On Proper Spaces of Some Positive Operators with the Property (*W*)

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1. Introduction

It is well known as a part of results by O. Perron and G. Frobenius that a non-negative matrix T is irreducible if and only if each one of the proper spaces of T and T', corresponding to the spectral radius $r(T)$, is a one-dimensional subspace spanned by a positive element [1]. This property has been extended to infinite-dimensional topological ordered vector spaces by many authors [2, 3, 5, 6, 8]. Among them, I. Sawashima gave the extension to the case of an operator T in a partially ordered Banach space, whose resolvent has the point $\lambda=r(T)$ as its pole [5]. Further, H. H. Schaefer has extended this property to an ergodic operator in a Banach lattice [8]. In this paper, the author investigates an extension of these results to the more general operators in a partially ordered Banach space, which are not necessarily ergodic. We examine operators with the property (W) in a general Banach space in Section 2 and give a result about irreducibility in Section 3 (Th. 1). Section 4 is devoted to an investigation of the property that the proper spaces of T and T', corresponding to $r(T)$, are one-dimensional (Th. 2 & 3). Theorem 2 is also a generalization of the results by H. H. Schaefer that if T is a Markov operator in $C(X)$ such that the proper space of T', corresponding to 1, is one-dimensional, then T is ergodic [7, p. 713].

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2. The property (W) and ergodicity

In this section, let E be a Banach space, $\mathfrak{L}(E)$ be the set of bounded linear operators in E, $r=r(T)$ be the spectral radius of $T\in\mathfrak{L}(E)$ and $R(\lambda, T)$ be the resolvent operator.

DEFINITION. Let the convex hull of the set $\{I, T, T^2, \cdots\}$[1] (resp. the

1) I is the identity mapping.

orbit $\{x, Tx, T^2x, \cdots\}$) be denoted by $\mathrm{co}(T)$ (resp. $\mathrm{co}(Tx)$). An operator $T\in\mathfrak{L}(E)$ is called *ergodic*[2], if the closure $\mathrm{co}(T)^a$ of $\mathrm{co}(T)$ in the strong operator topology contains an element $P\in\mathfrak{L}(E)$ such that $PT=TP=P=P^2$. An operator $T\in\mathfrak{L}(E)$ is said to have *the property* (W) if the set $\{\|(\lambda-r)R(\lambda, T)\| : \lambda>r\}$ is bounded.

LEMMA 1. *Let $T\in\mathfrak{L}(E)$ be an operator with $r(T)=1$ and the set $\{\|M_n\| : n\in N\}$ be bounded, where $M_n=(I+T+\cdots+T^{n-1})/n$. Then T has the property* (W).

PROOF. Suppose $\|M_n\|<C$ for every $n\in N$. Then by using the relations

$$(1-\mu)\sum_{k=0}^{\infty}\mu^k=1 \quad\text{and}\quad (1-\mu)^2\sum_{n=1}^{\infty}n\mu^{n-1}=1 \qquad\text{for}\quad 0<\mu<1,$$

and putting $\mu=1/\lambda$, we get

$$\begin{aligned}\|(\lambda-1)R(\lambda, T)\|&=\|(1-\mu)\sum_{k=0}^{\infty}T^k\mu^k\|\\&=\|(1-\mu)^2\sum_{j=0}^{\infty}\mu^j\sum_{k=0}^{\infty}T^k\mu^k\|=\|(1-\mu)^2\sum_{n=1}^{\infty}nM_n\mu^{n-1}\|\\&\leqq(1-\mu)^2\sum_{n=1}^{\infty}n\mu^{n-1}\|M_n\|<C, \qquad\text{for every } \lambda>1,\end{aligned}$$

which completes the proof.

If an operator $T\in\mathfrak{L}(E)$ is ergodic, there exists a projection to the space $\{x\in E : Tx=x\}$. When an operator $T\in\mathfrak{L}(E)$ has the property (W), the projection to the space $\{x\in E : Tx=rx\}$ does not necessarily exist, but there exists a projection P_* to the space $\{f\in E' : T'f=rf\}$, as shown in the following.

PROPOSITION 1. *Let an operator $T\in\mathfrak{L}(E)$ have the property (W) and fix a sequence $\{\lambda_n\}$ such that $\lambda_n>r$ and $\lambda_n\to r$ and an ultrafilter $\mathcal{U}$ on N containing no finite set. Then the bounded linear operator P_* defined by*

$$P_*f(x)=\mathcal{U}-\lim_{\lambda_n\downarrow r}(\lambda_n-r)R(\lambda_n, T')f(x)^{3)}$$

for every $f\in E'$ and every $x\in E$, is a projection to the space $\{f\in E' : T'f=rf\}$, satisfying the relation $P_*T'=T'P_*=rP_*$.

PROOF. The equation $P_*T'=rP_*$ is obtained by the equation, for every $f\in E'$ and every $x\in E$,

2) This definition of ergodicity is due to [8, p. 179]. When the set $\{\|T^n\| : n\in N\}$ is bounded, this is equivalent to the usual one, that is, $M_n=(I+T+\cdots+T^{n-1})/n$ converges strongly.

3) $\mathcal{U}-\lim\xi_n=\bigcap_{F\in\mathcal{U}}\{\xi_m : m\in F\}^a$.

$$
\begin{aligned}
P_* T' f(x) &= \mathcal{U}-\lim_{\lambda_n \downarrow r} (\lambda_n - r) R(\lambda_n, T') T' f(x) \\
&= \mathcal{U}-\lim_{\lambda_n \downarrow r} \{\lambda_n (\lambda_n - r) R(\lambda_n, T') f(x) - (\lambda_n - r) f(x)\} \\
&= r P_* f(x) .
\end{aligned}
$$

The equation $T'P_* = rP_*$ is easily seen by $T'P_*f(x) = P_*f(Tx)$ and therefore, P_*E' is contained in $\{f \in E' : T'f = rf\}$. If f is the proper vector of T', corresponding to r, then

$$
\begin{aligned}
P_* f(x) &= \mathcal{U}-\lim_{\lambda_n \downarrow r} (\lambda_n - r) R(\lambda_n, T') f(x) \\
&= \mathcal{U}-\lim_{\lambda_n \downarrow r} (\lambda_n - r) \sum_{k=0}^{\infty} \frac{r^k f(x)}{\lambda_n^{k+1}} \\
&= f(x) \qquad \text{for every } x \in E ,
\end{aligned}
$$

which implies that $\{f \in E' : T'f = rf\} \subset P_*E'$. Then we have $P_*E' = \{f \in E' : T'f = rf\}$ and $P_*^2 = P_*$.

REMARK. If T is a positive operator in a partially ordered Banach space[4], the projection P_* is a positive projection since the ultrafilter limit preserves the order relation.

A dual operator $P_*' \in \mathfrak{L}(E'')$ of P_* is not necessarily a projection to the space $\{\varphi \in E'' : T''\varphi = r\varphi\}$, although P_*E'' is contained in $\{\varphi \in E'' : T''\varphi = r\varphi\}$ by Prop. 1. However we can define a projection to the space $\{\varphi \in E'' : T''\varphi = r\varphi\}$, applying Prop. 1 to $T' \in \mathfrak{L}(E')$ instead of $T \in \mathfrak{L}(E)$. Let P_{**} be the projection defined by

$$
P_{**}\varphi(f) = \mathcal{U}-\lim_{\lambda_n \downarrow r} (\lambda_n - r) R(\lambda_n, T'') \varphi(f)
$$

for every $\varphi \in E''$ and every $f \in E'$, where $\{\lambda_n\}$ and $\mathcal{U}$ are the same sequence and the ultrafilter used in the definition of P_* in Prop. 1.

By the definition of P_* and P_{**}, the following lemma is obtained easily.

LEMMA 2. *If $T \in \mathfrak{L}(E)$ has the property (W), then $P_{**}|E$, the restriction of P_{**} to E, is equal to $P_*'|E$ and the space $P_*'E \cap E$ coincides with the space $\{x \in E : Tx = rx\}$.*

REMARK. In general, the restriction of P_*' to E is not in $\mathfrak{L}(E)$. If $P_*'E$ is contained in E, $P_*'|E$ is a projection to the space $\{x \in E : Tx = rx\}$ and T becomes ergodic if $r(T) = 1$, as shown in the following.

PROPOSITION 2. *Let $T \in \mathfrak{L}(E)$ be an operator with $r(T) = 1$.*

I) *If T has the property (W), the following* i)$\sim$v) *are equivalent.*

II) *The condition* vi) *implies* i)$\sim$v).

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4) In this paper, a partially ordered Banach space means a real partially ordered Banach space with a proper, closed, generating and normal cone.

i) *T is ergodic.*

ii) *There exists a projection P to the space $\{x\in E: Tx=x\}$ such that $PT=TP=P$ and its dual P' is a projection to the space $\{f\in E': T'f=f\}$.*

iii) *There exists a projection P such that*

$$\lim_{\lambda\downarrow 1}(\lambda-1)R(\lambda, T)x=Px, \qquad \text{for every } x\in E.$$

iv) *P_* is w^*-continuous.*

v) *$P_*'E\subset E$.*

vi) *$M_n=(I+T+\cdots+T^{n-1})/n$ converges strongly.*

PROOF. I) i) $\Rightarrow$ ii): By the definition of ergodicity, the equation $PT=TP=P=P^2$ is clear and implies the relations $PE\subset\{x\in E: Tx=x\}$ and $P'E'\subset\{f\in E': T'f=f\}$. If the relation $Tx=x$ holds, $\mathrm{co}(Tx)^a$ consists of $\{x\}$ and hence $Px=x$ and $PE=\{x\in E: Tx=x\}$. Suppose that the relation $T'f=f$ holds. Since f is continuous, $f(x)=f(y)$ for every element y of $\mathrm{co}(Tx)^a$ and so $f(x)=f(Px)$. Therefore $P'f(x)=f(Px)=f(x)$ for every $x\in E$. Then $P'E'=\{f\in E': T'f=f\}$.

ii) $\Rightarrow$ iii): For every element x of PE, $\lim_{\lambda\downarrow 1}(\lambda-1)R(\lambda, T)x=Px$ is obtained easily. For every element x of $(I-T)E$, $\lim_{\lambda\downarrow 1}(\lambda-1)R(\lambda, T)x=0=Px$ is obtained, since T has the property (W). By using Hahn-Banach theorem and the fact that the set $(I-P)E$ is closed, we get the relation $(I-P)E=((I-T)E)^a$, which shows that for every element x of $(I-P)E$, $\lim_{\lambda\downarrow 1}(\lambda-1)R(\lambda, T)x=0=Px$ since T has the property (W).

iii) $\Rightarrow$ iv): By the relation $P_*=P'$, P_* is w^*-continuous.

iv) $\Rightarrow$ v): For every $x\in E$ and every $\varepsilon>0$, consider a w^*-neighborhood $U(0, x, \varepsilon)$ of 0 in E', then there exists a w^*-neighborhood V such that $f\in V$ implies $P_*f\in U(0, x, \varepsilon)$, that is, $|P_*'x(f)|=|P_*f(x)|<\varepsilon$. This means that $P_*'x$ is $\sigma(E', E)$ continuous, hence $P_*'x\in E$.

v) $\Rightarrow$ ii) is obvious by Lemma 2.

iii) $\Rightarrow$ i): The relation $PT=TP=P=P^2$ is easily obtained. We have to show that P is contained in $\mathrm{co}(T)^a$, that is, for every $\varepsilon>0$ and a finite set $\{x_1, \cdots, x_n\}$ in E, there exists $S\in\mathrm{co}(T)$ satisfying $\|Sx_i-Px_i\|<\varepsilon$ $(i=1, \cdots, n)$. Put $a=\max\{\|x_1\|, \cdots, \|x_n\|\}$ and $\varepsilon'=\min\left\{1, \frac{\varepsilon}{(M+2)a+1}\right\}$, where M is such a number as $\sup_{\lambda>1}\|(\lambda-1)R(\lambda, T)\|<M$. Find λ_0 such that $2>\lambda_0>1$ and $\|(\lambda_0-1)R(\lambda_0, T)x_i-Px_i\|<\varepsilon'$ $(i=1, \cdots, n)$. Since $\lambda_0>1$, $S_n=\sum_{k=0}^{n}\frac{T^k}{\lambda_0^{k+1}}$ converges to $R(\lambda_0, T)$ in the uniform topology. Therefore there exists n_0 such that $\left\|R(\lambda_0, T)-\sum_{k=0}^{n_0}\frac{T^k}{\lambda_0^{k+1}}\right\|<\varepsilon'$ and $0<\frac{\lambda_0-1}{\lambda_0^{n_0}-1}<\varepsilon'$. Put $S=S_n\Big/\sum_{k=0}^{n_0}\frac{1}{\lambda_0^{k+1}}$. Then S is a desired one.

II) The condition vi) implies the set $\{\|M_n\|: n\in N\}$ is bounded by the principle of uniform boundedness. Hence T has the property (W)

by Lemma 1. ii) is easily obtained from vi), since

$$\frac{T^n}{n}x=\frac{I+\cdots+T^n}{n+1}x\cdot\frac{n+1}{n}-\frac{I+\cdots+T^{n-1}}{n}x$$

converges to 0 as $n\to\infty$ for every $x\in E$.

REMARK. If $\|T^n/n\|\to 0$, the equivalence between iii) and vi) in the uniform operator topology has been obtained in the appendix of [4]. If T is a positive operator with $r(T)=1$ in a partially ordered Banach space, the equivalence between i)$\sim$v) and vi) is obtained in a similar way as Th. 5 of [2] by a Tauberian theorem. When T is not a positive operator and the set $\{\|T^n\|: n\in N\}$ is unbounded, i)$\sim$v) does not necessarily imply vi).

COUNTER-EXAMPLE. Let

$$E=C(\{x_1, x_2, x_3\}) \quad\text{and}\quad T=\begin{pmatrix} 0 & -1 & 0\\ -1 & 0 & 0\\ 0 & -1 & -1\end{pmatrix}.$$

Then it is clear that

$$\lim_{\lambda\downarrow 1}(\lambda-1)R(\lambda, T)=\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0\\ -\frac{1}{2} & \frac{1}{2} & 0\\ \frac{1}{4} & -\frac{1}{4} & 0\end{pmatrix},$$

$$T^{2n}=\begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ n & n & 1\end{pmatrix} \quad\text{and}\quad T^{2n+1}=\begin{pmatrix} 0 & -1 & 0\\ -1 & 0 & 0\\ -n & -n-1 & -1\end{pmatrix}.$$

From these, we see

$$\lim M_{2n}=\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0\\ -\frac{1}{2} & \frac{1}{2} & 0\\ 0 & -\frac{1}{2} & 0\end{pmatrix}\neq\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0\\ -\frac{1}{2} & \frac{1}{2} & 0\\ \frac{1}{2} & 0 & 0\end{pmatrix}=\lim M_{2n+1},$$

which implies that M_n does not converge.

3. Properties of irreducible operators

Hereafter E denotes a real partially ordered Banach space with a positive cone K and T a positive operator of $\mathfrak{L}(E)$ (i. e. $TK\subset K$) with the spectral radius r.

DEFINITION. An element $x \in K$ is called *a non-support point of K*, if $f(x) > 0$ for every nonzero f in K'. A linear form $f \in E'$ (resp. An operator $U \in \mathfrak{L}(E)$) is *strictly positive* if $f(x) > 0$ (resp. $Ux \in K$ and $Ux \neq 0$) for every nonzero x in K. T is *irreducible*[5], if there exists a natural number $n = n(x, f)$ such that $f(T^n x) > 0$ for every nonzero x in K and for every nonzero f in K'.

PROPOSITION 3. *Let T be an irreducible operator with the property (W) and r belong to the point spectrum of T. Then r is positive and every proper vector, corresponding to r, of T in K (resp. of T' in K') is a non-support point of K (resp. a strictly positive linear form).*

PROOF. Since r belongs to the point spectrum of T, the operator P'_* is nonzero by Lemma 2. So P_* is nonzero projection to the proper space of T', corresponding to r, hence $P_* K'$ has nonzero element, since K' is generating. By using nonzero $f_0 \in P_* K'$, we can prove in the same way as Th. 2 of [5].

PROPOSITION 4. *Let T be nonzero operator with the property (W). If the following condition* (a) *is satisfied, the spectral radius r is positive and P_* is strictly positive. If the conditions* (a) *and* (b) *are satisfied, T is irreducible.*

(a) *The proper space of T, corresponding to r, contains a non-support point x_0 of K.*

(b) *Every proper vector of T', corresponding to r, in K' is a strictly positive linear form.*

PROOF. For every nonzero $f \in K'$, the relation $P_* f(x_0) = f(P'_* x_0) = f(x_0) > 0$ follows from Lemma 2 and the condition (a). So P_* is strictly positive. That r is positive comes from the relations $T'f(x_0) = f(Tx_0) = f(rx_0) = rf(x_0)$ and $T \neq 0$ and the fact that x_0 is a non-support point of K. We can prove that T is irreducible in a similar way as Th. 2 of [5], using the property of the ultrafilter limit instead of the C. Neumann's series.

By Prop. 3 and 4, we get the following theorem.

THEOREM 1. *Let nonzero positive operator $T \in \mathfrak{L}(E)$ have the property (W) and the proper space of T, corresponding to r, contain nonzero point of K. Then* i) *and* ii) *are equivalent.*

i) *T is irreducible.*

ii) (a) *The proper space of T, corresponding to r, contains a non-support point of K.*

5) This definition of irreducibility has been introduced by I. Sawashima in [5] as semi-non-supportness.

(b) *Every proper vector of T', corresponding to r, in K' is strictly positive.*

REMARK. The condition (a) in Theorem 1 can be replaced by the following:

(a$'$) Every proper vector of T, corresponding to r, in K is a non-support point.

4. The property that the proper spaces of T and T', corresponding to r, are one-dimensional

THEOREM 2. *Let T have the property (W) and $r(T)=1$ belong to the point spectrum of T. Then the following* i) *and* ii) *are equivalent.*

i) *T is ergodic and the proper space of T, corresponding to* 1, *is one-dimensional.*

ii) *The proper space of T', corresponding to* 1, *is one-dimensional.*

PROOF. i)$\Rightarrow$ii) is obvious.

ii)$\Rightarrow$i): Let the proper space of T', corresponding to 1, be one-dimensional. Since T has the property (W), $P_*E'=\{f\in E': T'f=f\}$ is one-dimensional by Prop. 1 and the assumption. Hence by fixing some nonzero $f_0\in E'$ and some nonzero $x_0\in E$ such that $T'f_0=f_0$, $Tx_0=x_0$ and $f_0(x_0)=1$, we have $P_*f=f(x_0)f_0$ for every $f\in E'$. So for every $x\in E$, the relation $P_*'x=f_0(x)x_0$ is obtained by $P_*'x(f)=f_0(x)x_0(f)$. Since x_0 is an element of E, $P_*'E$ is contained in E and $P_*'|E$ is a projection to the space $\{x\in E: Tx=x\}$ by Lemma 2. By Prop. 2, we get the statement i).

REMARK. In the proof, we did not use any order property in E. Therefore Theorem 2 can be applied to operators in arbitary Banach spaces.

THEOREM 3. *Let T be irreducible and r belong to the point spectrum of T. Then the following are equivalent.*

i) *The proper space of T, corresponding to r, is a one-dimensional subspace spanned by a non-support point of K.*

ii) *There exists a bounded positive projection Q to the proper space of T, corresponding to r.*

Moreover, if there exists a strictly positive linear form f_0 such that $T'f_0=rf_0$, the projection Q is strictly positive.

PROOF. When there exists a bounded positive projection Q to the proper space, we can prove that the proper space of T, corresponding to r, is one-dimensional in the similar way as Th. 1 of [5]. Conversely, suppose the proper space is one-dimensional and x_0 is a non-support point of K such that $Tx_0=rx_0$. Since $f(x_0)>0$ for every nonzero $f\in K'$, we can find $f_1 \in K'$ such that $f_1(x_0)=1$. Put $Qx=f_1(x)x_0$. Then Q is a desired one.

If there exists a strictly positive linear form f_0, then $f_0(x_0)=a>0$ and $Qx=(1/a)f_0(x)x_0 \neq 0$ for every nonzero $x \in K$. So Q is strictly positive.

The two results obtained by I. Sawashima [5] and H. H. Schaefer [8] follow Th. 1, 2 and 3. The following is the extension of Prop. 8.5 of [8] to operators in partially ordered Banach spaces by the above theorems.

COROLLARY. *Let T be ergodic, P be the projection associated with ergodicity and $r(T)=1$ belong to the point spectrum of T. Then the following are equivalent.*

(a) *T is irreducible.*

(b) *P is strictly positive, with range a one-dimensional subspace of E spanned by a non-support point of K.*

(c) *P' is strictly positive, with range a one-dimensional subspace of E' spanned by a strictly positive linear form.*

PROOF. (a)$\Rightarrow$(b): Since P is nonzero projection and the cone K is generating, PK has nonzero element. (b) is obtained by Prop. 3 and Th. 3. (b)$\Rightarrow$(c) is obtained by Prop. 4 and Th. 2. (c)$\Rightarrow$(a) is obtained by the following Lemma and Th. 1.

LEMMA 3. *Let T have the property (W) and $r(T)=1$ belong to the point spectrum of T. Then the following are equivalent.*

i) *The projection P_* is strictly positive, with range a one-dimensional subspace of E' spanned by a strictly positive linear form.*

ii) *The proper space of T', corresponding to 1, is a one-dimensional subspace of E' spanned by a strictly positive linear form and the proper space of T, corresponding to 1, contains a non-support point of K.*

PROOF. i)$\Rightarrow$ii): By the same way as the proof of ii)$\Rightarrow$i) of Th. 2, we can find nonzero $x_0 \in E$ and nonzero $f_0 \in K'$ such that $Tx_0=x_0$, $T'f_0=f_0$ and $P_*f=f(x_0)f_0$ for every $f \in E'$. Since P_* is strictly positive, $f(x_0)>0$ for every $f \in K'$. Then x_0 is a non-support point of K. ii)$\Rightarrow$i) is obtained by Prop. 4.

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