Natural Science Report, Ochanomizu University, Vol. 27, No. 2, 1976 77
お茶の水女子大学 自然科学報告 第27巻 第2号

Hypersurfaces Immersed in a Conformally Flat Riemannian Manifold

Dedicated to Professor S. Tachibana on his 50th Birthday

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(Received September 1, 1976)

In 1968, J. Simons [3] has established a formula for the Laplacian of the second fundamental form of a submanifold and has obtained some applications in the case of minimal hypersurfaces in the sphere. A formula of Simons' type has been improved by K. Nomizu and B. Smyth [2] in 1969. Based on the new formula of Simons' type, they have determined hypersurfaces of non-negative sectional curvature and constant mean curvature immersed in the Euclidean space or the sphere under the additional assumption which is satisfied if M is compact.

In the present paper, we first obtain a formula of Simons' type (2.11) in the case of a hypersurface M immersed with constant mean curvature in a conformally flat Riemannian manifold $\tilde{M}$ which satisfies some additional assumptions. These assumptions are described in terms of the Ricci operator of $\tilde{M}$ and naturally satisfied if $\tilde{M}$ is a space of constant sectional curvature. Our main results are the determination of the immersions of M into $\tilde{M}$ when M admits non-negative sectional curvature.

§ 1. Preliminaries and assumptions.

Let $\tilde{M}$ be an $(n+1)$-dimensional conformally flat Riemannian manifold with the metric $\langle\cdots,\cdots\rangle$. Let $\phi: M\to\tilde{M}$ be an isometric immersion of an n-dimensional Riemannian manifold M into $\tilde{M}$. Since our arguments are all local, we may consider ϕ as an imbedding on a neighbourhood and thus identify $x\in M$ with $\phi(x)\in\tilde{M}$. The tangent space $T_x(M)$ is identified with a subspace of the tangent space $T_x(\tilde{M})$, and the normal space $T_x^\perp$ is the 1-dimensional subspace of $T_x(\tilde{M})$ consisting of $\lambda\xi$, where $\lambda\in \boldsymbol{R}$ and ξ is a unit normal vector. As for the basic notations and formulas, we refer to Kobayashi-Nomizu [1].

Taking a neighbourhood U in M, we may choose a field of unit normal vector ξ. We denote by $\tilde{\nabla}$ and ∇ the covariant differentiations in $\tilde{M}$ and M, respectively, then we obtain

$$\tilde{\nabla}_X Y=\nabla_X Y+\langle AX, Y\rangle \xi , \tag{1.1}$$

$$\tilde{\nabla}_X \xi=-AX , \tag{1.2}$$

where X and Y are vector fields tangent to M and A is the second fundamental form with respect to ξ. The right hand side of (1.1) are the decomposition to the sum of the tangential and normal components of $\tilde{\nabla}_X Y$.

We denote by $\tilde{R}$, $\tilde{Q}$ and $\tilde{k}$ the Riemannian curvature tensor, the Ricci operator and the scalar curvature of $\tilde{M}$, respectively. Since $\tilde{M}$ is conformally fiat, the cuvature tensor $\tilde{R}$ satisfies the following equation by definition

$$\tilde{R}(X, Y)=\frac{1}{n-1}(\tilde{Q}X\wedge Y+X\wedge \tilde{Q}Y)-\frac{\tilde{k}}{n(n-1)}X\wedge Y , \tag{1.3}$$

where $X\wedge Y$ denotes the skew-symmetric endomorphism of $T_x(\tilde{M})$ defined by

$$(X\wedge Y)Z=\langle Y, Z\rangle X-\langle X, Z\rangle Y$$

for X, Y and Z of $T_x(\tilde{M})$.

Taking the orthogonal projection P of $T_x(\tilde{M})$ to $T_x(M)$, the Gauss' equation is given by

$$R(X, Y)=\frac{1}{n-1}((P\tilde{Q}X)\wedge Y+X\wedge (P\tilde{Q}Y)) -\frac{\tilde{k}}{n(n-1)}X\wedge Y+AX\wedge AY \tag{1.4}$$

for $X, Y\in T_x(M)$. Next the Codazzi's equation is expressed as

$$(\langle(\nabla_X A)Y, Z\rangle-\langle(\nabla_Y A)X, Z\rangle)\xi =\frac{1}{n-1}(\langle Y, Z\rangle P^\perp\tilde{Q}X-\langle X, Z\rangle P^\perp\tilde{Q}Y) ,$$

where $P^\perp=Id-P$ is the orthogonal projection of $T_x(\tilde{M})$ to $T_x^\perp$, and $X, Y, Z\in T_x(M)$.

We now assume that ξ is one of eigenvectors of $\tilde{Q}$, that is, there exists a function ρ on M such that $\tilde{Q}\xi=\rho\xi$. Then we have $P^\perp\tilde{Q}X=0$ for $X\in T_x(M)$, and the Codazzi's equation reduces to

$$(\nabla_X A)Y=(\nabla_Y A)X , \qquad X, Y\in T_x(M) . \tag{1.5}$$

Moreover we assume that $(\tilde{\nabla}_X\tilde{Q})\xi=0$ for any $X\in T_x(M)$. Then we have

(1.6) $$(P\widetilde{Q})A=A(P\widetilde{Q})=\rho A$$

and ρ is a constant function on M. Indeed, operating $\widetilde{\nabla}_X$ to $\widetilde{Q}\xi=\rho\xi$ we have

$$\widetilde{Q}AX=-(X\rho)\xi+\rho AX.$$

Sinec $\widetilde{Q}AX$ tangents to M, we obtain that $X\rho=0$ for any $X\in T_x(M)$ which means ρ is constant on M, and that $\widetilde{Q}AX=\rho AX$. By virtue of the symmetricity of $\widetilde{Q}$ and A, we have easily (1.6).

If $\widetilde{M}$ is a space of constant sectional curvature, then the Ricci tensor is proportional to the metric tensor with constant factor, and hence the above two assumptions are satisfied.

Finally we assume that M has constant mean curvature, that is, trace A=constant. Minimal immersion is a special case.

§ 2. Fundamental formula.

Setting $f=\langle A, A\rangle=\text{trace}\,A^2$ on M we proceed to a computation of Δf. It is known that

(2.1) $$\frac{1}{2}\Delta f=\langle\Delta' A, A\rangle+\langle\nabla A, \nabla A\rangle$$

is valid, where Δ' is the restricted Laplacian [2]. Under the assumptions stated in § 1, A satisfies the classical Codazzi's equation (1.5) and trace A=constant. Hence we have

(2.2) $$(\Delta' A)(X)=\sum_{i=1}^{n}[R(E_i, X), A]E_i$$

where $\{E_1, \cdots, E_n\}$ is an orthonormal frame on U (Nomizu-Smyth [2], pp. 369–371).

The right-hand side of (2.2) can be computed as follows. Making use of the Gauss' equation (1.4), we have

$$\begin{aligned}&\sum_{i=1}^{n}R(E_i, X)AE_i\\&=\frac{1}{n-1}(\widetilde{Q}AX-(\text{trace}\,(P\widetilde{Q}A))X+\widetilde{Q}AX-(\text{trace}\,A)\widetilde{Q}X)\\&\quad-\frac{\widetilde{k}}{n(n-1)}(AX-(\text{trace}\,A)X)+A^3X-(\text{trace}\,A^2)AX,\end{aligned}$$

and from (1.6), it becomes

(2.3) $$\begin{aligned}&\sum_{i=1}^{n}R(E_i, X)AE_i\\&=\frac{1}{n(n-1)}((2n\rho-\widetilde{k})AX-(n\rho-\widetilde{k})(\text{trace}\,A)X-n(\text{trace}\,A)\widetilde{Q}X)\\&\quad+A^3X-(\text{trace}\,A^2)AX.\end{aligned}$$

Similarly we get from (1.4)

$$\sum_{i=1}^{n} AR(E_i, X)E_i$$

$$=\frac{1}{n-1}(A\widetilde{Q}X-(\text{trace } P\widetilde{Q})AX+A\widetilde{Q}X-nA\widetilde{Q}X)$$

$$-\frac{\widetilde{k}}{n(n-1)}(AX-nAX)+A^3X-(\text{trace } A)A^2X .$$

since we have

$$\text{trace } P\widetilde{Q}=\text{trace } \widetilde{Q}-\rho=\widetilde{k}-\rho ,$$

we get using (1.6)

$$(2.4) \qquad \sum_{i=1}^{n} AR(E_i, X)E_i=-\frac{n(n-3)\rho+\widetilde{k}}{n(n-1)}AX+A^3X-(\text{trace } A)A^2X .$$

From (2.3) and (2.4), we obtain

$$(\Delta' A)(X)=\rho AX+\frac{\widetilde{k}-n\rho}{n(n-1)}(\text{trace } A)X-\frac{1}{n-1}(\text{trace } A)\widetilde{Q}X$$

$$+(\text{trace } A)A^2X-(\text{trace } A^2)AX .$$

Hence we have

$$(2.5) \qquad \Delta' A=\rho A+\frac{\widetilde{k}-n\rho}{n(n-1)}(\text{trace } A)Id-\frac{1}{n-1}(\text{trace } A)P\widetilde{Q}$$

$$+(\text{trace } A)A^2-(\text{trace } A^2)A ,$$

and from (2.1) we obtain

$$(2.6) \qquad \frac{1}{2}\Delta f=\rho \text{ trace } A^2+\sigma(\text{trace } A)^2$$

$$+(\text{trace } A)(\text{trace } A^3)-(\text{trace } A^2)^2+\langle \nabla A, \nabla A\rangle ,$$

where $\sigma=(\widetilde{k}-2n\rho)/n(n-1)$.

In the rest of this section, we shall transform (2.6) into a form which is convenient for our applications.

STEP 1. For each point x of U, let $\{e_1, \cdots, e_n\}$ be an orthonormal basis in $T_x(M)$ such that $Ae_i=\lambda_i e_i$, $1\leqq i\leqq n$. With respect to this basis, we have

$$(2.7) \qquad \lambda_i\langle\widetilde{Q}e_i, e_j\rangle=\rho\lambda_i\delta_{ij} \qquad (1\leqq i, j\leqq n)$$

which is obvious by (1.6).

STEP 2. Let K_{ij} $(1\leqq i, j\leqq n, i\neq j)$ be the sectional curvature in M of the plane spanned by e_i and e_j. Then we have

$$(2.8) \qquad (K_{ij}-\lambda_i\lambda_j)\lambda_i\lambda_j=-\sigma\lambda_i\lambda_j \qquad (1\leqq i, j\leqq n, i\neq j) .$$

PROOF. Let $\widetilde{K}_{ij}$ be the sectional curvature in $\widetilde{M}$ of the plane spanned by e_i and e_j. Then the Gauss' equation reads as

$$K_{ij}=\widetilde{K}_{ij}+\lambda_i\lambda_j .$$

Since $\widetilde{M}$ is conformally flat, we get

$$(2.9)\qquad K_{ij}-\lambda_i\lambda_j=\frac{1}{n-1}\left(\langle\widetilde{Q}e_i,\, e_i\rangle+\langle\widetilde{Q}e_j,\, e_j\rangle-\frac{\widetilde{k}}{n}\right)\qquad (1\leqq i,\ j\leqq n,\ i\neq j)\,,$$

from which we have (2.8) taking account of (2.7).

STEP 3. For any fixed number i $(1\leqq i\leqq n)$, we have

$$(2.10)\qquad \sum_{j\neq i}(K_{ij}-\lambda_i\lambda_j)\lambda_i=(\rho+\sigma)\lambda_i\qquad (1\leqq i\leqq n)\,.$$

PROOF. Noticing $\sum_{j=1}^{n}\langle\widetilde{Q}e_j,\, e_j\rangle=\widetilde{k}-\rho$ we sum up both sides of (2.9) for all $j\neq i$ $(1\leqq j\leqq n)$. Then we have

$$\sum_{j\neq i}(K_{ij}-\lambda_i\lambda_j)=\frac{1}{n-1}\left(\frac{\widetilde{k}}{n}-\rho+(n-2)\langle\widetilde{Q}e_i,\, e_i\rangle\right).$$

Making use of (2.7) again, we get (2.10).

STEP 4. Finally we have basic formula

$$(2.11)\qquad \frac{1}{2}\Delta f=\sum_{1\leqq i\leqq j\leqq n}(\lambda_i-\lambda_j)^2K_{ij}+\langle\nabla A,\, \nabla A\rangle\,.$$

PROOF. Using (2.8) and (2.10), we add all the terms of the right hand side in (2.6) except the last one.

$$\begin{aligned}
&\rho\,(\mathrm{trace}\,A^2)+\sigma\,(\mathrm{trace}\,A)^2+(\mathrm{trace}\,A)(\mathrm{trace}\,A^3)-(\mathrm{trace}\,A^2)^2\\
&=\rho\sum_i\lambda_i^2+\sigma(\sum_i\lambda_i)(\sum_j\lambda_j)+\sum_{i,j}\lambda_i\lambda_j^3-(\sum_i\lambda_i^2)(\sum_j\lambda_j^2)\\
&=(\rho+\sigma)\sum_i\lambda_i^2+\sigma\sum_{i\neq j}\lambda_i\lambda_j+\sum_{i\neq j}(\lambda_i\lambda_j^3-\lambda_i^2\lambda_j^2)\\
&=\sum_i\sum_{j\neq i}(K_{ij}-\lambda_i\lambda_j)\lambda_i^2-\sum_{j\neq i}(K_{ij}-\lambda_i\lambda_j)\lambda_i\lambda_j+\sum_{i<j}(\lambda_i-\lambda_j)^2\lambda_i\lambda_j\\
&=\sum_{i\neq j}(K_{ij}-\lambda_i\lambda_j)\lambda_i(\lambda_i-\lambda_j)+\sum_{i<j}(\lambda_i-\lambda_j)^2\lambda_i\lambda_j\\
&=\sum_{i<j}(K_{ij}-\lambda_i\lambda_j)(\lambda_i-\lambda_j)^2+\sum_{i<j}(\lambda_i-\lambda_j)^2\lambda_i\lambda_j\\
&=\sum_{i<j}(\lambda_i-\lambda_j)^2K_{ij}\,.
\end{aligned}$$

Hence we obtain (2.11).

§ 3. Eigenvalues of the second fundamental form.

Let M be a connected hypersurface immersed with constant mean curvature in a conformally flat Riemannian manifold $\widetilde{M}$ of dimension $n+1$, $n\geqq 3$. We assume that at every point $x\in M$ the Ricci operator $\widetilde{Q}$ of $\widetilde{M}$ leaves the normal space $T_x^\perp$ invariant and $\widetilde{\nabla}_X\widetilde{Q}$ annihilates $T_x^\perp$ for any $X\in T_x(M)$. We establish the following theorems.*)

*) After we have finished this paper, Messr. M. Kon and Y. Matsuyama announced to us that Theorem 1 is true without the assumption $(\widetilde{\nabla}_X\widetilde{Q})\xi=0$.

THEOREM 1. *If M is compact and has non-negative sectional curvature, then at every point of M we have*

$$\nabla A=0 \quad \text{and} \quad (\lambda_i-\lambda_j)^2K_{ij}=0 \qquad \text{for all} \quad i\neq j\,.$$

In paticular, the eigenvalues of A are constant.

PROOF. By assumption, $K_{ij}\geqq 0$. From (2.11) we have $\Delta f\geqq 0$. Since M is compact, we conclude that f is constant and $\Delta f=0$. Thus we get $\nabla A=0$ and $(\lambda_i-\lambda_j)^2K_{ij}=0$ for all i, j, $i\neq j$.

THEOREM 2. *If M has non-negative sectional curvature and if $f=\mathrm{trace}\,A^2$ is constant on M, then we have the same conclusions as Theorem* 1.

PROOF. This is evident from the formula (2.11) itself.

We can take such an integer q, $0\leqq q\leqq n$, that

$$\text{(3.1)} \qquad \begin{cases} \lambda_1\geqq\lambda_2\geqq\cdots\geqq\lambda_q\,, \qquad \lambda_r\neq 0 \quad (1\leqq r\leqq q)\,, \\ \text{and if } q<n\,, \quad \lambda_{q+1}=\cdots=\lambda_n=0\,. \end{cases}$$

Then $q=0$ means that all eigenvalues are zero. From now on we follow this arrangement.

LEMMA 1. *Let $\{e_1,\cdots,e_n\}$ be the orthonormal basis for $T_x(M)$ taken as Step* 1 *in section* 2. *Then under the assumptions of Theorem* 1 *or* 2, *we have*

(1) $K_{ra}=0$ $(1\leqq r\leqq q,\ q+1\leqq a\leqq n)$ *if* $q\geqq 1$,
(2) $K_{rs}=\lambda_r\lambda_s-\sigma$ $(1\leqq r,\ s\leqq q,\ r\neq s)$ *if* $q\geqq 2$,
(3) $\langle \tilde{Q}e_r, e_r\rangle=\rho$ $(1\leqq r\leqq q)$ *if* $q\geqq 1$,
(4) $\langle \tilde{Q}e_a, e_a\rangle=\dfrac{\tilde{k}}{n}-\rho$ $(q+1\leqq a\leqq n)$ *if* $q\leqq n-1$.

PROOF. We get (1) from Theorem 1 or 2 and $\lambda_r\neq 0$, $\lambda_a=0$ $(1\leqq r\leqq q<a\leqq n)$. Putting $i=j=r$ $(1\leqq r\leqq q)$ in (2.7), we get $\langle Qe_r, e_r\rangle\lambda_r=\rho\lambda_r$. Because $\lambda_r\neq 0$ we have (3). Putting $i=r, j=s$ $(1\leqq r,\ s\leqq q,\ r\neq s)$ in (2.9) and using (3) we get (2). Finally we have (4) by means of (2.9), (1), (3) and $\lambda_a=0$ $(q+1\leqq a\leqq n)$.

THEOREM 3. *Under the assumptions of Theorem* 1 *or* 2, *A has at most three distinct constants as eigenvalues at every point.*

PROOF. By virtue of Theorem 1 or 2, the eigenvalues of A remain constant. Thus the set of umbilics is open in M. Since it is obviously a closed set, either M is totally umbilic or M has no umbilic. In the latter case, we show that A has at most three eigenvalues at any point of M.

First we recall the arrangement (3.1). We may assume that the largest eigenvalue λ_1 is positive by the following reason: Since $q=0$ implies $\lambda_1=\cdots=\lambda_n=0$, which is contrary to the fact of no umbilic, we have $1\leqq q\leqq n$. Hence $\lambda_1\neq 0$. If $\lambda_1<0$, then $\lambda_q<0$. We then change the field of unit normals ξ into $-\xi$ thus changing A into $-A$, whose largest eigenvalue $-\lambda_q$ is positive. Having assumed that

$$\lambda_1\geqq\cdots\geqq\lambda_q\,,\quad \lambda_1>0\,,\quad \lambda_r\neq 0\qquad (1\leqq r\leqq q)\,,$$

we have from (1) of Lemma 1 that

$$K_{1j}=0\quad (2\leqq j\leqq n)\qquad \text{if}\quad q=1\,,$$
$$K_{12}\geqq\cdots\geqq K_{1q}\geqq 0=K_{1,q+1}=\cdots=K_{1n}\qquad \text{if}\quad q\geqq 2\,.$$

When $q=1$, our conclusion of Theorem is trivial, and hence we have only to consider the case $q\geqq 2$. We take the largest integer p such that $K_{1p}>0$ and $K_{1,p+1}=0$, $2\leqq p\leqq n$. If there does not exist such an integer, that is, if $K_{1j}=0$ for all $j=2,\cdots,n$, then we define $p=1$. The case $K_{1n}>0$ does not arise because of (1) of Lemma 1 and the assumption of no umbilic, and hence $p\leqq n-1$. Suppose $p\geqq 2$, then from Theorem 1 or 2, we get

$$(\lambda_1-\lambda_t)^2K_{1t}=0\qquad (2\leqq t\leqq p)$$

which implies that

$$\lambda_1=\cdots=\lambda_p=\lambda\,,\quad \text{say}\,. \tag{3.2}$$

In addition, from $K_{1,p+1}=\cdots=K_{1n}=0$ and (2) of Lemma 1, we have

$$\lambda_{p+1}=\cdots=\lambda_q=\frac{\sigma}{\lambda}=\mu\,,\quad \text{say}\,. \tag{3.3}$$

Thus it is proved that the possible eigenvalues are only λ, 0 and $\mu=\sigma/\lambda$, if $p\geqq 2$. μ may coincide with λ or 0. If $p=1$, then using (2) of Lemma 1 again, (3.3) becomes true under $p=1$, too, ((3.2) holds trivially if $p=1$) and we reach the same conclusion as the case $p\geqq 2$. Hence Theorem 3 is proved.

In the following, we describe in detail the eigenvalues of A.

(I) If M is a totally umbilical hypersurface, then it has only one eigenvalue λ. If $\lambda=0$, then M is totally geodesic.

(II) Suppose that M has no umbilical point. As is shown in the proof of Theorem 3, it is true that $1\leqq p\leqq q\leqq n$ and

$$\begin{cases} \lambda_1=\cdots=\lambda_p=\lambda>0\,, & \lambda_{p+1}=\cdots=\lambda_q=\mu \\ \lambda_{q+1}=\cdots=\lambda_n=0\,, & (\lambda\mu=\sigma)\,. \end{cases} \tag{3.4}$$

(II-1) When $\sigma=0$, then we have $p=q$; for if $p<q$, then $(0\neq)\lambda_q=\sigma/\lambda=0$ is a contradiction. Thus there exist just two eigenvalues $\lambda(>0)$ and 0, such that

$$\lambda_1=\cdots=\lambda_q=\lambda\,,\quad \lambda_{q+1}=\cdots=\lambda_n=0\qquad (1\leqq q\leqq n-1)\,. \tag{3.5}$$

(II-2) When $\sigma=\lambda^2$, then $\lambda=\mu$ is valid, and hence we have (3.5) again. We remark that in this case $p=1$, that is, $K_{1j}=0$ $(2\leqq j\leqq n)$. Indeed, if $p\geqq 2$, then $K_{1p}=\lambda_1\lambda_p-\sigma=\lambda^2-\sigma=0$ which contradicts $K_{1p}>0$.

(II-3) When the function σ is not constant on M, then we have $p=q$; for if $p<q$, then λ_q is equal to σ/λ contradicting the constancy of eigenvalues of A. Thus we have (3.5) in this case.

(II-4) When σ is constant and $\sigma\neq\lambda^2$, 0, then we have (3.4) with non-zero constants λ and μ, $\lambda\neq\mu$.

The sectional curvature of M will be useful for our discussion below. Now we shall express the sectional curvature of M in terms of eigenvalues λ, $\mu=\sigma/\lambda$ of A and σ.

LEMMA 2. *Under the assumptions of Theorem* 1 *or* 2, *we account the case in which* (3.4) *holds for* $2\leqq p<p+1<q\leqq n-2$. *Then we have*

(1) $K_{rs}=\lambda^2-\sigma$ $(1\leqq r,\ s\leqq p,\ r\neq s)$,

(2) $K_{tu}=\mu^2-\sigma$ $(p+1\leqq t,\ u\leqq q,\ t\neq u)$,

(3) $K_{ab}=\sigma$ $(q+1\leqq a,\ b\leqq n,\ a\neq b)$.

In the case (3.5) *with* $2\leqq q\leqq n-2$, (2) *and* (3) *hold.*

PROOF. We get (1) and (2) from (2) of Lemma 1, (3.2) and (3.3). Combining (2.9) with (4) of Lemma 1, we have (3). The latter case is similar to the above.

Making use of Lemma 2 we have some estimation of multiplicities of eigenvalues of A.

LEMMA 3. *Under the assumptions of Theorem* 1 *or* 2, *suppose that* A *has mutually distinct three eigenvalues* λ, $\mu=\sigma/\lambda$ *and* 0. *Then at least one of them has multiplicity one.*

PROOF. As A has three eigenvalues λ, μ and 0, it occurs only in the case (II-4) and they satisfy

$$\lambda\mu=\sigma\,,\quad \lambda\neq\mu \quad\text{and}\quad \lambda\mu\neq 0\,.$$

Assume that their multiplicities are $p\geqq 2$, $q-p\geqq 2$ and $n-q\geqq 2$. Then we can take sectional curvatures K_{sr} $(1\leqq r,\ s\leqq p)$, K_{tu} $(p+1\leqq t,\ u\leqq q)$ and K_{ab} $(q+1\leqq a,\ b\leqq n)$ on M, which are all non-negative. Then we have from Lemma 2

$$\lambda(\lambda-\mu)\geqq 0\,,\quad \mu(\mu-\lambda)\geqq 0 \quad\text{and}\quad \lambda\mu\geqq 0\,.$$

These three inequalities contradict either $\lambda-\mu\neq 0$ or $\lambda\mu\neq 0$.

LEMMA 4. *Under the assumptions of Theorem* 1 *or* 2, *suppose that* A *has the eigenvalue* λ (*respectively* 0). *If* $\sigma>\lambda^2$ (*respectively* $\sigma<0$), *then the multiplicity of* λ (*respectively* 0) *is one.*

PROOF. Assume that the multiplicity p of λ is greater than one. Then we have $K_{rs}=\lambda^2-\sigma<0$ $(1\leqq r,\ s\leqq p,\ r\neq s)$, which is impossible. As for the eigenvalue 0, the proof is similar.

LEMMA 5. *Under the assumptions of Theorem* 1 *or* 2, *suppose that* A *has the non-zero eigenvalues* λ *and* μ $(\lambda>0,\ \lambda>\mu=\sigma/\lambda)$. *If* σ *is positive, then the multiplicity of* μ *is one.*

PROOF. Assume that the multiplicity $q-p$ of μ is greater than one. Then taking the sectional curvature K_{tu} $(p+1\leqq t,\ u\leqq q)$, we have

$$K_{tu}=\mu^2-\sigma=\frac{\sigma}{\lambda^2}(\sigma-\lambda^2)\,.$$

Since $0<\sigma<\lambda^2$, we have $K_{tu}<0$, which is impossible.

§ 4. Classification of immersion.

In this section, we shall determine the immersion of M into $\tilde{M}$ under the assumptions stated in § 3. First of all, we define distribution T^{ν}, ν being one of eigenvalues of A, as follows:

$$T^{\nu}(x)=\{X\in T_x(M)\,;\ AX=\nu X\}\,,\qquad x\in M\,.$$

The distribution T^{ν} is differentiable, involutive and totally geodesic on M because A is parallel.

LEMMA 6. *Let* M *be a connected hypersurface immersed in a Riemannian manifold* $\tilde{M}$, *and assume that* $\nabla A=0$. *Then the maximal integral manifold* M_{ν} *of* T^{ν} *is totally geodesic in* M *and totally umbilic in* $\tilde{M}$ (*If* $\nu=0$, M_0 *is totally geodesic in* M *and* $\tilde{M}$).

PROOF. For any $X,\ Y\in T_x(M)$, $x\in M_{\nu}$, we have

$$\tilde{\nabla}_X Y=\nabla_X Y+\langle AX,\ Y\rangle\xi=\nabla_X Y+\nu\langle X,\ Y\rangle\xi\,.$$

Since $\nabla_X Y\in T_x(M_{\nu})$, our assertion follows immediately.

LEMMA 7. *If* $\tilde{M}$ *is conformally flat,* $\nabla A=0$ *and*

$$\langle\tilde{Q}X,\ X\rangle=\rho\qquad \textit{far any}\quad X\in T_x(M_{\nu})\,,\qquad \nu\neq 0\,,$$

$$\langle\tilde{Q}X,\ X\rangle=\frac{\tilde{k}}{n}-\rho\qquad \textit{for any}\quad X\in T_x(M_0)\,,$$

then the sectional curvature of M_{ν} *is* $\nu^2-\sigma$ *for* $\nu\neq 0$ *and* σ *for* $\nu=0$.

PROOF. Let $K(X\wedge Y)$ be the sectional curvature of M for the plane spanned by $X,\ Y\in T_x(M_{\nu})$, $x\in M_{\nu}$. Since M_{ν} is totally geodesic in M, $K(X\wedge Y)$ is the sectional curvature of M_{ν}. From (2.9) we have

$$K(X\wedge Y)=\nu^2+\frac{1}{n-1}\left(\langle\widetilde{Q}X,\ X\rangle+\langle\widetilde{Q}Y,\ Y\rangle-\frac{\widetilde{k}}{n}\right).$$

Hence we get

$$K(X\wedge Y)=\nu^2-\sigma \qquad \text{for any} \quad X,\ Y\in T_x(M_\nu), \qquad \nu\neq 0$$

and

$$K(X\wedge Y)=\sigma \qquad \text{for any} \quad X,\ Y\in T_x(M_0).$$

Finally we shall summarize our results. Let M be a connected hypersurface immersed with constant mean curvature in a conformally flat Riemannian manifold $\widetilde{M}$ of dimension $n+1$, $n\geqq 3$. We assume that at every point $x\in M$ the Ricci operator $\widetilde{Q}$ of $\widetilde{M}$ leaves the normal space $T_x^\perp$ invariant and $\widetilde{\nabla}_X\widetilde{Q}$ annihilates $T_x^\perp$ for any $X\in T_x(M)$. Moreover we assume that either M is compact and has non-negative sectional curvature or M has non-negative sectional curvature and trace A^2 is constant on M. We choose a basis $\{e_1, \cdots, e_n\}$ for $T_x(M)$, $x\in M$, such that

$$Ae_i=\lambda_i e_i \qquad (1\leqq i\leqq n),$$

$$\lambda_1\geqq \cdots \geqq\lambda_q, \qquad \lambda_{q+1}= \cdots =\lambda_n=0.$$

In the following, we represent the matrix corresponding to the operator A with respect to this basis.

Case (I-1).

$$A=\begin{bmatrix} 0 & & \\ & \ddots & \\ & & 0 \end{bmatrix}.$$

M is totally geodesic in $\widetilde{M}$.

Case (I-2).

$$A=\begin{bmatrix} \lambda & & \\ & \ddots & \\ & & \lambda \end{bmatrix},$$

where λ is positive constant. Since $n\geqq 3$, M has constant sectional curvature $\lambda^2-\sigma$ (hence σ is constant on M, $\sigma\leqq\lambda^2$) and is totally umbilic in $\widetilde{M}$.

Case (II-1, 2 and 3).

$$A=\begin{bmatrix} \overbrace{\lambda \ \ddots \ \lambda}^{q} & \\ & \overbrace{0 \ \ddots \ 0}^{n-q} \end{bmatrix},$$

where λ is positive constant. The multiplicity q $(1\leqq q\leqq n-1)$ of λ is as follows:

q is arbitrary if $0 \leqq \sigma \leqq \lambda^2$;

$q=n-1$ if $\sigma<0$; $\quad q=1$ if $\lambda^2<\sigma$.

Hence M is locally isometric to $M_\lambda \times M_0$, $M_\lambda \times \boldsymbol{R}$ or $\boldsymbol{R} \times M_0$.

Case (II-4-1).

$$A=\begin{bmatrix} \overbrace{\lambda \cdot \cdot \cdot \lambda}^{p} & \\ & \overbrace{\mu \cdot \cdot \cdot \mu}^{n-p} \end{bmatrix},$$

where λ is positive constant, $\lambda\mu=\sigma$ and $\mu<\lambda$. Since $\mu=\sigma/\lambda$ is an eigenvalue of A, σ is constant on M. The multiplicity p $(1 \leqq p \leqq n-1)$ of λ is as follows:

p is arbitrary if $\sigma<0$;

$p=n-1$ if $\sigma \geqq 0$ (This case occurs only if $\mu^2<\sigma<\lambda^2$).

Hence M is locally isometric to $M_\lambda \times M_\mu$ or $M_\lambda \times \boldsymbol{R}$.

Case (II-4-2).

$$A=\begin{bmatrix} \overbrace{\lambda \cdot \cdot \cdot \lambda}^{p} & & \\ & \overbrace{\mu \cdot \cdot \cdot \mu}^{q-p} & \\ & & \overbrace{0 \cdot \cdot \cdot 0}^{n-q} \end{bmatrix},$$

where λ is positive constant, $\lambda\mu=\sigma$ and $\mu<\lambda$. σ is constant on M as above. Multiplicities of eigenvalues are as follows:

$n-q=1$ if $\sigma<0$;

$q-p=1$ if $\sigma \geqq 0$ (This case occurs only if $\mu^2<\sigma<\lambda^2$).

Hence M is locally isometric to $M_\lambda \times M_\mu \times \boldsymbol{R}$ or $M_\lambda \times \boldsymbol{R} \times M_0$.

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