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Some Remarks on Theorems of Korovkin type for Adapted Spaces

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§ 1. Introduction.

Korovkin's theorem [3], arisen from the study of the role of Bernstein polynomials in the proof of Weierstrass' approximation theorem, has been generalised in the case where X is a compact Hausdorff space and $\mathfrak{F}$ a separating subset of $C(X)$ containing constant 1.

Bauer has shown, making use of the theory of adapted spaces, that theorems of Korovkin type hold also for locally compact Hausdorff spaces.

In this paper we shall show that a theorem of Korovkin type in case of a lattice, instead of an algebra, will be obtained for locally compact Hausdorff spaces. Further, when P is an adapted cone, we shall give a sufficient condition for a sublattice of the vector lattice H_P of all P-bounded continuous functions to be dense in H_P.

§ 2. Linearly separating families in $R^{\times}$.

Throught this paper, let X be a locally compact Hausdorff space. Let $\mathfrak{F}$ be a family of real-valued functions defined on X. The family $\mathfrak{F}$ is said to be linearly separating if for any two different elements x, y of X and any $\lambda \geqq 0$, there is $f \in \mathfrak{F}$ such that $f(x) \neq \lambda f(y)$. Put

$$J\mathfrak{F} = \{nf\,;\ n \in \boldsymbol{N}, f \in \mathfrak{F}\} \cup \{-nf\,;\ n \in \boldsymbol{N}, f \in \mathfrak{F}\}\,.$$

For $f, g \in \mathfrak{F}$, the function:

$$x \longmapsto \max\{f(x), g(x)\} \qquad (\text{resp. } x \longmapsto \min\{f(x), g(x)\})$$

is denoted as usual by $f \vee g$ (resp. $f \wedge g$). Put

$$\mathfrak{F}_1 \vee \mathfrak{F}_2 = \{f \vee g\,;\ f \in \mathfrak{F}_1,\ g \in \mathfrak{F}_2\}$$

for two families $\mathfrak{F}_1$ and $\mathfrak{F}_2$ of real-valued functions. Then holds the following proposition.

PROPOSITION 1. *Let $\mathfrak{F}$ be a linearly separating family of real-valued functions. Then for any two different points x and y, there is* *a function* $v \in \hat{\mathfrak{F}} = \{J\mathfrak{F} \vee J\mathfrak{F} + J\mathfrak{F}\}$ *such that*

$$v(x)=0\,, \quad v(y)>0 \qquad \text{and} \qquad v \geqq 0 \text{ on } X\,. \tag{1}$$

PROOF. By the assumption there are $u \in \mathfrak{F}$ and $w \in \mathfrak{F}$ satisfying $u(y) \neq 0$ and $w(x) \neq \frac{u(x)}{u(y)} w(y)$. Assume that $u(x)=0$ and put

$$v_1 = u \vee 0 \qquad \text{and} \qquad v_2 = (-u) \vee 0\,.$$

Then both v_1 and v_2 belong to $\hat{\mathfrak{F}}$, and one of them has the property (1). Next, consider the case where $u(x) \neq 0$. Then, since $\frac{w(x)}{u(x)} \neq \frac{w(y)}{u(y)}$, there is a rational number $\frac{n}{m}$ between $\frac{w(x)}{u(x)}$ and $\frac{w(y)}{u(y)}$. Let $v = mw - nu$. Then holds

$$v(x)v(y) < 0\,.$$

Thus, one of two functions $v_1 = v \vee 0$ and $v_2 = (-v) \vee 0$ has the property (1).

§ 3. Adapted spaces and the Choquet boundaries.

Let P be a cone of $C^+(X)$. A real-valued function f defined on X is said to be P-bounded if there are functions h_1 and h_2 in P such that $-h_1 \leqq f \leqq h_2$. The set H_P of all P-bounded continuous functions of X is a linear subspace of $C(X)$. A cone P of $C^+(X)$ is said to be adapted if P has the following two properties;

(a_1) for each $x \in X$ there is $f \in P$ such that $f(x) > 0$,

(a_2) $P \subset o(P)$.*)

A linear subspace B of $C(X)$ is said to be adapted if $B^+ = B \cap C^+(X)$ is an adapted cone satisfying $B = B^+ - B^+$.

Let P be an adapted cone and C be a min-stable cone with $P \subset C \subset H_P$. A positive Radon measure μ on X is said to be P-integrable if each function in P is μ-integrable. The set of all P-integrable positive measures is denoted by $\mathfrak{M}_P^+$.

Let x be a point in X. Then a measure $\mu \in \mathfrak{M}_P^+$ is said to be C-representing measure for x if $\mu(h) \leqq h(x)$ holds for all $h \in C$. The set of all C-representing measures for x is denoted by $M_x = M_x(C)$. The set $\delta_C(X)$ of all points x satisfying $M_x = \{\varepsilon_x\}$ is said to be the Choquet boundary of x with respect to C.

Let B be an adapted space. Put

$$C(B) = \{f_1 \wedge \cdots \wedge f_n\,;\ f_i \in B,\ n \in \boldsymbol{N}\}\,.$$

*) A real-valued function f on X is said to be dominated by a real-valued function $g \geqq 0$ on X, in notation $f \in o(g)$, if for any $\varepsilon > 0$ there is a compact set $K \subset X$ such that $|f(x)| \leqq \varepsilon g(x)$ holds on $X \setminus K$. The notation $o(P)$ also is used for the set $\bigcup_{h \in p} o(h)$.

Then $C(B)$ is a min-stable cone with $B \subset C(B) \subset H_{B^+}$. The Choquet boundary with respect to $C(B)$ is denoted by $\delta_B(X)$. Remark that $\mu \in \mathfrak{M}_P^+$ is a $C(B)$-representing measure if and only if holds

$$\mu(f) = f(x) \qquad \text{for all} \quad f \in B .$$

PROPOSITION 2. *Assume that a subset* $\mathfrak{F}$ *of* $C(X)$ *has the following properties;*

(f_1) $\qquad \mathfrak{F} \subset o(\mathfrak{F}^+)$,

(f_2) $\qquad$ *for each* $x \in X$ *there is* $f \in \mathfrak{F}^+$ *such that* $f(x) > 0$,

(f_3) $\qquad \mathfrak{F}$ *is linearly separating.*

Then the linear space B *generated by* $J\mathfrak{F} \vee J\mathfrak{F}$ *is an adapted space and* $\delta_B(X) = X$.

PROOF. First it is proved for B to be adapted. By the assumption (f_1), for every $f \in \mathfrak{F}$ there is a function $g \in \mathfrak{F}^+$ such that $|f(x)| \leqq g(x)$ holds in the complement $X \setminus K$ of a compact set K. By the assumption (f_2), there is $g_0 \in \mathfrak{F}^+$ such that $g_0 > 0$ on K and then there is a positive integer m satisfying $f \leqq g + mg_0$. Hence the equality

$$f = f + g + mg_0 - (g + mg_0)$$

shows that f is written as a difference of two non-negative functions in the linear span of $\mathfrak{F}$. Let f and g be two elements of $\mathfrak{F}$. From $f = f_1 - f_2$ and $g = g_1 - g_2$ with $f_i \in B$, $g_i \in B$, follows

$$|\pm(mf) \vee \pm(ng)| \leqq |mf| + |ng| \leqq m(f_1 + f_2) + n(g_1 + g_2) .$$

Also from the definition of B follows $B = B^+ - B^+$. Further from $f \in o(f_1)$ and $g \in o(g_1)$, it follows that $\alpha f + \beta g \in o(|\alpha| f + |\beta| g)$. Hence $\pm(mf) \vee \pm(ng) \in o(mf_1 + ng_1)$. Thus $B \subset o(B^+)$. By the assumption (f_2), for every $x \in X$ there is $f \in \mathfrak{F}^+ \subset B^+$ such that $f(x) > 0$. Hence B is an adapted space. Second consider $x \in X$ and $\mu \in M_x$. Further assume that the support S_μ of μ contains y with $x \neq y$. From Proposition 1, it follows that there is $v \in B^+$ such that $v(x) = 0$ and $v(y) > 0$. Hence holds

$$\mu(v) = v(x) = 0 .$$

But this is a contradiction since $v \geqq 0$ and $v(y) > 0$ at $y \in S_\mu$. Thus, the support of μ is $\{x\}$. Hence $\mu = \alpha \varepsilon_x$ for $\alpha > 0$. Consequentry if $g \in B^+$ with $g(x) > 0$, then

$$\alpha g(x) = \mu(g) = g(x) .$$

Hence α must be equal to 1, which shows that x is a point of the Choquet boundary.

Let P be an adapted cone and C be a min-stable cone with $P \subset C \subset H_P$. A function f in H_P is said to be C-concave (resp. C-affine) if for each

$x \in X$ and each $\mu \in M_x(C)$

$$\mu(f) \leqq f(x) \quad (\text{resp. } \mu(f) = f(x)) .$$

Let f be a function in H_P. Put

$$\bar{Q}_x f = \inf \{h(x) ; h \geqq f, h \in C\}$$

and

$$\underline{Q}_x f = -\bar{Q}(-f) .$$

Since

$$\bar{Q}_x f = \sup \{\mu(f) ; \mu \in M_x(C)\}$$

for each $f \in H_P$ ([8], [1]), f is C-concave (resp. C-affine) if and only if $\bar{Q}_x f = f(x)$ (resp. $f(x) = \bar{Q}_x f = \underline{Q}_x f$) for all $x \in X$, from which follows

PROPOSITION 3. *$\delta_c(X) = X$ if and only it $\bar{Q}_x f = \underline{Q}_x f$ holds for all $x \in X$.*

PROPOSITION 4. *Assume that a linear space B of C-affine functions in H_P is min-stable and linearly separating. Then $\delta_C(X) = X$.*

PROOF. This proof is similar as in the proof of Proposition 2.

§ 4. Korovkin-type theorems.

Let B be an adapted space in $C(X)$ and $(L_i)_{i \in I}$ be a net of increasing maps $L_i : H_{B^+} \to R^\times$. The net $(L_i)_{i \in I}$, according to Bauer, is said to converge pointwise on X (resp. locally uniformly on X) to the identity on a subset D of H_{B^+} if

$$\lim L_i h(x) = h(x) \tag{2}$$

for all $h \in D$ and all $x \in X$ (resp. if for all $h \in D$ the convergence (2) is uniform on every compact subset K of X). Bauer proved the following theorem ([1], Corollary 2.8).

THEOREM 1 (Bauer). *Let $(L_i)_{i \in I}$ be a net of increasing maps $L_i : H_{B^+} \to R^\times$ and assume that $\delta_B(X) = X$. Then pointwise (resp. locally uniform) convergence on X of the net (L_i) to the identity on B implies pointwise (resp. locally uniform) convergence on X of the net (L_i) to the identity on H_{B^+}.*

Applying this theorem, the following Korovkin type theorem will be obtained.

THEOREM 2. *Let $\mathfrak{F}$ be a subset of $C(X)$ having the property (f_1), (f_2) and (f_3) in Proposition 2. Consider a net $(L_i)_{i \in I}$ of positive linear maps of H_{B^+} into $R^\times$ where B is the linear space generated by $J\mathfrak{F} \vee J\mathfrak{F}$*

on X pointwise (resp. locally uniformly) to f for all functions on $J\mathfrak{F}\vee J\mathfrak{F}$. Then $(L_i f)$ converges on X pointwise (resp. locally uniformly) to f for all functions $f\in H_{B^+}$.

PROOF. From Proposition 2, it follows that B is an adapted space and that the Choquet boundary with respect to $C(B)$ is equal to X. Since each L_i is linear, the convergence of $(L_i f)$ to f for all $f\in J\mathfrak{F}\vee J\mathfrak{F}$ implies the convergence of $(L_i f)$ to f for all $f\in B$. Thus the conclusion follows from Theorem 1.

REMARK. If $\mathfrak{F}$ contains constant 1, $(L_i f)$ converges to f for all bounded continuous functions f.

§ 5. A dense subspace of H_P.

Let P be an adapted cone in $C^+(X)$. For each $u\in P$, let H_u be the Banach space of continuous functions f on X such that $|f|\leqq \lambda u$ for some $\lambda\geqq 0$ with the norm $||f||_u=\{\inf ; |f|\leqq \lambda u\}$ and consider $H_P=\bigcup_{u\in P} H_u$ with the topology of inductive limits of Banach spaces $\{H_u\}_{u\in P}$. Then the dual of H_P is the set $\mathfrak{M}_P$ of all P-integrable measures on X and holds $\mathfrak{M}_P=\mathfrak{M}_P^+-\mathfrak{M}_P^+$.

PROPOSITION 5. *Let ν be a measure in $\mathfrak{M}_P$ and (f_α) be a decreasing family in H_P with $f=\inf f_\alpha=\lim f_\alpha$ contained in H_P. Then*

$$\nu(f)=\lim \nu(f_\alpha)\,.$$

PROOF. Write $\nu=\nu_1-\nu_2$ with $\nu_i\in\mathfrak{M}_P^+$. Then

$$\lim_\alpha \nu_i(f_\alpha)=\nu_i(f) \qquad (i=1,\,2)\ ([7])\,.$$

Hence

$$\begin{aligned}\nu(f)=\nu_1(f)-\nu_2(f)&=\lim \nu_1(f_\alpha)-\lim \nu_2(f_\alpha)\\ &=\lim \nu(f_\alpha)\,.\end{aligned}$$

Let B be a linearly separating and inf-stable adapted space in $C(X)$. Then, from Proposition 2, follows $\delta_B(X)=X$ and every function in H_P is a $C(B)$-affine function. Further, consider H_{B^+} with the inductive limits of $\{H_u\}_{u\in B^+}$. Then B is dense in H_{B^+} ([5]). More generally, let P be an adapted cone and C a min-stable cone with $P\subset C\subset H_P$. Assume that a linear space B of C-affine functions in H_P is min-stable and linearly separating. Then holds $\delta_C(X)=X$ and every function in H_P is a C-affine function. Under some additional assumptions, the linear space B will be shown to be dense in H_P.

THEOREM 3. *Let P be an adapted cone and C be a min-stable cone with $P\subset C\subset H_P$. Assume that a linear subspace B of C-affine functions* *in H_P is min-stable and linearly separating. Further, assume that every $f \in -C$ satisfies*

$$\bar{Q}_x f = \inf\{h(x)\,;\ h \geqq f,\ h \in B\}.$$

Then B is dense in H_P.

PROOF. From Proposition 2 it follows that the Choquet boundary with respect to C is X. Hence

$$\bar{Q}_x f = f(x)$$

for all $f \in H_P$ and all $x \in X$. Let ν be any measure in $\mathfrak{M}_P^+$ satisfying $\nu(h)=0$ for all $h \in B$. By the assumption of the present theorem holds

$$f(x) = \bar{Q}_x f = \inf\{h(x)\,;\ h \geqq f,\ h \in B\}$$

for every $f \in -C$. Put

$$I = \{h \in B\,;\ h \geqq f\}.$$

Since B is min-stable, I is a decreasing family. Consequently, from Proposition 5 follows

$$\nu(f) = \nu(\bar{Q} f) = \lim_I \nu(h) = 0.$$

Thus $\nu(f)=0$ for all $f \in -C$ so that for all $f \in H_P$, since $C-C$ is dense in H_P ([5]). Hence $\bar{B} = H_P$.

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