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A Generalization of Lagrange's Theorem on the Expansion of Inverse Functions

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Lagrange's theorem on the expansion of inverse functions is generalized for functions of two variables.

§ 1. The case of one variable

The classical theorem of Lagrange[1] states that if x is only one zero of the equation

$$u=a+sf(u)$$

for given a and s in a certain domain, a function $F(x)$ can be expanded by the formula

$$F(x)=F(a)+\sum_{n=1}^{\infty}\frac{s^n}{n!}\left(\frac{d}{da}\right)^{n-1}[F'(a)f^n(a)] \tag{1}$$

The theorem has been proved with the aid of the theory of functions of a complex variable. We shall prove the theorem with the aid of the Dirac delta function and generalize it for functions of two or more variables. Using the unit function $\varepsilon(x)$

$$\varepsilon(x)=\begin{cases}1 & x>0\\ 0 & x<0\end{cases}$$

and its derivative, that is, the delta function

$$\delta(x)=\varepsilon'(x)$$

we have for any differentiable function $G(x)$

$$\int_{-\infty}^{\infty}G(u)\delta(u-a-sf(u))du=\int_{-\infty}^{\infty}G(u)\delta(u-x)du/(1-sf'(u))$$
$$=G(x)/(1-sf'(x))$$

under the assumption that $u-a-sf(u)$ has only one zero and

$1-sf'(x)>0$ for $-\infty<u<\infty$.

Expanding $\delta(u-a-sf(u))$ in powers of s

$$\delta(u-a-sf(u))=\sum_{n=0}^{\infty}\frac{(-s)^n f^n(u)}{n!}\delta^{(n)}(u-a)$$

where $\delta^{(n)}(u-a)=(d/du)^n\delta(u-a)$, we have

$$\begin{aligned} G(x)/(1-sf'(x)) &= \sum_{n=0}^{\infty}\frac{(-s)^n}{n!}\int_{-\infty}^{\infty}G(u)f^n(u)\delta^{(n)}(u-a)du \\ &= \sum_{n=0}^{\infty}\frac{s^n}{n!}\left(\frac{d}{da}\right)^n[G(a)f^n(a)]\,. \end{aligned}$$

Setting $G(x)/(1-sf'(x))=F(x)$ and rearranging the expression, we have

$$F(x)=\sum_{n=0}^{\infty}\frac{s^n}{n!}\left(\frac{d}{da}\right)^{n-1}[F'(a)f^n(a)] \tag{3}$$

which reduces to (1) if the term $n=0$ is separated.

§ 2. The case of two variables

We assume that two equations

$$\begin{aligned} u&=a+sf(u, v) \\ v&=b+tg(u, v) \end{aligned}$$

admits only one solution $u=x$ and $v=y$, and the Jacobian $\partial(u-sf, v-tg)/\partial(u, v)$ never vanishes for $-\infty<u<\infty$ and $-\infty<v<\infty$, s, t being sufficiently small. If two functions $\varphi(u, v)$ and $\psi(u, v)$ vanish simultaneously only at $u=x$, $v=y$, and the Jacobian $\partial(\varphi, \psi)/\partial(u, v)$ does not vanish there, we see that

$$\delta(\varphi)\delta(\psi)\,\frac{\partial(\varphi, \psi)}{\partial(u, v)}=\delta(u-x)\delta(v-y)$$

and

$$\begin{aligned} F(x, y) &= \iint_{-\infty}^{\infty}F(u, v)\delta(u-x)\delta(v-y)dudv \\ &= \iint_{-\infty}^{\infty}F(u, v)\delta(\varphi)\delta(\psi)\frac{\partial(\varphi, \psi)}{\partial(u, v)}dudv\,. \end{aligned}$$

Setting

$$\begin{aligned} \varphi(u, v)&=u-a-sf(u, v) \\ \psi(u, v)&=v-b-tg(u, v) \end{aligned}$$

and expanding $\delta(\varphi)$, $\delta(\psi)$ in powers of s, t respectively

$$\delta(\varphi)=\sum_{m=0}^{\infty}\frac{(-s)^m}{m!}\delta^{(m)}(u-a)f^m(u, v)$$

$$\delta(\psi)=\sum_{n=0}^{\infty}\frac{(-t)^n}{n!}\delta^{(n)}(v-b)g^n(u, v)$$

we have

$$\begin{aligned}F(x, y)&=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-s)^m}{m!}\frac{(-t)^n}{n!}\\&\quad\times\iint_{-\infty}^{\infty}F(u, v)J(u, v)f^m(u, v)g^n(u, v)\delta^{(m)}(u-a)\delta^{(n)}(v-b)dudv\\&=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{s^m}{m!}\frac{t^n}{n!}\left(\frac{\partial}{\partial a}\right)^m\left(\frac{\partial}{\partial b}\right)^n[F(a, b)J(a, b)f^m(a, b)g^n(a, b)]\,. \qquad (4)\end{aligned}$$

This is a generalized theorem of Lagrange. The Jacobian, however, involves s and t

$$J(u, v)=\frac{\partial(\varphi, \psi)}{\partial(u, v)}=\begin{vmatrix}1-sf_u & -sf_v\\ -tg_u & 1-tg_v\end{vmatrix},$$

so a rearrangement is needed. We have finally

$$\begin{aligned}F(x, y)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{s^m}{m!}\frac{t^n}{n!}\left(\frac{\partial}{\partial a}\right)^{m-1}\left(\frac{\partial}{\partial b}\right)^{n-1}\Big[&\frac{\partial^2 F}{\partial a\partial b}f^m g^n\\&+\frac{\partial F}{\partial a}\frac{\partial f^m}{\partial b}g^n+\frac{\partial F}{\partial b}f^m\frac{\partial g^n}{\partial a}\Big]\,. \qquad (5)\end{aligned}$$

An alternative form may be

$$F(x, y)/J(x, y)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{s^m}{m!}\frac{t^n}{n!}\left(\frac{\partial}{\partial a}\right)^m\left(\frac{\partial}{\partial b}\right)^n[F(a, b)f^m(a, b)g^n(a, b)]\,. \qquad (6)$$

In this form, extension to the case of three or more variables is straightforward. It may be noted that a Jacobian $\partial(x, y)/\partial(a, b)$ is inverse to the Jacobian $J(a, b)$ since we have

$$1=\frac{\partial(a, b)}{\partial(a, b)}=\frac{\partial(a+\varphi(r, y),\ b+\psi(x, y))}{\partial(a, b)}=\frac{\partial(\varphi, \psi)}{\partial(x, y)}\frac{\partial(x, y)}{\partial(a, b)}$$

§ 3. Examples

Setting $F(x, y)=x$ and then y, we have

$$x=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{s^m}{m!}\frac{t^n}{n!}\left(\frac{\partial}{\partial a}\right)^{m-1}\left(\frac{\partial}{\partial b}\right)^{n-1}\left[\frac{\partial f^m}{\partial b}g^n\right]$$

$$y=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{s^n}{m!}\frac{t^n}{n!}\left(\frac{\partial}{\partial a}\right)^{m-1}\left(\frac{\partial}{\partial b}\right)^{n-1}\left[f^m\frac{\partial g^n}{\partial a}\right].$$

If we put

$$x=a+sy^2 \qquad f=y^2$$
$$y=b+txy\,, \qquad g=xy$$

we have

$$x=a\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(2m+n-1)!}{m!(n-m+1)!(2m-1)!}\left(\frac{sb^2}{a}\right)^m(ta)^n$$

$$y=b\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(2m+n)!}{m!(n-m)!(2m+1)!}\left(\frac{sb^2}{a}\right)^m(ta)^n\,.$$

If we put more generally

$$x=a+sx^\lambda y^\mu \qquad f=x^\lambda y^\mu$$
$$y=b+tx^\sigma y^\tau \qquad g=x^\sigma y^\tau$$

we have

$$x=a\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\frac{l\mu(l\lambda+m\sigma)!(l\mu+m\tau-1)!}{l!m!(l\lambda+m\sigma-l+1)!(l\mu+m\tau-m)!}(sa^{\lambda-1}b^\mu)^l(ta^\sigma b^{\tau-1})^m$$

$$y=b\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\frac{m\sigma(l\lambda+m\sigma-1)!(l\mu+m\tau)!}{l!m!(l\lambda+m\sigma-l)!(l\mu+m\tau-m+1)!}(sa^{\lambda-1}b^\mu)^l(ta^\sigma b^{\tau-1})^m$$

For $\lambda=\mu=\sigma=\tau=1$, x, y may be expressed by Appell's hypergeometric functions of two variables[2),3)]

$$x=a\sum\sum\frac{(l+m)!(l+m-1)!}{(l-1)!l!m!(m+1)!}(sb)^l(ta)^m=aF_4(0,\,1;\;0,\,2;\;sb,\,ta)$$

$$y=b\sum\sum\frac{(l+m-1)!(l+m)!}{l!(l+1)!m!(m-1)!}(sb)^l(ta)^m=bF_4(0,\,1;\,2,\,0;\,sb,\,ta)\,.$$

Incidentally we have

$$\frac{\partial(x,y)}{\partial(a,b)}=\sum\sum\frac{s^l}{l!}\frac{t^m}{m!}\left(\frac{\partial}{\partial a}\right)^l\left(\frac{\partial}{\partial b}\right)^m(ab)^{l+m}$$

$$=\sum\sum\frac{(l+m)!(l+m)!}{l!l!m!m!}(sb)^l(ta)^m=F_4(1,\,1;\;1,\,1;\;sb,\,ta)\,.$$

References

[1] E. T. Whittaker and G. N. Watson, *Modern Analysis*, p. 132, Cambridge Univ. Press. 1935.
[2] W. N. Bailey, *Generalized Hypergeometric Series*, p. 73, Stechert-Hafner, New York, 1964.
[3] A. Erdélyi, *Higher Transcendental Functions*, Vol. 1, p. 224, McGraw-Hill, New York, 1953.