t, x)dt$, as a function of x, belongs to the equivalence class of $\int_a^b T_tf\,dt$.

For every $f\supset L_1$ and $p>0$, put

$$U_pf=\int_0^\infty e^{-pt}T_tf\,dt.$$

Then U_p is a bound linear operator on L_1 with $\|U_p\|_1\leqq\frac{1}{p}$. If V_p is the adjoint operator of U_p, V_p is a bounded linear operator on L_∞ with $\|V_p\|_\infty\leqq\frac{1}{p}$. We remark that V_p is monotonically continuous on L_∞ for every $p>0$. The operator V_p can be extended to a positive linear operator on F. For $f\in F$, define

$$V_pf=\lim_{n\to\infty}V_pf_n$$

where (f_n) is an increasing sequence of L_∞^+ convergent to f. Clearly the definition is independent of the particular sequence (f_n). The positive linear operator V_p on F is monotonically continuous and $(V_p)_{p>0}$ satisfies (2.1). If the conservative part (resp. the dissipative part) of X with respect to the resolvent $(V_p)_{p>0}$ is C(resp. D), we have the following theorem.

THEOREM 3.4. *Let f be an element of L_1^+. Then*

$$\lim_{b\to\infty}\int_0^b T_tf\,dt=0 \text{ or } \infty \text{ on } C$$

and

$$\lim_{b\to\infty}\int_0^b T_tf\,dt<\infty \text{ on } D.$$

PROOF. Let $f\in L_1^+$ and $A\in\mathfrak{A}$, Then, by Fatou's lemma and Fubini's theorem,

$$\int_A\left(\lim_{b\to\infty}\int_0^b T_tf\,dt\right)dm=\lim_{b\to\infty}\int 1_A\left(\int_0^b T_tf\,dt\right)dm$$

$$= \lim_{b\to\infty} \int_0^b \left(\int (T_t f)\, 1_A dm \right) dt.$$

Also, using Proposition 2. 3 we have

$$\begin{aligned}\int fV(A)\, dm &= \lim_{p\to 0} \int fV_p(1_A)\, dm = \lim_{p\to 0} \int (U_p f) 1_A\, dm \\ &= \lim_{p\to 0} \lim_{b\to\infty} \int 1_A \left(\int_0^b e^{-pt}\, T_t f\, dt \right) dm \\ &= \lim_{b\to\infty} \lim_{p\to 0} \int_0^b e^{-pt} \left(\int (T_t f)\, 1_A\, dm \right) dt \\ &= \lim_{b\to\infty} \int_0^b \left(\int (T_t f)\, 1_A\, dm \right) dt.\end{aligned}$$

Hence

$$\int_A \lim_{b\to\infty} \left(\int_0^b T_t f\, dt \right) dm = \int fV(A)\, dm \quad \text{for each } f \in L_1^+ \text{ and } A \in \mathfrak{A}.$$

Let A be an arbitrary measurable subset of C. Then, by Corollary 3. 1, we have

$$a \int_A \lim_{b\to\infty} \int_0^b T_t f\, dt \, \Big) dm = \int f\, (aV(A))\, dm = \int f\, V(A) dm = \int_A \lim_{b\to\infty} \int_0^b T_t\, f\, dt \, \Big)$$

for any real number $a>0$. If $\int_A \left(\lim_{b\to\infty} \int_0^b T_t f \right) dm < \infty$, then $\int_A \left(\lim_{b\to\infty} \int T_t f\, dt \right)$ $dm = 0$ and hence $\lim_{b\to\infty} \int_0^b T_t f\, dt = 0$ on A.

Thus it follows that

$$\lim_{b\to\infty} \int_0^b T_t f\, dt = 0 \text{ or } \infty \text{ on } C.$$

Further, let A be a subset of D with $V(1_A) < \infty$. It follows from Corollary 2. 3 that there exists an increasing sequence (A_n) of $\mathfrak{A}$ satisfying $\bigcup_{n=1}^{\infty} A_n = A$ and $V(A_n) \leqq n$.

$$\int_{An} \left(\lim_{b\to\infty} \int_0^b T_t f\, dt \right) dm = \int fV(A_n)\, dm \leqq n\| f \|_1$$

implies

$$\lim_{b\to\infty} \int_0^b (T_t f)\, dt < \infty \text{ on } A_n.$$

Hence

$$\lim_{b\to\infty} \int_0^b (T_t f) dt < \infty \text{ on } A.$$

Consequently

$$\lim_{b\to\infty} \int_0^b (T_t f) dt < \infty \text{ on } D.$$

REMARK. From Theorem 3. 4 it follows that under the previous assumptions our decomposition with respect to the resolvent $(V_p)_{p>0}$ is identical with the well-known ergodic decomposition (cf. [4, Theorem 2. 1]).

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