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On Differential forms in Differentiable Manifolds Admitting a Vector Field

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Introduction. Let M^n be an n dimensional differentiable manifold, and assume that M^n admits a vector field E and a differential 1-form α such that

(0.1) $$i(E)\alpha=1\,,$$

(0.2) $$i(E)d\alpha=0\,,$$

where $i(E)$ and d denote the interior and the exterior derivation respectively.

First we shall give some examples of such manifolds.

EXAMPLE 1. A differential 1-form α in a $2m+1$ dimensional differentiable manifold M^{2m+1} is called a contact form if it satisfies $\alpha\wedge(d\alpha)^m\neq 0$. It is known [1] that a vector field E can be associated to the contact form α so as to satisfy (0.1) and (0.2).

EXAMPLE 2. Consider a Riemannian manifold M^n with metric $\langle\,,\rangle$ and suppose that M^n admits a unit Killing vector field E. Let α be the differential 1-form associated to E by

$$\alpha(X)=\langle E, X\rangle$$

for any vector X. These E and α satisfy (0.1) and (0.2).

EXAMPLE 3. This is a special case of Example 2. Suppose that a Riemannian manifold M^{2m+1} admits a unit Killing vector field E whose associated 1-form α satisfies

$$\nabla_X d\alpha=\alpha\wedge X^\flat$$

for any vector X, where ∇_x denotes the covariant differential and $X^\flat$ means the 1-form naturally associated to X. Such an E is called a Sasakian structure and the M^{2m+1} is called a Sasakian space. It is known [1] that the α of a Sasakian structure E is a contact form. Thus the Sasakian structure is a special case of Example 1 too.

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Henceforth, M^n is an n dimensional differentiable manifold which admits α and E satisfying (0.1) and (0.2), and all the differentiabilities are C^∞. A p-form means a differential p-form and all vector spaces are over the real field.

Let $e(\alpha)$ and L be the operators given by

$$e(\alpha)u=\alpha\wedge u\,, \qquad Lu=d\alpha\wedge u$$

for any p-form u. $\theta(E)$ denotes the Lie derivative with respect to E. A p-form u is said to be *horizontal* if it satisfies

$$i(E)u=0\,.$$

By definition, $i(E)f=0$ holds good for any function f. We call a p-form u *invariant* if

$$\theta(E)u=0$$

is satisfied. By a *basic* form we mean a horizontal and invariant form.

We denote by $\boldsymbol{I}^p(M)$ resp. $\boldsymbol{B}^p(M)$ the vector space of all invariant resp. basic p-forms on M^n. Now let us consider an operator d_E defined by

$$d_E=d-i(E)L$$

which was first introduced in the case of Sasakian manifolds by Tachibana-Ogawa [2].

We shall call a p-form u *E-closed* if it satisfies $d_E u=0$. Let $\boldsymbol{I}_E{}^p(M)$ resp. $\boldsymbol{B}_c{}^p(M)$ be the vector space of all E-closed invariant resp. closed basic p-forms on M^n. If we define a relation $\overset{p}{\sim}$ in $\boldsymbol{I}_E{}^p(M)$ by

$$u \overset{i}{\sim} v \Longleftrightarrow \exists w\in \boldsymbol{I}^{p-1}(M):\ u-v=d_E w\,,$$

it is an equivalence relation in $\boldsymbol{I}_E{}^p(M)$. Thus the set

$$\boldsymbol{H}_E{}^p(M)=\boldsymbol{I}_E{}^p(M)/\overset{i}{\sim}$$

of the equivalence classes $[u]_E$ becomes a vector space in the natural way. Similarly, a relation $\overset{b}{\sim}$ in $\boldsymbol{B}_c{}^p(M)$ is introduced by

$$u \overset{b}{\sim} v \Longleftrightarrow \exists w\in \boldsymbol{B}^{p-1}(M):\ u-v=dw$$

and we have a vector space of the equivalence classes $[u]_b$,

$$\boldsymbol{H}_b{}^p(M)=\boldsymbol{B}_c{}^p(M)/\overset{b}{\sim}\,.$$

Now we are at the position to state a theorem which is the purpose of this paper.

THEOREM. *$\boldsymbol{H}_E{}^p(M)$ is isomorphic to the direct sum of $\boldsymbol{H}_b{}^{p-1}(M)$ and $\boldsymbol{H}_b{}^p(M)$, i.e., we have*

$$\boldsymbol{H}_E{}^p(M)\cong \boldsymbol{H}_b{}^{p-1}(M)\oplus \boldsymbol{H}_b{}^p(M)\,, \qquad p\geqq 0\,,$$

where $\boldsymbol{H}_b{}^{-1}(M)=\{0\}$

G. Reeb [3] has discussed the cohomology of differential forms in a dynamical system. Our theorem is a generalization of one of theorems in [3] in a sense.

1. Preliminaries. It is well known that the following identities about d, $i(E)$ and $\theta(E)$ hold good:

(1.1) $$dd=0, \quad i(E)i(E)=0,$$

(1.2) $$\theta(E)=i(E)d+di(E),$$

(1.3) $$\theta(E)d=d\theta(E), \quad \theta(E)i(E)=i(E)\theta(E);$$

We also have the following identities taking account of (0.1) and (0.2):

(1.4) $$dL=Ld, \quad e(\alpha)L=Le(\alpha),$$

(1.5) $$i(E)L=Li(E),$$

(1.6) $$i(E)e(\alpha)+e(\alpha)i(E)=I, \quad \text{(identity)}.$$

For a basic u we have from (1.2)

(1.7) $$i(E)du=0, \quad u\in \boldsymbol{B}^p(M),$$

and hence

$$u\in \boldsymbol{B}^p(M) \quad \text{implies} \quad du\in \boldsymbol{B}^{p+1}(M).$$

As the operator d_E is defined by

$$d_E u=(d-i(E)L)u,$$

we have

(1.8) $$i(E)d_E=i(E)d$$

and taking account of (1.2)

(1.9) $$i(E)d_E u=-di(E)u, \quad u\in \boldsymbol{I}^p(M).$$

It is seen for a basic u that

(1.10) $$d_E u=du, \quad u\in \boldsymbol{B}^p(M)$$

and

(1.11) $$d_E e(\alpha)u=-e(\alpha)du, \quad u\in \boldsymbol{B}^p(M)$$

by means of (1.5) and (1.6).

From (1.10) we know that "E-closed" coincides with "closed" for the basic form. Hence we have

$$\boldsymbol{B}_c{}^p(M)\subset \boldsymbol{I}_E{}^p(M).$$

On the other hand, as the equation

$$d_E d_E u=-L\theta(E)u$$

holds for any p-form u, we get

(1.12) $$d_E d_E u=0, \quad u\in \boldsymbol{I}^p(M),$$

which justifies the naturality of d_E.

It is easy to see that

$$\alpha\in \boldsymbol{I}_E{}^1(M)\,,\qquad d\alpha\in \boldsymbol{B}_c{}^2(M)\,,$$

i. e., we have

$$\begin{aligned}&\theta(E)\alpha=0\,, && d_E\alpha=0\,,\\ &\theta(E)d\alpha=0\,, && i(E)d\alpha=0\,.\end{aligned}\tag{1.13}$$

Now let us associate to a p-form u a $(p-1)$-form $\phi(u)$ by

$$\phi(u)=i(E)u\,.$$

Then we have

LEMMA 1.1. *ϕ is linear and*

$$\begin{aligned}&\phi(\boldsymbol{I}^p(M))=\boldsymbol{B}^{p-1}(M)\,, && (\mathbf{ker}\,\phi)\cap\boldsymbol{I}^p(M)=\boldsymbol{B}^p(M)\,,\\ &\phi(\boldsymbol{I}_E{}^p(M))=\boldsymbol{B}_c{}^{p-1}(M)\,, && (\mathbf{ker}\,\phi)\cap\boldsymbol{I}_E{}^p(M)=\boldsymbol{B}_c{}^p(M)\,.\end{aligned}$$

This is proved directly taking account of (1.1)~(1.3). To see "onto" we take $u=e(\alpha)v$ for $v\in\boldsymbol{B}^{p-1}(M)$ (or $\boldsymbol{B}_c{}^{p-1}(M)$) and make use of (1.6) and (1.11).

From (1.6) any $u\in\boldsymbol{I}^p(M)$ is written as

$$u=e(\alpha)i(E)u+i(E)e(\alpha)u\,.$$

Thus by virtue of (1.1), (1.5) and (1.13) we can get

LEMMA 1.2. *For any $u\in\boldsymbol{I}^p(M)$ there exists $u_0\in\boldsymbol{B}^p(M)$ such that*

$$u=e(\alpha)\phi(u)+u_0\,.$$

2. Proof of Theorem. Consider the onto linear map

$$\phi:\ \boldsymbol{I}_E{}^p(M)\longrightarrow\boldsymbol{B}_c{}^{p-1}(M)$$

in Lemma 1.1. Let $u\overset{i}{\sim}v$ in $\boldsymbol{I}_E{}^p(M)$, then there exists $w\in\boldsymbol{I}^{p-1}(M)$ such that

$$u-v=d_Ew\,.$$

Hence we have

$$\begin{aligned}\phi(u)-\phi(v)&=\phi(d_Ew)=i(E)d_Ew\\ &=-di(E)w=-d\phi(w)\end{aligned}$$

by virtue of (1.9). This equation implies $\phi(u)\overset{b}{\sim}\phi(v)$ because of $\phi(w)\in\boldsymbol{B}^{p-2}(M)$. Thus ϕ induces the natural onto linear map

$$\phi^*:\ \boldsymbol{H}_E{}^p(M)\longrightarrow\boldsymbol{H}_b{}^{p-1}(M)\,.$$

Consequently, the proof of our theorem is completed if we show that $\mathbf{ker}\,\phi^*\cong\boldsymbol{H}_b{}^p(M)$.

LEMMA 2.1. *Let $K_E{}^p(M)$ be the vector subspace of $H_E{}^p(M)$ defined by*

$$K_E{}^p(M)=\{[u]_E \mid v\in B_c{}^p(M):\ u \overset{i}{\sim} v\}\,.$$

Then we have

$$\ker \phi^*=K_E{}^p(M)\,.$$

PROOF. Let $[u]_E\in \ker \phi^*$. Then there exists $w\in B^{p-2}(M)$ such that $\phi(u)=dw$, and applying Lemma 1.2 to u, we have

$$u=e(\alpha)\phi(u)+v=e(\alpha)dw+v$$

for some $v\in B^p(M)$. As $e(\alpha)dw=-d_E e(\alpha)w$ by (1.11), it follows that

$$u=-d_E e(\alpha)w+v\,.$$

Taking acount of $e(\alpha)w\in I^{p-1}(M)$, (1.12) and (1.10) we get

$$u \overset{i}{\sim} v\,,\qquad v\in B_c{}^p(M)\,.$$

Conversely, suppose that $[u]_E\in K_E{}^p(M)$. Then there exists $v\in B_c{}^p(M)$ and $w\in I^{p-1}(M)$ such that $u=v+d_E w$. If we take account of (1.9), it follows

$$\phi(u)=i(E)d_E w=-di(E)w\,,$$

from which we get $[\phi(u)]_b=0$. q. e. d.

LEMMA 2.2.

$$K_E{}^p(M)\cong H_b{}^p(M)\,.$$

PROOF. Let us define a linear map f by

$$f:\ B_c{}^p(M)\longrightarrow K_E{}^p(M)\,,\qquad u\longrightarrow [u]_E\,.$$

Then it is seen that f is onto and induces a linear map

$$f^*:\ H_b{}^p(M)\longrightarrow K_E{}^p(M)\,,\qquad [u]_b\longrightarrow [u]_E\,.$$

We shall show that f^* is an isomorphism. Consider $[u]_b\in \ker f^*$. Then $u\in B_c{}^p(M)$ and $[u]_E=0$. Hence there exists $w\in I^{p-1}(M)$ and $w_0\in B^{p-1}(M)$ such that

$$u=d_E w=d_E(e(\alpha)\phi(w)+w_0)\,.$$

If we take account of (1.11) and (1.10), it follows that

$$u=-e(\alpha)d\phi(w)+dw_0\,. \tag{2.1}$$

On the other hand, we have $i(E)u=0$ and by (1.7) $i(E)dw_0=0$. Hence we have

$$i(E)e(\alpha)d\phi(w)=0$$

and by means of (1.6)

$$d\phi(w)=e(\alpha)i(E)d\phi(w)\,.$$

If we take account of $\phi(w)\in \boldsymbol{B}^{p-2}(M)$ and (1.7), the last equation becomes $d\phi(w)=0$. Consequently we get from (2.1) $u=dw_0$ which means $[u]_b=0$. q. e. d.

Thus the proof of our theorem is completed.

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