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On a Characterization of Quasi-Complete p-Groups

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In this paper, we investigate some abelian p-groups. Notation and terminology follow [1] and [2].

Let us recall the definition of subdirect sum of two groups. Let A be an abelian group which has a direct decomposition $A=A_1\oplus A_2$. A subgroup G of A is said to be a subdirect sum of A_1 and A_2 if the projections $\pi_1|G\to A_1$ and $\pi_2|G\to A_2$ are epimorphisms. The kernels of π_1 and π_2 are called the kernels of this subdirect sum. Then the following isomorphisms hold:

$$G/(A_1\cap G\oplus A_2\cap G)\cong A_1/A_1\cap G\cong A_2/A_2\cap G\cong (A_1\oplus A_2)/G,$$

$$G/A_1\cap G\cong A_2 \quad \text{and} \quad G/A_2\cap G\cong A_1.$$

Let G be an abelian p-group with no elements of infinite height. We consider the following two properties on G.

(I) Let B be a basic subgroup of G such that $B\subset G\subset \overline{B}$ and $B=B_1\oplus B_2$ a direct decomposition of B into unbounded direct summands. Then G is a subdirect sum of $\overline{B}_1$ and $\overline{B}_2$.

(II) Let B, B_1 and B_2 be the same as in (I). Then G is a subdirect sum of $\overline{B}_1$ and $\overline{B}_2$ with pure kernels.

Clearly the property (II) implies the property (I). The property (II) will give us a characterization of quasi-complete p-groups. A quasi-complete p-group is a reduced p-group in which the closure (with respect to p-adic topology) of every pure subgroup is again pure.

PROPOSITION 1. *Let G be an abelian p-group with no elements of infinite height. For the quasi-completeness of G, it is necessary and sufficient that G satisfies the property* (II).

PROOF. The necessity follows from Prop. 74.7 in [2]. Suppose G has the property (II), and let P be a pure subgroup of G. If P is a group of bounded order, then P is a direct summand of G. Hence the closure of P coincides with P itself which is pure in G. If P is an unbounded pure subgroup of G, P has an unbounded basic subgroup B_1 such that $B_1\subset P\subset \overline{B}_1$ where the closure of P in G is $\overline{B}_1\cap G$. B_1 can be enlarged to $B_1\oplus B_2$ which is a basic subgroup of G, since P is pure in G. If B_2 is a group of bounded order, then $G=\bar{B}_1\cap G\oplus B_2$. Hence the closure of P is again pure in G since $\bar{B}_1\cap G$ is a direct summand of G. Let B_1 and B_2 be unbounded. From the property (II), $\bar{B}_1\cap G$ is pure in G. We have just covered all cases and our assertion has been proved.

Next proposition tells us that a group G which satisfies the property (I) is very close to its torsion-completion.

PROPOSITION 2. *Let G be an abelian p-group with no elements of infinite height and B a basic subgroup of G such that $B\subset G\subset \bar{B}$. If G satisfies the property* (I), *then* $|\bar{B}/G|\leqq 2^{\aleph_0}$.

PROOF. If G is a group of bounded order, then $\bar{B}=G$. Suppose G is unbounded and let $B=B_1\oplus B_2$ be a decomposition of B into two unbounded summands such that $|B_1|=\aleph_0$. From the property (I), it follows that G is a subdirect sum of $\bar{B}_1$ and $\bar{B}_2$. Hence $(\bar{B}_1\oplus\bar{B}_2)/G\cong\bar{B}_1/\bar{B}_1\cap G$. Since $|\bar{B}_1|=2^{\aleph_0}$, $|\bar{B}/G|\leqq 2^{\aleph_0}$.

COROLLARY. *Let G be a quasi-complete p-group and B its basic subgroup. Then* $|\bar{B}/G|\leqq 2^{\aleph_0}$.

References

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[3] P. Hill and C. Megibben: Quasi-closed primary groups, Acta Math. Acad. Sci. Hungar., 16 (1965), 271-274.