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A Remark on the Kerr Metric

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Schiffer et al[1] have simplified the derivation of the Kerr metric[2] by assuming a degenerate metric

$$g_{\alpha\beta}=\eta_{\alpha\beta}-2ml_\alpha l_\beta$$

$\eta_{\alpha\beta}$ being the Lorentz metric and l_α being a null vector with respect to the Lorentz metric, and by demanding that $g_{\alpha\beta}$ satisfy the Einstein free field equations for any value of m. This demand splits the field equations into 3 sets of equations according to the order in m, so that the solution of the field equations becomes easy.

We remark here that the demand is unnecessary. Without the demand, the Einstein field equations yield 3 sets of equations.

Generalizing the Einstein free field equations a little, we set up the field equations with the cosmological term

$$R_{\mu\nu}-\Lambda g_{\mu\nu}=0 \tag{1}$$

which reduce to the free field equations discussed by Schiffer et al. when $\Lambda=0$. We list here basic relations

$$\left.\begin{aligned} &g_{\alpha\beta}=\eta_{\alpha\beta}-Ml_\alpha l_\beta\,, \qquad g^{\alpha\beta}=\eta^{\alpha\beta}+Ml^\alpha l^\beta\,, \qquad M=2m \\ &R_{\mu\nu}=-\partial_\alpha\Gamma^\alpha_{\mu\nu}+\Gamma^\alpha_{\beta\mu}\Gamma^\beta_{\alpha\nu}\,. \end{aligned}\right\} \tag{2}$$

The relations

$$v^\alpha=l^\beta\partial_\beta l^\alpha\,, \qquad v^\alpha l_\alpha=0\,, \qquad l^\alpha l_\alpha=0 \tag{3}$$

remain intact, while we have

$$\begin{aligned}\Gamma^\alpha_{\mu\nu}&=\frac{1}{2}g^{\alpha\beta}(\partial_\mu g_{\nu\beta}+\partial_\nu g_{\mu\beta}-\partial_\beta g_{\mu\nu})\\ &=-\frac{1}{2}M(\partial_\mu l_\nu l^\alpha+\partial_\nu l_\mu l^\alpha-\partial^\alpha l_\mu l_\nu)+\frac{1}{2}M^2l^\alpha(v_\mu l_\nu+v_\nu l_\mu)\end{aligned} \tag{4}$$

and

$$l^\nu\Gamma^\alpha_{\mu\nu}=-M/2\cdot(v_\mu l^\alpha+v^\alpha l_\mu)\,. \tag{5}$$

Therefore we have, by virtue of the relations (3), (4) and (5),

$$\begin{aligned}l^\mu l^\nu R_{\mu\nu}&=-l^\mu l^\nu\partial_\alpha\Gamma^\alpha_{\mu\nu}=-M/2\cdot l^\mu l^\nu\partial_\alpha\{\partial_\mu l_\nu l^\alpha+\partial_\nu l_\mu l^\alpha-\partial^\alpha l_\mu l_\nu\}\\ &=-M/2\cdot l^\mu l^\nu\partial_\alpha\{l^\alpha(\partial_\mu l_\nu+\partial_\nu l_\mu)\}\end{aligned}$$

$$=M/2\cdot\partial_\alpha l^\mu l^\nu\cdot l^\alpha(\partial_\mu l_\nu+\partial_\nu l_\mu)$$
$$=Mv^\nu v_\nu\,. \tag{6}$$

Because of the field equations (1), we have

$$v^\nu v_\nu=0 \quad \text{or} \quad v^\nu=-Al^\nu\,. \tag{7}$$

If we use this relation, we have

$$R_{\mu\nu}=MK_{\mu\nu}-M^2Sl_\mu l_\nu \tag{8}$$

$$K_{\mu\nu}=\frac{1}{2}\cdot\partial_\alpha(\partial_\mu l_\nu l^\alpha+\partial_\nu l_\mu l^\alpha-\partial^\alpha l_\mu l_\nu)$$
$$=-\frac{1}{2}\cdot\{\partial_\mu(A+L)l_\nu+\partial_\nu(A+L)l_\mu+\partial_\alpha\partial^\alpha l_\mu l_\nu\} \tag{9}$$

$$S=\frac{1}{2}\cdot\{A^2-2\partial_\alpha Al^\alpha+\partial_\alpha l^\beta\cdot\partial^\alpha l_\beta-\partial_\alpha l^\beta\cdot\partial_\beta l^\alpha\}\,. \tag{10}$$

Hence we have

$$R_{\mu\nu}l^\nu=MK_{\mu\nu}l^\nu=MSl_\mu$$

while the field equations (1) give

$$R_{\mu\nu}l^\nu=\Lambda l_\mu\,.$$

Therefore we have

$$MS=\Lambda\,. \tag{11}$$

Using this relation in (8), we have

$$R_{\mu\nu}=MK_{\mu\nu}-M\Lambda l_\mu l_\nu$$

while the field equations (1) give

$$R_{\mu\nu}=\Lambda g_{\mu\nu}=\Lambda(\eta_{\mu\nu}-Ml_\mu l_\nu)\,.$$

Therefore we have

$$MK_{\mu\nu}=\Lambda\eta_{\mu\nu}\,. \tag{12}$$

In short the field equations (1) reduce to (7), (11) and (12).

If Λ vanishes, (11) and (12) reduce to

$$S=0$$
$$K_{\mu\nu}=0$$

respectively. In other words, the $R_{\mu\nu}$ (8) vanish in each order in m. It is to be noted that the relation (7) is valid when the cosmological term is present.

References

1) M.M. Schiffer, R.J. Alder, J. Mark and C. Sheffield: J. Math. Phys. 14 (1973) 52.
2) R.P. Kerr: Phys. Rev. Letters, 11 (1963) 237.