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On the Isometric Deformation Vector of the Hypersurface in Riemannian Manifolds

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Introduction

In their paper [1], Goldstein and Ryan formulated the theory of infinitesimal rigidity for submanifolds, and then specialized to the case where the ambient space has constant curvature to obtain some interesting results concerning infinitesimal rigidity of spheres. The notion of infinitesimal rigidity occurs in connection with the deformation of submanifolds. Let $\tilde{M}$ be a Riemannian manifold with metric $\tilde{g}$ and M a manifold. If γ_t be a deformation of an immersion $\gamma_0: M \rightarrow \tilde{M}$, a vector field $z=(d\gamma_t/dt)_{t=0}$ on $\gamma_0(M)$ is associated to γ_t naturally. When the family of induced metrics $g_t=\gamma_t^* \tilde{g}$ on M satisfies $(dg_t/dt)_{t=0}=0$, γ_t is called infinitesimal isometric. In this case with codimension 1 the corresponding z satisfies a nice equation ((2) in § 2), which enables us to make use of Yano-Bochner's technics [2]. Especially we get Theorem 4 as a replica of the famous following theorem due to Yano: *In a compact orientable Riemannian manifold, an infinitesimal affine transformation is necessarily an isometry.*

In this paper, all manifolds, tensors and maps are assumed to be C^∞. All manifolds are assumed connected. $\tilde{M}^{n+1}$ is an $n+1$ dimensional Riemannian manifold with positive definite metric $\tilde{g}=(\tilde{g}_{\lambda\mu})$. M^n is an n dimensional manifold. A submanifold $S=(M^n, r)$ in $\tilde{M}^{n+1}$ consists of M^n and an immersion $r: M^n \rightarrow \tilde{M}^{n+1}$. If M^n is compact orientable, we shall say that S is compact orientable. The ranges of indices are as follows:

$$\alpha, \beta, \cdots, \lambda, \mu, \cdots = 1, \cdots, n+1 ,$$
$$a, b, \cdots, r, s, \cdots = 1, \cdots, n .$$

§ 1. Preliminaries

Consider a submanifold $S=(M^n, r)$ in $\tilde{M}^{n+1}$. For simplicity, we shall identify $r(M^n)$ with M^n locally. Let $x^\lambda = x^\lambda(u^a)$ be the local expression of M^n in $\tilde{M}^{n+1}$ in terms of local coordinates $\{u^a\}$ of M^n and $\{x^\lambda\}$ of $\tilde{M}^{n+1}$. The induced metric $g=r^*\tilde{g}=(g_{ab})$ on M^n is given by

$$\tilde{g}_{\lambda\mu}B_a^\lambda B_b^\mu=g_{ab} ,$$

where we have put

$$B_a^\lambda=\frac{\partial x^\lambda}{\partial u^a} .$$

Let $N=(N^\lambda)$ be a unit normal vector (local) field and $N_\lambda=\tilde{g}_{\lambda\mu}N^\mu$ be its covariant components. Denoting by (g^{ab}) the inverse of (g_{ab}) and putting

$$\begin{gathered} B_\lambda^a=g^{ab}B_b^\mu\tilde{g}_{\mu\lambda} , \\ B_a^\lambda B_\lambda^b=\delta_a^b , \qquad N^\lambda N_\lambda=1 , \\ B_a^\lambda N_\lambda=0 , \qquad N^\lambda B_\lambda^a=0 , \end{gathered}$$

viz. the metrix (B_λ^a, N_λ) is the inverse of the matrix (B_a^λ, N^λ). Hence, $B_a^\lambda B_\mu^a=\delta_\mu^\lambda-N^\lambda N_\mu$ are valid. If we put $B^{a\lambda}=g^{ab}B_b^\lambda$ and $B_{a\lambda}=\tilde{g}_{\lambda\mu}B_a^\mu$, the following relations hold good:

$$B^{a\lambda}B_{b\lambda}=\delta_b^a , \qquad B^{a\lambda}N_\lambda=0 , \quad \text{etc.}$$

Let $\tilde{\nabla}$ and ∇ be the Riemannian connections with respect to $\tilde{g}$ and g respectively. ∇ operates not only on the quantities in M^n but the ones defined along M^n, [3]. For example, it is well known that

(i) for $\tilde{g}$ along M^n, we have

$$\begin{aligned} \nabla_a\tilde{g}_{\lambda\mu}&=\frac{\partial\tilde{g}_{\lambda\mu}}{\partial u^a}-B_a^\nu\widetilde{\begin{Bmatrix}\varepsilon\\ \nu\lambda\end{Bmatrix}}\tilde{g}_{\varepsilon\mu}-B_a^\nu\widetilde{\begin{Bmatrix}\varepsilon\\ \nu\mu\end{Bmatrix}}\tilde{g}_{\lambda\varepsilon} \\ &=B_a^\nu\tilde{\nabla}_\nu g_{\lambda\mu}=0 , \end{aligned}$$

(ii) for a vector field η^λ defined in a neighbourhood of M^n in $\tilde{M}^{n+1}$, we have

$$\nabla_a\eta^\lambda=\frac{\partial\eta^\lambda}{\partial u^a}+B_a^\mu\widetilde{\begin{Bmatrix}\lambda\\ \mu\nu\end{Bmatrix}}\eta^\nu=B_a^\mu\tilde{\nabla}_\mu\eta^\lambda ,$$

(iii) for B_a^λ, we have

$$\nabla_aB_b^\lambda=\frac{\partial B_b^\lambda}{\partial u^a}+B_a^\mu\widetilde{\begin{Bmatrix}\lambda\\ \mu\nu\end{Bmatrix}}B_b^\nu-\begin{Bmatrix}c\\ ab\end{Bmatrix}B_c^\lambda=\nabla_bB_a^\lambda .$$

The Euler-Schouten tensor $H_{ab}{}^\lambda$ of M^n is defined by $H_{ab}{}^\lambda=\nabla_aB_b^\lambda$. As we have

$$\nabla_a g_{bc} = 0 = \nabla_a(\tilde{g}_{\mu\nu} B_b^\mu B_c^\nu) = \tilde{g}_{\mu\nu}(H_{ab}{}^\mu B_c^\nu + B_b^\mu H_{ac}{}^\nu)$$
$$= H_{ab}{}^\mu B_{c\mu} + H_{ac}{}^\mu B_{b\mu} ,$$

$H_{ab}{}^\mu B_{c\mu}$ is skew-symmetric with respect to b and c. On the other hand, it being symmetric with respect to a and b, we have $H_{ab}{}^\mu B_{c\mu} = 0$ and $H_{ab}{}^\lambda$ can be written in the form

$$H_{ab}{}^\lambda = h_{ab} N^\lambda .$$

h_{ab} is called the second fundamental tensor of M^n. It is easy to see that the following equation holds:

$$\nabla_a N^\lambda = -h_a^e B_e^\lambda .$$

§ 2. Infinitesimal affine deformation

Consider a submanifold $S = (M^n, r)$ in $\tilde{M}^{n+1}$. Let $I = (-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$. A map

$$\gamma : I \times M^n \longrightarrow \tilde{M}^{n+1} , \qquad (t, u) \longrightarrow \gamma_t(u) ,$$

is called a deformation of S if $\gamma_0 = r$ and γ_t is an immersion for each $t \in I$.

For each t, the induced metric on M^n will be denoted by $g(t) = \gamma_t^* \tilde{g}$ and its Riemannian connection by $\nabla(t)$.

DEFINITION. *A deformation γ is called*
(i) *affine, if $\nabla(t) = \nabla(0)$ for $t \in I$,*
(ii) *infinitesimal affine, if $\nabla'(0) = 0$,*
(iii) *isometric, if $g(t) = g(0)$ for $t \in I$,*
(iv) *infinitesimal isometric, if $g'(0) = 0$,*
where dash means the differentiation with respect to t.

It is evident that an (infinitesimal) isometric deformation is (infinitesimal) affine.

For each $u \in M^n$, let $z_{t,u}$ be the tangent vector to the curve $t \longrightarrow \gamma_t(u)$ at t, viz. $z_{t,u} = \partial \gamma_t(u)/\partial t$. For simplicity, we shall denote $z_{t,u}$ by z *at* t, and $z_{0,u}$ by z.

In the rest of this section we find the condition for γ in order to be infinitesimal affine or isometric in terms of z.

We shall use the same notations as in § 1 for $r(M^n) = M^n$ (local identification) and for $\gamma_t(M^n)$ the corresponding notations with *at* t.

First we have

$$\begin{aligned}\frac{\partial}{\partial t} g_{ab} &= \frac{\partial}{\partial t}(\tilde{g}_{\lambda\mu} B_a^\lambda B_b^\mu) \\ &= z^\nu \frac{\partial \tilde{g}_{\lambda\mu}}{\partial x^\nu} B_a^\lambda B_b^\mu + \tilde{g}_{\lambda\mu}\left(\frac{\partial z^\lambda}{\partial u^a} B_b^\mu + B_a^\lambda \frac{\partial z^\mu}{\partial u^b}\right) \\ &= B_{b\lambda} \nabla_a z^\lambda + B_{a\mu} \nabla_b z^\mu \quad at\ t\ , \\ \frac{\partial}{\partial t} g^{ab} &= -g^{ac} g^{be} \frac{\partial}{\partial t} g_{ce} \\ &= -(B_\lambda^a \nabla^b z^\lambda + B_\lambda^b \nabla^a z^\lambda) \quad at\ t\ ,\end{aligned}$$

where g and ∇ mean (of course) $g(t)$ and $\nabla(t)$, etc.

For each t, let

$$z^\lambda = B_e^\lambda \xi^e + \psi N^\lambda \qquad at\ t$$

be the decomposition of z at t into the tangential and the normal components. Then, as we have

$$\nabla_a z^\lambda = B_e^\lambda (\nabla_a \xi^e - \psi h_a^e) + (h_{ae} \xi^e + \nabla_a \psi) N^\lambda \qquad at\ t\ ,$$

the following equation holds good:

$$(1) \qquad \frac{\partial}{\partial t} g_{ab} = \nabla_a \xi_b + \nabla_b \xi_a - 2\psi h_{ab} \qquad at\ t\ .$$

Consequently we have

THEOREM 1. (Goldstein-Ryan). *Let γ be a deformation of a submanifold $S=(M^n, r)$ in $\tilde{M}^{n+1}$. In order that γ be infinitesimal isometric, it is necessary and sufficient that $z^\lambda = B_e^\lambda \xi^e + \psi N^\lambda$ on $r(M^n)$ satisfies*

$$B_{b\lambda} \nabla_a z^\lambda + B_{a\lambda} \nabla_b z^\lambda = 0\ ,$$

or equivalently

$$(2) \qquad \nabla_a \xi_b + \nabla_b \xi_a = 2\psi h_{ab}\ .$$

REMARK. As is seen in [1], we know from (2) that the domain where $\xi^e = 0$ but $\psi \neq 0$ is totally geodesic.

Now we put

$$A_{bc} = \nabla_b \xi_c + \nabla_c \xi_b - 2\psi h_{bc} \qquad at\ t$$

for a deformation γ. By (1) and a computation we can get

$$\begin{aligned}\frac{\partial}{\partial t}\left\{ {a \atop bc} \right\} &= \frac{1}{2} g^{ae} (\nabla_b A_{ce} + \nabla_c A_{be} - \nabla_e A_{bc}) \\ &= \nabla_b \nabla_c \xi^a + R_{ebc}{}^a \xi^e - \nabla_b(\psi h_c^a) - \nabla_c(\psi h_b^a) + \nabla^a(\psi h_{bc}) \qquad at\ t\ ,\end{aligned}$$

where $R_{ebc}{}^{a}$ is the curvature tensor of $g(t)$.

Thus we get

THEOREM 2. *In order that a deformation γ of $S=(M^n, r)$ in $\tilde{M}^{n+1}$ be infinitesimal affine, it is necessary and sufficient that $z^\lambda = B_e^\lambda \xi^e + \psi N^\lambda$ on $r(M^n)$ satisfies*

$$\nabla_b \nabla_c \xi^a + R_{ebc}{}^{a} \xi^e = \nabla_b(\psi h_c^a) + \nabla_c(\psi h_b^a) - \nabla^a(\psi h_{bc}) .$$

REMARK. A deformation γ is called infinitesimal volume-preserving, if

$$(\sqrt{\det (g_{ab}(t))})'_{t=0} = 0 .$$

The condition in terms of z is $B_\lambda^a \nabla_a z^\lambda = 0$ on $r(M^n)$, which is equivalent to $\nabla_a \xi^a = \psi h$, where we have put

$$h = h_a^a .$$

§ 3. Non-existence of isometric deformation

We assume that $\tilde{M}^{n+1}$ is orientable, and consider a deformation γ of a compact orientable submanifold $S=(M^n, r)$. Then, by the local identification we can associate to $z^\lambda = B_e^\lambda \xi^e + \psi N^\lambda$ on $r(M^n)$ the pair (ξ, ψ) which consists of a vector field $\xi = (\xi^a)$ and a function ψ, defined globally on M^n. If we take account of Theorem 1, 2 and Remark, the pair (ξ, ψ) in the following definition would be worthy to be studied.

DEFINITION. *Let $S=(M^n, r)$ be a compact orientable submanifold in an orientable Riemannian manifold $\tilde{M}^{n+1}$. By a deformation vector we mean (ξ, ψ) which is a pair of a vector field ξ and a function ψ on M^n. A deformation vector (ξ, ψ) is called*

(i) *isometric, if it satisfies*

$$\nabla_a \xi_b + \nabla_b \xi_a = 2\psi h_{ab} , \tag{3}$$

(ii) *affine, if it satisfies*

$$\nabla_b \nabla_c \xi^a + R_{ebc}{}^{a} \xi^e = \nabla_b(\psi h_c^a) + \nabla_c(\psi h_b^a) - \nabla^a(\psi h_{bc}) , \tag{4}$$

(iii) *volume-preserving, if it satisfies*

$$\nabla_e \xi^e = \psi h , \tag{5}$$

with respect to the induced metric $g = r^* \tilde{g}$ and a fixed unit normal vector field on $r(M^n)$, where we regard h_{ab} as a tensor field on M^n through the local identification of $r(M^n)$ with M^n.

It is easy to see that an isometric deformation vector is necessarily affine and volume-preserving.

Now we shall prove the following

THEOREM 3. *Let $S=(M^n, r)$ be a compact orientable submanifold in an orientable Riemannian manifold $\tilde{M}^{n+1}$. If S satisfies*

(i) $h^2-2h_{ab}h^{ab}>0$,

(ii) *the Ricci form is negative definite*

with respect to the induced metric, then there does not exist an isometric deformation vector except the zero.

PROOF. Let (ξ, ψ) be an isometric deformation vector. It satisfies (3), (4) and (5). From (3) we have

$$h^{ab}\nabla_a\xi_b=\psi h_{ab}h^{ab} \tag{6}$$

and from (4)

$$\nabla_b\nabla^b\xi^a+R^a_b\xi^b=2\nabla^b(\psi h^a_b)-\nabla^a(\psi h)\ , \tag{7}$$

where $R_{be}=R_{cbe}{}^c$ denotes the Ricci tensor. By (7) we have

$$\begin{aligned}\frac{1}{2}\nabla^b\nabla_b(\xi_a\xi^a)&=\xi_a\nabla^b\nabla_b\xi^a+\nabla_b\xi_a\nabla^b\xi^a\\ &=-R_{ab}\xi^a\xi^b+2\xi^a\nabla_b(\psi h^b_a)-\xi^a\nabla_a(\psi h)+\nabla_b\xi_a\nabla^b\xi^a\end{aligned}$$

and by integration with respect to the volume element $d\sigma$

$$\int_M R_{ab}\xi^a\xi^b d\sigma=\int_M \{2\xi^a\nabla_b(\psi h^b_a)-\xi^a\nabla_a(\psi h)+\nabla_b\xi_a\nabla^b\xi^a\}d\sigma\ . \tag{8}$$

On the other hand, if we take account of (6) and (5), it follows that

$$\int_M \xi^a\nabla_b(\psi h^b_a)d\sigma=-\int_M \psi h^{ab}\nabla_b\xi_a d\sigma=-\int_M \psi^2 h_{ab}h^{ab}d\sigma\ ,$$

$$\int_M \xi^a\nabla_a(\psi h)d\sigma=-\int_M \psi h\nabla_a\xi^a d\sigma=-\int_M \psi^2 h^2 d\sigma\ .$$

Substituting these equations into (8) we obtain

$$\int_M R_{ab}\xi^a\xi^b d\sigma=\int_M \{\psi^2(h^2-2h_{ab}h^{ab})+\nabla_b\xi_a\nabla^b\xi^a\}d\sigma$$

and hence complete the proof, taking account of the assumptions.

COROLLARY. *Under the assumptions in Theorem 3, there does not exist an isometric deformation of S other than the identity.*

§ 4. Isometric deformation vector

As in § 3, let us suppose that $S=(M^n, r)$ is a compact orientable submanifold in an orientable Riemannian manifold $\tilde{M}^{n+1}$.

We consider an affine deformation vector (ξ, ψ). It satisfies

$$\nabla_b \nabla_c \xi_a + R_{ebca} \xi^e = \nabla_b(\psi h_{ca}) + \nabla_c(\psi h_{ba}) - \nabla_a(\psi h_{bc}) ,$$

and we can get the following three equations:

$$\nabla^b \nabla_b \xi_a + R_{ea} \xi^e = 2\nabla^b(\psi h_{ba}) - \nabla_a(\psi h) , \tag{9}$$

$$\nabla_a \xi^a = \psi h + C , \qquad C: \text{const.} , \tag{10}$$

$$h^{ca} \nabla_b(\nabla_c \xi_a - \psi h_{ca}) = 0 . \tag{11}$$

Now we need the following

DEFINITION. *A deformation vector (ξ, ψ) is called normal (or tangential), if ξ (or ψ) vanishes identically.*

First we shall prove the following

THEOREM 4. *Let $S=(M^n, r)$ be a compact orientable submanifold in an orientable Riemannian manifold $\tilde{M}^{n+1}$. Let us assume that $r(M^n)$ has the parallel second fundamental tensor and is not minimal, or equivalently*

$$\nabla_a h_{bc} = 0 \quad \textit{and} \quad h \neq 0 .$$

Then if M^n admits a non-normal affine deformation vector, it also admits an isometric deformation vector which is different from the zero.

PROOF. Let (ξ, ψ) be a non-normal affine deformation vector. It satisfies (9), (10) and (11). First we have

$$h^{ca}(\nabla_c \xi_a - \psi h_{ca}) = C_1 \quad \text{(const.)} \tag{12}$$

by virtue of (11) and $\nabla_a h_{bc} = 0$. Next, let us define f by

$$hf = \nabla_a \xi^a , \tag{13}$$

then it follows from (10) that

$$\psi = f - \frac{C}{h} ,$$

and we have from (9) and (12)

(14) $$\nabla^b \nabla_b \xi_a + R_{ba}\xi^b = 2f_b h_a^b - f_a h\ ,$$

(15) $$h^{ca}(\nabla_c \xi_a - f h_{ca}) = C_2 \quad \text{(const.)}\ ,$$

where $f_a = \nabla_a f$.

Now we put

(16) $$A_{ab} = \nabla_a \xi_b + \nabla_b \xi_a - 2f h_{ab}\ ,$$

where f is the one given by (13). Then, taking account of (13), (14) and (15) we have

(17) $$\begin{aligned} \| A_{ab} \|^2 = A_{ab}A^{ab} &= 2A_{ab}\nabla^a \xi^b - 4f h^{ab}(\nabla_a \xi_b - f h_{ab}) \\ &= 2A_{ab}\nabla^a \xi^b - 4C_2 f\ , \end{aligned}$$

(18) $$\nabla^a A_{ab} = \nabla^a \nabla_a \xi_b + R_{ba}\xi^a - 2f^a h_{ab} + \nabla_b \nabla_a \xi^a = 0\ .$$

If we substitue these equations into

(19) $$\nabla^a(A_{ab}\xi^b) = \nabla^a A_{ab}\xi^b + A^{ab}\nabla_a \xi_b$$

and integrate over M^n, it follows that $\| A_{ab} \|^2 = 0$ and (ξ, f) is isometric. If $(\xi, f) = (0, 0)$, then (ξ, ψ) is normal, which contradicts the assumption.

From the above proof we have

COROLLARY. *Under the assumption in Theorem* 4, *if* (ξ, ψ) *is a volume-preserving affine deformation vector, then it is isometric.*

Next we shall consider a case of $h = 0$.

THEOREM 5. *Let* $S = (M^n, r)$ *be a compact orientable submanifold in an orientable Riemannian manifold* $\tilde{M}^{n+1}$. *Let us assume that* $r(M^n)$ *has the parallel second fundamental tensor and is minimal but not totally geodesic, or equivalently*

$$\nabla_a h_{bc} = 0\ , \qquad h = 0 \quad \text{and} \quad h^{ab}h_{ab} \neq 0\ .$$

Then, if M^n *admits a non-normal affine deformation vector, it also admits an isometric deformation vector which is different from the zero.*

PROOF. This case, let us define f by

(20) $$h^{ca}(\nabla_c \xi_a - f h_{ca}) = 0\ .$$

Then we have

$$\psi = f - \frac{C_1}{h_{ca}h^{ca}}$$

by virtue of (12), and from (9) and $h=0$ it follows

$$\nabla^b \nabla_b \xi_a + R_{ba} \xi^b = 2 f_b h^b_a .$$

As in the proof of Theorem 4, let us define A_{ab} by (16) with f of (20). Then we obtain

$$|| A_{ab} ||^2 = 2 A_{ab} \nabla^a \xi^b ,$$
$$\nabla^a A_{ab} = \nabla^a \nabla_a \xi_b + R_{ba} \xi^b - 2 f^a h_{ab}$$

taking account of (17), (18) and (10). Thus we can complete the proof making use of (19).

REMARK. After formulating this paper, I was informed from Prof. K. Yano that he had made a similar work [4]. Then a cooperated work [5] was done about complex hypersurfaces in Kählerian manifolds.

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