A Proof to Dieudonné's Theorem on Infinite Abelian p-groups

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Our purpose in this paper is to present a simple and shorter proof to Dieudonné's theorem (see [1]). Notation and terminology follow [2].

THEOREM. *Let G be an abelian primary group for some prime p. If G contains a subgroup A such that G/A is a direct sum of cyclic groups and that A is the union of an ascending chain $A_1 \subset A_2 \subset \cdots \subset A_i \subset \cdots$ of subgroups of bounded height taken in G, then G is a direct sum of cyclic groups.*

PROOF. Let $G[p]$ be the p-socle of G. It suffices to show that $G[p]$ is the union of an ascending chain of subgroups of bounded height (see Theorem 12 in [3]).

Since G/A is a direct sum of cyclic groups, G/A is the union of an ascending chain $\bar{G}_1 \subset \bar{G}_2 \subset \cdots \subset \bar{G}_i \subset \cdots$ of its subgroups of bounded height. Consequently, there exists an ascending chain

$$G_1 \subset G_2 \subset \cdots \subset G_i \subset \cdots$$

of subgroups of G containing A such that $G = \cup G_i$ and that the set of elements of G_i not belonging to A is of bounded height.

Since $G[p]$ is a vector space, there exists an ascending chain

$$G'_1 \subset G'_2 \subset \cdots \subset G'_i \subset \cdots$$

of subgroups of $G[p]$ such that $G_i[p] = A[p] \oplus G'_i$.

Consider following ascending chain

$$A_1[p] \oplus G'_1 \subset A_2[p] \oplus G'_2 \subset \cdots \subset A_i[p] \oplus G'_i \subset \cdots .$$

Then $\cup (A_i[p] \oplus G'_i) = G[p]$.

Since the set of elements of G_i not belonging to A is of bounded height as stated above, the set of elements of $A_i[p] \oplus G'_i$ not belonging to A is of bounded height. On the other hand, $(A_i[p] \oplus G'_i) \cap A = A_i[p]$ is of bounded height. This completes the proof.

References

[1] J. Dieudonné: Sur les p-groupes abéliens infinis, Portugal. Math. 11 (1952), 1–5.
[2] L. Fuchs: Infinite abelian groups, volume 1. Academic Press, 1970.
[3] I. Kaplansky: Infinite abelian groups. The University of Michigan Press, 1969.