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On Simple Groups of Order $2^a 3^b p^c$ with a Cyclic Sylow Subgroup

Keiko Sentoh

Department of Mathematics, Faculty of Science
Ochanomizu University, Tokyo

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1. Introduction

The problem of classification of all simple groups whose order is divisible by exactly three primes, is the well-known open question in group theory.

The only known simple groups with this property are $PSL(2, q)$, $q=5, 7, 8, 9$ and 17, $PSL(3, 3)$, $U(3, 3)$ and $Q(5, 3)$.

J. Thompson's classification [7] of minimal simple groups shows the following result; If the order of a non-solvable finite group is divisible by three primes only, then the primes are 2, 3, and p, where p belongs to the set $\{5, 7, 13, 17\}$.

There has been the conjecture that every simple group has a cyclic Sylow subgroup.

We note here some known results on a simple group G of order $2^a 3^b p^c$ with a cyclic Sylow subgroup S.

R. Brauer [1] proved that the cyclic Sylow subgroup is self-centralizing. The Burnside's theorem implies that $[N_G(S):C_G(S)]\neq 1$. M. Herzog proved in [3] that if $[N_G(S):C_G(S)]=2$, then G is isomorphic to one of the groups; $PSL(2, q)$, $q=5, 7, 8, 9$, and 17.

The classification of simple groups of order $2^a 3^b p$ is completely known. (cf. [1], [8], [9] and [10]). Using this result, J. Leon [6] showed that a simple group of order $2^a 3^b 5^c$ which contains a cyclic Sylow subgroup is isomorphic to $PSL(2, 5)$.

In this paper, we prove the following theorem.

THEOREM.

Let G be a simple group of order $2^a 3^b p^c$, where p is a prime. If G has a cyclic Sylow subgroup S such that $[N_G(S):C_G(S)]=3$, then G is isomorphic to $PSL(2, 7)$ or $U(3, 3)$.

In order to prove the Theorem, we solve in Section 4 the degree equations for the principal block, applying the results of M. Herzog

[5] which relies on the paper of E. C. Dade [2] on blocks with cyclic defect groups.

In Section 2, we derive the preliminary results. In Section 3, we prove the Theorem assuming the Lemma 1. In Section 4, we complete the proof by solving the degree equations (Lemma 1).

2. Preliminary results

G will denote a simple group of order $2^a 3^b p^c$ containing a cyclic Sylow subgroup S such that $[N_G(S): C_G(S)]=3$. Since S is cyclic, $[N_G(S): C_G(S)]$ is relatively prime to the order of S. Hence S cannot be a Sylow 3-subgroup.

As mentioned in Section 1, the cyclic Sylow subgroup is self-centralizing. It is clear that the Sylow 2-subgroup cannot be cyclic from the Burnside's theorem on normal complements. Hence S is a Sylow p-subgroup P. $N_G(P)/C_G(P)$ is isomorphic to a p'-subgroup of $\mathrm{Aut}(P)$ of order $p^{c-1}(p-1)$ so that the order of $N_G(P)/C_G(P)$ divides $p-1$. Therefore $p=7$ or 13.

3. Proof of Theorem

E. C. Dade [2] described the characters in blocks with a cyclic defect group. Using Dade's result, Herzog [4] obtained the following description of the principal p-block of G. We note here the relevant results in [5, Prop. 1].

Let G be a finite group with a cyclic Sylow p-subgroup P of order p^c, $q=[N_G(P): C_G(P)]$ and B is the principal p-block of G. Then the following holds;

(1) B contains $q+(p^c-1)/q$ ordinary irreducible characters divided into two families:
the exceptional characters: $\{X_\lambda : \lambda \in \Lambda\}$ and
the non-exceptional characters: $\{X_i : i=1, \cdots, q\}$.

(2) $f_0=X_\lambda(1)\equiv -q\varepsilon_0 \pmod{p^c}$ for $\lambda \in \Lambda$ and
$f_i=X_i(1)\equiv \varepsilon_i \pmod{p^c}$ for $i=1, \cdots, q$,
where $\varepsilon_j=\pm 1$ for $j=0, 1, \cdots, q$.

(3) $\sum_{j=0}^{q} \varepsilon_j f_j=0$

In our situation, the principal p-block contains $3+(p^c-1)/3$ ordinary irreducible characters.

The characters in the first family are $(p^c-1)/3$ characters $\{X_\lambda : \lambda \in \Lambda\}$, all of the same degree. The characters in the second family are 3 ordinary irreducible characters X_1 (the identity character), X_2 and X_3. Let f_i denote the degree of X_i, for $i=2, 3$ and f_0 denote the common degree of X_λ.

It follows from Herzog's results that;

$$f_i \equiv \varepsilon_i \pmod{p^c} \qquad \text{for } i=2, 3 \quad \text{and}$$
$$f_0 \equiv -3\varepsilon_0 \pmod{p^c}\,, \quad \text{where } \varepsilon_0, \varepsilon_2, \varepsilon_3 = \pm 1\,.$$
$$1+\varepsilon_2 f_2+\varepsilon_3 f_3+\varepsilon_0 f_0 = 0\,.$$

The f_i are not divisible by p; hence we may write f_i as $2^{s_i}3^{t_i}$ for $i=2, 3, 0$, where the s_i and t_i are non-negative integers.

In both cases ($p=7$ and $p=13$), we obtain the following:

LEMMA 1.
If $c \geqq 2$, the equations

$$1+\varepsilon_2 2^{s_2}3^{t_2}+\varepsilon_3 2^{s_3}3^{t_3}+\varepsilon_0 2^{s_0}3^{t_0}=0$$
$$\varepsilon_i=\pm 1,\ 2^{s_i}3^{t_i} \equiv \varepsilon_i \pmod{p^c} \qquad \text{for } i=2, 3$$
$$\varepsilon_0=\pm 1,\ 2^{s_0}3^{t_0} \equiv -3\varepsilon_0 \pmod{p^c}$$

have no solution other than the following one in either case, $p=7$ or $p=13$;

$$\varepsilon_2 2^{s_2}3^{t_2}=\varepsilon_3 2^{s_3}3^{t_3}=1\,, \quad \varepsilon_0 2^{s_0}3^{t_0}=-3\,.$$

The proof of this lemma will be given in Section 4. The Lemma implies that G has a non-identity character X of degree 1 and hence the kernel of X contains $G'=G$ (because of the simplicity of G). This is a contradiction. Hence we shall have the following:

If G is a simple group of order $2^a3^bp^c$ containing a cyclic Sylow p-subgroup P such that $[N_G(P): C_G(P)]=3$, then $c=1$.

Taking into account of this restriction on the order of G and using the results of [8] and [9], we can determine the type of G of order $2^a3^bp^c$.

If $p^c=7$, we have the following possibilities: $PSL(2, 7)$ of order $2^3\cdot 3\cdot 7$, $PSL(2, 8)$ of order $2^3\cdot 3^2\cdot 7$ and $U(3, 3)$ of order $2^5\cdot 3^3\cdot 7$.

Since P is self-centralizing and $[N_G(P): C_G(P)]=3$, the number of Sylow 7-subgroup of G is $[G: N_G(P)]=2^a\cdot 3^{b-1}$, which must be congruent to 1 modulo 7. This eliminates $PSL(2, 8)$.

The other two cases satisfy the condition that $[N_G(P): C_G(P)]=3$. So G is isomorphic to $PSL(2, 7)$ or $U(3, 3)$.

If $p^c=13$, we have the only one possibility; $PSL(3, 3)$ of order $2^4\cdot 3^3\cdot 13$. In this case, the centralizer of a Sylow 13-subgroup has index 12 in its normalizer (by D. Wales [9]). This case is eliminated. Hence we obtain the Theorem.

4. Solving the degree equations

We discuss the proof of the Lemma 1 in this section. As it is impossible to write down the whole of actual calculations in the limited space, so an outline of the procedure will be given.

In the equation $1+\varepsilon_2 2^{s_2}3^{t_2}+\varepsilon_3 2^{t_3}3^{t_3}+\varepsilon_0 2^{s_0}3^{t_0}=0$, there must be some integers i and j $(i\neq j)$ such that $2^{s_i}3^{t_i}$ is odd and $2^{s_j}3^{t_j}$ is relatively prime to 3; so that $s_i=0$ and $t_j=0$. (Because if $i=j$, then $2^{s_i}3^{t_i}=2^0\cdot 3^0=1$; hence $\varepsilon_i=1$. So there must be some integer $k(\neq i=j)$ such that $2^{s_k}3^{t_k}$ is relatively prime to 3, that is, $t_k=0$. Thus, without loss of generality, $i\neq j$.)

We treat separately two cases: Case $A(p=7)$ and Case $B(p=13)$. In both cases, it follows from our assumptions that one of the following cases holds;

(I) $1+\varepsilon_2 2^{s_2}+\varepsilon_3 3^{t_3}+\varepsilon_0 2^{s_0}3^{t_0}=0,\ 2^{s_2}\equiv\varepsilon_2,\ 3^{t_3}\equiv\varepsilon_3,\ 2^{s_0}3^{t_0}\equiv-3\varepsilon_0 \pmod{p^c},$ $c\geqq 2$

(II) $1+\varepsilon_2 2^{s_2}+\varepsilon_3 2^{s_3}3^{t_3}+\varepsilon_0 3^{t_0}=0,\ 2^{s_2}\equiv\varepsilon_2,\ 2^{s_3}3^{t_3}\equiv\varepsilon_3,\ 3^{t_0}\equiv-3\varepsilon_0 \pmod{p^c},$ $c\geqq 2$

(III) $1+\varepsilon_2 3^{t_2}+\varepsilon_3 2^{s_3}3^{t_3}+\varepsilon_0 2^{s_0}=0,\ 3^{t_2}\equiv\varepsilon_2,\ 2^{s_3}3^{t_3}\equiv\varepsilon_3,\ 2^{s_0}\equiv-3\varepsilon_0 \pmod{p^c},$ $c\geqq 2$

We show here the procedure in Case A-(I). The other cases can be treated similarly.

Case A-(I) we solve the equation

$$1+\varepsilon_2 2^{s_2}+\varepsilon_3 3^{t_3}+\varepsilon_0 2^{s_0}3^{t_0}=0 \quad \text{(Eq. (I))}$$

with conditions $2^{s_2}\equiv\varepsilon_2,\ 3^{t_3}\equiv\varepsilon_3,\ 2^{s_0}3^{t_0}\equiv-3\varepsilon_0 \pmod{p^c}$ $c\geqq 2$.

The procedure for solving the equation divides into the following steps.

Step. 1. The ε_2, ε_3 and ε_0 take on the value -1 or $+1$, but not all ε_i can be equal. We have $\varepsilon_2=+1$ because $2^{s_2}\not\equiv -1 \pmod 7$. Hence there are three possible cases corresponding to different values for ε_2, ε_3, and ε_0. Since the exponents of 2 and 3 modulo 7 are 3 and 6, s_2 is computed modulo 3 and t_3 is computed modulo 6. Each case is

Table A-(I)

	ε_2	ε_3	ε_0	$\tilde{s}_2$	$\tilde{t}_3$	$\tilde{s}_0$	$\tilde{t}_0$
(1)	1	1	−1	0	0	0	1
(2)	1	1	−1	0	0	1	5
(3)	1	1	−1	0	0	2	3
(4)	1	−1	1	0	3	0	4
(5)	1	−1	1	0	3	1	2
(6)	1	−1	1	0	3	2	0
(7)	1	−1	−1	0	3	0	1
(8)	1	−1	−1	0	3	1	5
(9)	1	−1	−1	0	3	2	3

subdivided into three subcases according to the values of s_0 modulo 3. The t_0 is then computed modulo 6 from the condition $2^{s_0}3^{t_0} \equiv -3\varepsilon_0 \pmod 7$. Thus finally we have nine different cases, corresponding to different values for the ε_i, the s_i modulo 3 and the t_i modulo 6. Let the $\tilde{s}_i$ denote the values of the s_i modulo 3 and the $\tilde{t}_i$ denote the values of the t_i modulo 6. Table A-(I) gives these nine cases.

Step 2. Some cases of Table A-(I) can be eliminated by finding the cases in which the Eq. (I) is satisfied together with the above conditions. In Case (1), if $t_3=0$, then s_0, t_0 and s_2 are determined absolutely; $s_0=0$, $t_0=1$ and $s_2=0$. This is clearly one of the solutions of the Eq. (I). If $t_3 \neq 0$, $s_0=1$ and hence Case (3) is eliminated.

In the similar manner, it is not difficult to know that (4), (6), (7) and (9) can be eliminated. The remaining cases are (2), (5) and (8).

Step 3. In this step, we show that none of these three cases can be solutions of the Eq. (I). Here we use congruences modulo 73 to determine the possible values of the s_i and t_i modulo higher powers of 2 and 3. The exponents of 2 and 3 modulo 73 are 9 and 12. Then the left side of the Eq. (I) can be computed from the values of ε_i

Table A-(II)

	ε_2	ε_3	ε_0	$\tilde{s}_2$	$\tilde{s}_3$	$\tilde{t}_3$	$\tilde{t}_0$
(1)	1	1	−1	0	0	0	1
(2)	1	1	−1	0	1	4	1
(3)	1	1	−1	0	2	2	1
(4)	1	−1	1	0	0	3	4
(5)	1	−1	1	0	1	1	4
(6)	1	−1	1	0	2	5	4
(7)	1	−1	−1	0	0	3	1
(8)	1	−1	−1	0	1	1	1
(9)	1	−1	−1	0	2	5	1

Table A-(III)

	ε_2	ε_3	ε_0	$\tilde{t}_2$	$\tilde{s}_3$	$\tilde{t}_3$	$\tilde{s}_0$
(1)	1	−1	1	0	0	3	2
(2)	1	−1	1	0	1	1	2
(3)	1	−1	1	0	2	5	2
(4)	−1	1	1	3	0	0	2
(5)	−1	1	1	3	1	4	2
(6)	−1	1	1	3	2	2	2
(7)	−1	−1	1	3	0	3	2
(8)	−1	−1	1	3	1	1	2
(9)	−1	−1	1	3	2	5	2

and the values of the s_i modulo 9 and the t_i modulo 12. In Case (2) we determine s_2 modulo 9 and t_3 modulo 12. Then we know that $s_2 \equiv 0$ or 3 or 6 (mod 9), $t_3 \equiv 0$ or 6, $t_0 \equiv 5$ or 11 (mod 12) and $s_0 = 1$. There are 12 possibilities, according to the values of s_2, t_3, and t_0. In each case, we compute the left side of the Eq. (I) modulo 73 and discard

Table B-(I)

	ε_2	ε_3	ε_0	$\tilde{s}_2$	$\tilde{t}_3$	$\tilde{s}_0$	$\tilde{t}_0$
(1)	1	1	−1	0	0	4	0
(2)	1	1	−1	0	0	0	1
(3)	1	1	−1	0	0	8	2
(4)	−1	1	1	6	0	10	0
(5)	−1	1	1	6	0	6	1
(6)	−1	1	1	6	0	2	2
(7)	−1	1	−1	6	0	4	0
(8)	−1	1	−1	6	0	0	1
(9)	−1	1	−1	6	0	8	2

Table B-(II)

	ε_2	ε_3	ε_0	$\tilde{s}_2$	$\tilde{s}_3$	$\tilde{t}_3$	$\tilde{t}_0$
(1)	1	−1	−1	0	6	0	1
(2)	1	−1	−1	0	2	1	1
(3)	1	−1	−1	0	10	2	1
(4)	−1	1	−1	6	0	0	1
(5)	−1	1	−1	6	8	1	1
(6)	−1	1	−1	6	4	2	1
(7)	1	1	−1	0	0	0	1
(8)	1	1	−1	0	8	1	1
(9)	1	1	−1	0	4	2	1

Table B-(III)

	ε_2	ε_3	ε_0	$\tilde{t}_2$	$\tilde{s}_3$	$\tilde{t}_3$	$\tilde{s}_0$
(1)	1	1	−1	0	0	0	4
(2)	1	1	−1	0	8	1	4
(3)	1	1	−1	0	4	2	4
(4)	1	−1	1	0	6	0	10
(5)	1	−1	1	0	2	1	10
(6)	1	−1	1	0	10	2	10
(7)	1	−1	−1	0	6	0	4
(8)	1	−1	−1	0	2	1	4
(9)	1	−1	−1	0	10	2	4

all the cases for which we obtain a non-zero quantity modulo 73. There is no case in which the left side of the Eq. (I) is zero modulo 73. So (2) is discarded.

In Case (5), $s_2 \equiv 0$ or 3 or 6 (mod 9), $t_3 \equiv 3$ or 9, $t_0 \equiv 2$ or 8 (mod 12) and $s_0 = 1$. Except the following; $s_2 \equiv 0$ (mod 9), $t_3 \equiv 3$, $t_0 \equiv 2$ (mod 12) and $s_0 = 1$, there is no case in which the left side of the Eq. (I) is zero modulo 73. For this remaining case, we check the conditions that these ε_i, s_i, and t_i can satisfy the Eq. (I) with the above conditions. We find easily that s_2 is determined absolutely: $s_2 = 3$. On the other hand, $2^{s_2} \equiv 1 \pmod{7^2}$ because $2^{s_2} \equiv 1 \pmod{7^c}$ $c \geqq 2$. Since the exponent of 2 modulo 7^2 is 21 and $s_2 > 0$, $s_2 \geqq 21$, which is a contradiction. Thus (5) is discarded. In Case (8), $s_2 \equiv 0$ or 3 or 6 (mod 9), $t_3 \equiv 3$ or 9, $t_0 \equiv 5$ or 11 (mod 12) and $s_0 = 1$. The remaining case is the following: $s_2 \equiv 0$ (mod 9), $t_3 \equiv 3$, $t_0 \equiv 5$ (mod 12) and $s_0 = 1$. On the other hand, $3^{t_3} \equiv -1 \pmod{7^2}$ because $3^{t_3} \equiv -1 \pmod{7^c}$ $c \geqq 2$. Since the exponent of 3 modulo 7^2 is 42, $3^{21} \equiv -1 \pmod{7^2}$. Hence $t_3 \equiv 0$ (mod 21) and t_3 is odd.

Considering all these conditions we know that we have no solution in the remaining case. Hence (8) is discarded.

From the result of this procedure, the Eq. (I) with above conditions has only one solution: $\varepsilon_2 2^{s_2} = \varepsilon_3 3^{t_3} = 1$ and $\varepsilon_0 2^{s_0} 3^{t_0} = -3$.

We have shown here the procedure of solving the equation in Case A-(I). We can deal with both cases A-(II) and A-(III) similarly as in Case A-(I). The result is that the Eq. (II) has the only one solution ($\varepsilon_2 2^{s_2} = \varepsilon_3 2^{s_3} 3^{t_3} = 1$ and $\varepsilon_0 3^{t_0} = -3$) and the Eq. (III) has no solution. Thus in Case A, the Lemma is proved.

Proceeding similarly, the Lemma can be proved in Case B, where the s_i is computed modulo 12 and t_i is computed modulo 3 because the exponents of 2 and 3 modulo 13 are 12 and 3.

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