Natural Science Report, Ochanomizu University, Vol. 26, No. 1, 1975
お茶の水女子大学 自然科学報告 第26巻 第1号

A Characterization of the Suzuki Groups

Yoko Usami

Department of Mathematics, Faculty of Science
Ochanomizu University, Tokyo
(Received April 15, 1975)

§ 1. Introduction.

Let G be a finite group. A subgroup A is called a strongly self-centralizing subgroup if the centralizer of any nonidentity element of A is A. In this paper we shall consider a group G with a set $A_1, A_2, \cdots, A_r$ of non-conjugate strongly self-centralizing subgroups. If $[N_G(A_i): A_i]=d_i>1$ $(1\leqq i\leqq r)$, we shall describe this as a self-centralization system of type $(d_1(|A_1|), \cdots, d_r(|A_r|))$ or $(d_1, \cdots, d_r)$ for short.

The following are the known simple groups with self-centralization system of type (2).

1. $PSL(2, 2^n)$ $|A_1|=2^n+1$, $|A_2|=2^n-1$, (Specially these groups are of type (2, 2).)
2. $PSL(2, q)$ q : odd $|A|=q\pm 1/2$, the sign being chosen to make $|A|$ odd.
3. $Sz(q)$ $q=2^{2n+1}|A|=2^n-1$.
4. $PSL(3, 4)$ $|A|=5$.

The known results about groups with self-centralization system are as follows.

THEOREM A (Stewart [2]). *A finite group with a self-centralization system of type* $(2(|A|))$, *where* 3 *divides* $|A|$, *is isomorphic to* $PSL(2, q)$ *for some* q.

THEOREM B (Harada [1]). *A finite simple group with a self-centralization system of type* (2, 2) *is isomorphic to* $PSL(2, 2^n)$, *where* $2^n>2$.

THEOREM C (Harada [1]). *Let* G *be a finite simple group with a self-centralization system of type* $(2(|A|))$. *If* $G\leqq 4\ (|A|+1)^3$, *then* G *is isomorphic to* $PSL(2, q)$, *where* $q=p^n>3$.

Here is a problem whether there are finite simple groups other than the Suzuki groups with self-centralization system of type (2, 4, 4). This problem seems difficult but if we add some more conditions to G, we can solve it. In this paper we shall prove this problem under special conditions.

§ 2. Irreducible characters of a finite group with a strongly self-centralizing subgroup.

Let G be a finite group and A be a strongly self-centralizing subgroup of G. Then A has following special properties.

LEMMA 1.
(1) *A is a* T.I. *set in G.*
(2) *A is a Hall subgroup of G.*
(3) *$N_G(B) \subseteqq N_G(A)$ holds for any subgroup $B \neq 1$ of A.*

PROOF. See Harada [1].

By these special properties of A, we can apply the theory of exceptional characters to compute a part of character table of G and moreover the order of G.

We first remark A is an abelian subgroup and $N_G(A)$ is a Frobenius group and A is a Frobenius kernal in $N_G(A)$. Let n denote the order of A, d the index of $[N_G(A):A]$ and put $w=n-1/d$. $N_G(A)/A$ induces a permutation of the $n-1$ nonprincipal irreducible characters of A and they are distributed into w orbits under the action of $N_G(A)/A$. Nonprincipal irreducible characters of A induce the same irreducible characters of $N_G(A)$ if and only if they are in a same orbit. And these w induced irreducible characters of $N_G(A)$ (we write them $\theta_1, \cdots, \theta_w$) vanish on elements not conjugate to an element of A and have same degrees. Here we apply the following theorem (See p. 61 [4]).

THEOREM. *Let A be a* T.I. *set in the finite group G. Assume that $\theta_1, \cdots, \theta_w$ are distinct irreducible complex characters of $N_G(A)$ $(w \geqq 2)$, such that all θ_i vanish outside A and $\theta_1(1)= \cdots =\theta_n(1)$. Then there exists a* sign $\varepsilon = \pm 1$, *and distinct irreducible complex characters $X_1, \cdots, X_w$ of G such that*

$$\theta_i{}^G - \theta_j{}^G = \varepsilon(X_i - X_j) \qquad \textit{all} \quad 1 \leqq i,\ j \leqq n\,.$$

$X_1, \cdots, X_w$ are called the exceptional characters associated with $\theta_1, \cdots, \theta_w$ or simply A-characters.

By this theorem we can obtain w irreducible characters of our group G ; $X_1, \cdots, X_w$. Moreover we can obtain the following results.

(i) X_i is not constant on $A^{\#}$ for $i=1, \cdots, w$ (Let $B^{\#}$ denote $B-\{1\}$ for any subgroup B).
(ii) If an element x is not conjugate to any element of $A^{\#}$, then we have $X_i(x)=X_1(x)$.
(iii) If an irreducible character Y is non-exceptional, Y takes a constant value c on $A^{\#}$ and degree $Y \equiv c \pmod n$. Especially if Y vanishes on $A^{\#}$, Y has a degree divisible by n.

Now let x_i denote a nonprincipal irreducible character of A which induces θ_i and let φ_i denote the character of G induced by $1_A - x_i$. Then φ_i vanishes on elements not conjugate to an element of $A^\#$ and

$$\varphi_i = 1_G - \varepsilon X_i + a \sum_{j=1}^{w} X_j + c \sum Y .$$

Here $\varepsilon = \pm 1$, a is a certain integer and the second summation ranges over all the non-exceptional characters except 1_G. We have moreover

$$(1) \qquad (\varphi_i, \varphi_i) = 1 + (a-\varepsilon)^2 + (w-1)a^2 + \sum c^2 = d+1 .$$

So by (1) we can determine the decomposition of φ_i if d is small.

First we apply above results to the case $d=2$ and $w \geqq 2$. In this case

$$\varphi_i = 1_G - \varepsilon X_i + \varepsilon Y \qquad (\varepsilon = \pm 1) ,$$

where Y is a nonprincipal non-exceptional irreducible character. Let $Dg(X)$ denote the degree of a character X of G. By (iii) and the decomposition of φ_i we get the following lemma.

LEMMA 2. *$Dg(Y) = kn + \varepsilon$, $Y(x) = \varepsilon$ if $x \in A^\#$ and $Dg(X_i) = kn + 2\varepsilon$, where k is a non-negative integer. All the nonprincipal non-exceptional irreducible characters of G other than Y have degrees divisible by n and vanish on $A^\#$. The following lemma is obtained by Harada* [1] *using the argument on the group algebra which essentially due to Brauer and Fowler* [6].

LEMMA 3. *Under the same notation as above if G is a finite group with a self-centralization system of type* $(2(|A|))$ *and* $|A|-1/d \geqq 2$, *then the order of G is written in the form*

$$|G| = n(kn+2\varepsilon)(kn+\varepsilon)|C_G(\tau)|^2/(kn+\varepsilon - Y(\tau))^2 ,$$

where τ is an involution in $N_G(A)$. If we put $m = |C_G(\tau)|/kn + \varepsilon - Y(\tau)$, then m is an integer and

$$m^2 \equiv 1 \pmod{n} .$$

Throughout the rest of this section we assume G is a group with a self-centralization system of type $(2(|A_1|), 4(|A_2|), 4(|A_3|))$, where $|A_1| = n_1 \geqq 5$, $|A_2| = n_2 > 5$ and $|A_3| = n_3 > 5$. For short we call G is a (♯) group if G satisfies above conditions. We remark n_1, n_2 and n_3 are relatively prime to one another and we can assume $n_2 < n_3$. Since G is a group with a self-centralization system of type $(2(|A_1|))$, we can apply above argument to get A_1-characters $\{X_{1i} : 1 \leqq i \leqq n_1 - 1/2\}$ and Y_1 which vanishes on $A_1^\#$ and lemma 2, 3 holds for A_1, n_1 and Y_1.

Next we apply the results of third and forth paragraphs in this section to the case $d=4$ and $w \geqq 2$ since G is now also a group with a self-centralization system of type $(4(|A_2|))$ and $|A_2| = n_2 > 5$. From (1) we get an equality

$$1+(a-\varepsilon_2)^2+(w-1)a^2+\sum c^2=5\,.$$

So we have following possibilities:

$$\varphi_{2j}=1_G-\varepsilon_2X_{2j}+c_{21}Y_{21}+c_{22}Y_{22}+c_{23}Y_{23} \qquad 1\leqq j\leqq n_2-1/4$$

$$\begin{cases}\varphi_{21}=1_G+\varepsilon_2X_{22}+c_{21}Y_{21}+c_{22}Y_{22}+c_{23}Y_{23} \\ \varphi_{22}=1_G+\varepsilon_2X_{21}+c_{21}Y_{21}+c_{22}Y_{22}+c_{23}Y_{23} & \text{if}\quad n_2=9\end{cases}$$

$$\begin{cases}\varphi_{21}=1_G+\varepsilon_2(X_{22}+X_{23})+c_{21}Y_{21}+c_{22}Y_{22} \\ \varphi_{22}=1_G+\varepsilon_2(X_{21}+X_{23})+c_{21}Y_{21}+c_{22}Y_{22} \\ \varphi_{23}=1_G+\varepsilon_2(X_{21}+X_{22})+c_{21}Y_{21}+c_{22}Y_{22} & \text{if}\quad n_2=13\end{cases}$$

$$\begin{cases}\varphi_{21}=1_G+\varepsilon_2(X_{22}+X_{23}+X_{24})+c_{21}Y_{21} \\ \varphi_{22}=1_G+\varepsilon_2(X_{21}+X_{23}+X_{24})+c_{21}Y_{21} \\ \varphi_{23}=1_G+\varepsilon_2(X_{21}+X_{22}+X_{24})+c_{21}Y_{21} \\ \varphi_{24}=1_G+\varepsilon_2(X_{21}+X_{22}+X_{23})+c_{21}Y_{21} & \text{if}\quad n_2=17\end{cases}$$

$$\varepsilon_2=\pm1 \qquad \text{each}\quad c_{2i}=\pm1\,.$$

Here $\{X_{2j}:1\leqq j\leqq n_2-1/4\}$ are A_2-characters and Y_{21}, Y_{22} and Y_{23} are irreducible characters which vanish on $A_2^\sharp$. But in case $n_2=9$, two possibilities can be regarded the same by an exchange of X_{21} and X_{22}.

As to A_3 we also apply the same argument to get A_3-characters and irreducible characters Y_{31}, Y_{32}, Y_{33} which vanish on $A_3^\sharp$ and similar decomposition of φ_{3l} $(1\leqq l\leqq n_3-1/4)$.

Therefore we have obtained irreducible characters 1_G, $\{X_{1i}:1\leqq i\leqq n_1-1/2\}$, $\{X_{2j}:1\leqq j\leqq n_2-1/4\}$, $\{X_{3l}:1\leqq l\leqq n_3-1/4\}$, Y_1, Y_{21}, Y_{22}, Y_{23}, Y_{31}, Y_{32}, Y_{33}. We remark that Y_{22}, Y_{23}, Y_{32} and Y_{33} may not exist if $n_2, n_3=13$ or 17.

LEMMA 4. *Let G be a (♯) group. If $n_1\neq7$ or $n_2\neq9$, G has distinct irreducible characters 1_G, $\{X_{1i}:1\leqq i\leqq n_1-1/2\}$, $\{X_{2j}:1\leqq j\leqq n_2-1/4\}$, $\{X_{3l}:1\leqq l\leqq n_3-1/4\}$, Y_1, Y_2, Y_3 and*

$$\varphi_{1i}=1_G-\varepsilon X_{1i}+\varepsilon Y_1 \qquad 1\leqq i\leqq n_1-1/2$$

$$\varphi_{2j}=1_G-\varepsilon_2X_{2j}-\varepsilon Y_1+c_2Y_2+c_3Y_3 \qquad 1\leqq j\leqq n_2-1/4$$

$$\varphi_{3l}=1_G-\varepsilon_3X_{3l}-\varepsilon Y_1-c_2Y_2-c_3Y_3 \qquad 1\leqq l\leqq n_3-1/4\,.$$

If $n_1=7$ and $n_2=9$, there is one more possibility that G has distinct irreducible characters 1_G, $\{X_{1i}:1\leqq i\leqq n_1-1/2\}$, $\{X_{2j}:1\leqq j\leqq n_2-1/4\}$, $\{X_{3l}:1\leqq l\leqq n_3-1/4\}$, Y_1 and

$$\varphi_{1i}=1_G-\varepsilon X_{1i}+\varepsilon Y_1 \qquad i=1,2,$$
$$\varphi_{2j}=1_G-\varepsilon X_{2j}+\varepsilon(X_{11}+X_{12}+X_{13}) \qquad j=1,2$$
$$\varphi_{3l}=1_G-\varepsilon_3 X_{3l}-\varepsilon Y_1+\varepsilon(X_{21}+X_{22}) \qquad 1\leqq l\leqq n_3-1/4\,.$$

In any case degrees of other irreducible characters are divisible by $n_1n_2n_3$.

PROOF. Clearly A_1-, A_2- and A_3-characters are all distinct. Now we proceed in a series of steps.

Step 1. If $n_1\neq 7$ or $n_2\neq 9$, $Y_1=Y_{21}$ or Y_{22} or Y_{23}.

(Proof.) We first remark if an irreducible character W of G has same degree as X_{1i}, then W is one of A_1-characters, and if W has same degree as Y_1, then $W=Y_1$ and if W has same degree as Y_{21} (Y_{22} or Y_{23}), then $W=Y_{21}$ or Y_{22} or Y_{23}. Suppose $Y_1\neq Y_{21}, Y_{22}, Y_{23}$. If $\{X_{1i}: 1\leqq i\leqq n_1-1/2\}\subsetneqq\{Y_{21}, Y_{22}, Y_{23}\}$, then $Dg(X_{1i})$ and $Dg(Y_1)$ are divisible by n_2 which is a contradiction by lemma 2. Hence $n_1=5$ or 7. If $n_1=5$, we may assume $\{X_{11}, X_{12}\}=\{Y_{21}, Y_{22}\}$. In this case since $Dg(\varphi_{2j})=0$, the exceptional decomposition of φ_{2j} can not occur and

$$\varphi_{2j}=1_G-\varepsilon_2X_{2j}+c_{21}Y_{21}+c_{22}Y_{22}+c_{23}Y_{23}\,.$$

Since $Dg(Y_{22})=Dg(Y_{21})=Dg(X_{11})=Dg(Y_1)+\varepsilon\equiv\varepsilon \pmod{n_2}$, we get $c_{21}=c_{22}=\varepsilon$. Hence

$$\begin{aligned}\varphi_{2j}&=1_G-\varepsilon_2X_{2j}+\varepsilon Y_{21}+\varepsilon Y_{22}+c_{23}Y_{23}\\&=1_G-\varepsilon_2X_{2j}+\varepsilon X_{11}+\varepsilon X_{12}+c_{23}Y_{23}\,.\end{aligned}$$

In this case if Y_1 does not coincide with Y_{31} or Y_{32} or Y_{33}, we obtain by the same way as above

$$\varphi_{3l}=1_G-\varepsilon_3X_{3l}+\varepsilon X_{11}+\varepsilon X_{12}+c_{33}Y_{33}\,.$$

Since $(\varphi_{2j}, \varphi_{3l})=0$, this is a contradiction. If $n_1=7$, then $\{X_{11}, X_{12}, X_{13}\}=\{Y_{21}, Y_{22}, Y_{23}\}$. Since $Dg(Y_{21})=Dg(Y_{22})=Dg(Y_{23})=Dg(X_{1i})=Dg(Y_1)+\varepsilon\equiv\varepsilon \pmod{n_2}$, we get $c_{21}=c_{22}=c_{23}=\varepsilon$. Hence

$$Dg(X_{2j})=3Dg(X_{1i})+\varepsilon \quad\text{and}\quad \varepsilon_2=\varepsilon\,.$$

In this case if Y_1 dose not coincide with Y_{31} or Y_{32} or Y_{33}, we obtain by the same argument as above

$$Dg(X_{3l})=3Dg(X_{1i})+\varepsilon\,.$$

But this is a contradiction since $Dg(X_{3l})$ is divisible by n_2 and $Dg(X_{2l})$ is not so. Hence we may assume $Y_{31}=Y_1$. Now it follows that A_1-characters, A_2-characters, $Y_{31}=Y_1, Y_{32}, Y_{33}$ are distinct characters if $n_2>9$. Hence n_3 must divide both $Dg(X_{1i})$ and $Dg(X_{2j})$. This is a contradiction since $Dg(X_{2j})=3Dg(X_{1i})+\varepsilon$.

Step 2. If $n_1=7$, $n_2=9$ and $Y_1 \neq Y_{21}, Y_{22}, Y_{23}$, then

$$\varphi_{1i}=1_G-\varepsilon X_{1i}+\varepsilon Y_1 \qquad i=1, 2, 3$$

$$\varphi_{2j}=1_G-\varepsilon X_{2j}+\varepsilon(X_{11}+X_{12}+X_{13}) \qquad j=1, 2$$

$$\varphi_{3l}=1_G-\varepsilon_3 X-\varepsilon Y_1+\varepsilon(X_{21}+X_{22}) \qquad 1\leqq l\leqq n_3-1/4 .$$

(Proof.) From the last part of the proof of step 1, we may assume $\{X_{11}, X_{12}, X_{13}\}=\{Y_{21}, Y_{22}, Y_{23}\}$, $c_{21}=c_{22}=c_{23}=\varepsilon$, $\varepsilon_2=\varepsilon$, $Y_{31}=Y_1$ and $\{X_{21}, X_{22}\}=\{Y_{32}, Y_{33}\}$. Since $Dg(Y_{32})=Dg(Y_{33})=Dg(X_{21})=3Dg(X_{1i})+\varepsilon\equiv\varepsilon \pmod{n_3}$, we get $c_{32}=c_{33}=\varepsilon$. Since $(\varphi_{1i}, \varphi_{3l})=0$, we get $c_{31}=-\varepsilon$. Then the assurtion follows.

Step 3. $Y_1=Y_{31}$ or Y_{32} or Y_{33}.

(Proof.) Since $n_3>n_2\geqq 9$, we can use the same argument as step 1. Then if $n_1\neq 7$ or $n_2\neq 9$, we may assume $Y_1=Y_{21}=Y_{31}$.

Step 4. If $n_1\neq 7$ or $n_2\neq 9$, $c_{21}=c_{31}=-\varepsilon$.

(Proof.) This assurtion follows from the equalities and congruences:

$$Dg(X_{1i})=kn_1+2\varepsilon\equiv 0 \pmod{n_2 n_3}$$

$$Dg(Y_1)=kn_1+\varepsilon\equiv c_{21} \pmod{n_2}$$

$$Dg(Y_1)=kn_1+\varepsilon\equiv c_{31} \pmod{n_3} .$$

Step 5. If $n_1\neq 7$ or $n_2\neq 9$, there exist Y_{22}, Y_{23}, Y_{32} and Y_{33} and we may assume $Y_{22}=Y_{32}$ and $Y_{23}=Y_{33}$, $c_{22}=-c_{32}$, $c_{23}=-c_{33}$.

(Proof). By $(\varphi_{2j}, \varphi_{3l})=0$ and step 4.

By (iii) we can easily see degrees of other irreducible characters are divisible by $n_1 n_2 n_3$. Thus the lemma is proved.

§ 3. Sylow groups of odd prime and their normalizers.

In this paper we shall prove the following theorem.

THEOREM. *If G is a finite simple group with a self-centralization system of type* $(2(|A_1|), 4(|A_2|), 4(|A_3|))$, *where* $|A_1|=n_1\geqq 5$, $|A_2|=n_2>5$, $|A_3|=n_3>5$ *and satisfies the conditions:*

(i) $|G|\leqq 3n_1^2 n_2^2 n_3^2$,

(ii) *if r is an odd prime relatively prime to $n_1 n_2 n_3$ then a r-Sylow group is an abelian,*

then G is isomorphic to one of the Suzuki groups.

From now on we assume G is a simple (♯) group satisfying (i) and (ii).

First of all we remark that G has at most two irreducible characters other than the mentioned characters in lemma 4, since $|G|\leqq$

$3n_1^2n_2^2n_3^2$ and since the sum of square of degrees of all irreducible characters of G is $|G|$.

LEMMA 5. *The involutions of G form a single conjugate class.*

PROOF. See Harada [1]. (This assurtion holds for a simple group with a self-centralization system of type (2).)

Let τ be an involution in G. If $C_G(\tau)$ is a 2-group, G is isomorphic to a Suzuki group by the study of the classification of C. I. T. group. (A group of even order is called a C. I. T. group if it satisfies the condition that the centralizer of any involution is a 2-group. c. f. Suzuki [7]) So throughout the rest of this paper we suppose $C_G(\tau)$ is not a 2-group and we shall show this is a contradiction.

Now since $C_G(\tau)$ is not a 2-group, G has elements of order p and $2p$ for some odd prime p which is relatively prime to $n_1n_2n_3$. And $N_G(A_i)/A_i$ $(i=2, 3)$ are cyclic group, since G has no involution which acts trivially on A_2 or A_3. Now we recall the fact that the number of irreducible characters coincides with the number of conjugate classes. In the case $n_1=7$ and $n_2=9$, if the exceptional possibility of lemma 4 occurs, G has at most three conjugate classes other than the classes which has at least an element of A_1 or A_2 or A_3. This is a contradiction since there exist elements of order 2, 4, p and $2p$. Then the exceptional possibility of lemma 4 can not occur.

LEMMA 6. *G has four conjugate classes of order 2, 4, p and $2p$ other than the classes each of which has at least an element of A_1 or A_2 or A_3. G has at most one conjugate class other than the above mentioned classes, and if G has such a conjugate class, then it's order is one of 4, p, $2p$, $4p$, r (an odd prime relatively prime to $pn_1n_2n_3$), p^2, and 8. $|C_G(\tau)|$ is divisible only by 2 and p since G has no element of order $2r$.*

LEMMA 7. *A Sylow p group P of G is an elementary abelian group.*

PROOF. By condition (ii) P is an abelian. Hence by the theorem of Burnside (p. 240 [5]) we can say two elements of P are conjugate in G if and only if they are conjugate in $N_G(P)$. Suppose P is not an elementary abelian. Then by lemma 6 the elements of order p^2 form a single conjugate class. Let S be the set of the elements of order p^2 in P. Then $N_G(P)$ acts transitively on S. Hence we have

$$|N_G(P)|=|S|\cdot|C_{N_G(P)}(\rho)| \qquad \text{for} \quad \rho\in S\,.$$

The highest power of p dividing the left side is $|P|$ and the highest power of p dividing the right side is larger than $|P|$ since $|S|$ is divisible by p. This is a contradiction. The lemma is proved.

LEMMA 8. *If the elements of order p form a single conjugate class, then* $p=3$ *or* 5.

PROOF. Let x be an element of order p. Then $N_G(\langle x\rangle)/C_G(x)$ is a subgroup of a cyclic group of order $p-1$. But now a generator of Aut $(\langle x\rangle)$ is contained in $N_G(\langle x\rangle)/C_G(x)$ by the assumption. Hence $N_G(\langle x\rangle)/C_G(x)$ is exactly a cyclic group of order $p-1$. Possibilities of the order of a cyclic subgroup of even order is 2, 4 and 8 in this case. Hence $p-1=2$ or 4 or 8. Therefore $p=3$ or 5. The lemma is proved.

LEMMA 9. *If G has an element of order r, a r-Sylow group R of G has a order of* 5 *or* $3^?$ *or* 7 *or* 11 *or* 5^2.

PROOF. In this case G has only five classes of order 2, 4, p, $2p$, r other than the classes each of which has at least an element of A_1 or A_2 or A_3. Hence R is an elementary abelian and $C_G(R)=R$. $N_G(R)$ has r-complement L by the theorem of Schur-Zassenhaus and $N_G(R)$ is a Frobenius group, where R is a Frobenius kernal. Moreover L acts on $R^\#$ transitively by the theorem of Burnside (p. 240 [5]). Hence we have $|L|=|R^\#|$.

Since L is a Frobenius complement, 2-Sylow group of L is cyclic or generalized quaternion and a Sylow group of odd prime of L is cyclic. In this case the highest power of 2 dividing L is 2 or 4 or 8. If an odd prime r' divides $|L|$, then $r'=p$, since L has a cyclic subgroup of order $2r'$ (See theorem II, p. 39 [9]). Hence

$$|L|=2,\ 4,\ 8,\ 2p,\ 4p,\ 8p\ .$$

Since $p=3$ or 5 by lemma 8,

$$|R|=3,\ 5,\ 9,\ 7,\ 11,\ 13,\ 5^2,\ 41\ .$$

On the other hand let y be an element of order r. Then $N_G(\langle y\rangle)/C_G(y)$ is a cyclic group of order $r-1$ since the elements of order r form a single class. Since the possibilities of the order of cyclic subgroup of even order in G are 2, 4 and $2p$, $r=3$ or 5 or 7 or 11.

If $|R|=3$, G is a finite simple group with a self-centralization system of type $(2(n_1), 2(3))$ and therefore G is isomorphic to $PSL(2, 2^n)$ by theorem B. But $PSL(2, 2^n)$ is not of type (2, 4, 4), so $|R|\neq 3$.

Hence we have the assertion. The lemma is proved.

Next, we consider the structure of $N_G(P)$. $N_G(P)$ has a p-complement K by the theorem of Schur-Zassenhaus; namely $N_G(P)=PK$, $P\cap K=\{1\}$.

LEMMA 10. $|K|$ *is even.*

PROOF. Suppose $|K|$ is odd.

Case 1 : the elements of order p form a single class.

In this case K acts on $P^{\#}$ transitively by the theorem of Burnside (p. 240 [5]). Hence $|K|$ is divisible by $|P^{\#}|$. This is a contradiction since $|P^{\#}|$ is even.

Case 2 : the elements of order p form two classes.

In this case G has no element of order r by lemma 6. Hence if r' is a prime divisor of $|K|$, r' is a prime divisor of n_1 or n_2 or n_3. Moreover from solvability of K and lemma 1 (3) it follows $|K|=m_i$, $m_i|n_i$ ($i=1$ or 2 or 3). Here K acts on $P^{\#}$ regularly and distributes the elements of $P^{\#}$ into two orbits, where one orbit has the inverse elements of the elements of another orbit. By the theorem of Burnside (p. 240 [5]) these two classes are not conjugate to each other in G and hence G has only two classes containing elements of order p. Hence it follows that any element x of order p has an involution which commutes with x. And in this case any two elements of order $2p$ are conjugate to each other by lemma 6. Hence there exists an element g in G such that

$$\tau x = g^{-1}\tau x^{-1} g \qquad \text{for} \quad \tau \in C_G(x)\,.$$

Then $x^2 = g^{-1}x^{-2}g$ and so $x=g^{-1}x^{-1}g$. This is a contradiction.

LEMMA 11. *If $C_G(P)=P$, then we have following assertions:*

(i) $p=3$ *or* 5.

(ii) *if K is not a 2 group, then $O_2(K)=1$. ($O_2(K)$ is unique maximal normal 2 subgroup of K.)*

PROOF. (i) Case 1 : the elements of order p form a single class.

The assertion follows from lemma 8.

Case 2 : the elements of order p form two classes.

In this case G doesn's have elements of order 8 or r. Since $|K|$ is even we can write

$$P = C_P(\tau)\times[P, \tau] \qquad \text{for an involution } \tau \text{ in } K\,.$$

From the assumption $C_G(P)=P$ it follows $[P, \tau]\neq 1$. So let $x\neq 1$ be an element in $[P, \tau]$. Then $x^{\tau}=x^{-1}$. Hence $N_G(\langle x\rangle)/C_G(x)$ is a cyclic group of even order, so the order is 2 or 4. Since $N_G(\langle x\rangle)$ distributes $\langle x\rangle^{\#}$ into two orbits both of which have same length, or one orbit, we obtain $p-1=2$ or 4 or 8. Thus $p=3$ or 5.

(ii) If K is not a 2 group, an element of odd prime order r' in K induces a fixed-point-free automorphism of $O_2(K)\cdot P$. From the theorem of Thompson on the nilpotency of Frobenius kernal (p. 83-92 [9]) it follows

$$O_2(K)P = O_2(K)\times P\,.$$

Hence if $O_2(K)\neq 1$, this is a contradiction since $C_G(P)=P$. The lemma is proved.

K is clearly a C. I. T. group. If $C_G(P)=P$, we can use the known classification of C. I. T. groups ([7], [8]) and lemma 11 to obtain the following possibilities of K.

(1) K is a 2 group.

(2) A 2-Sylow group of K is cyclic or generalized quaternion.

(3) $K \cong PSL(2, p')$ or $PSL(2, 9)$ or $PSL(2, q)$ or $Sz(q)$ or $PSL(3, 4)$ or M_q. (p' is a Fermat or Mersenne prime, q is a power of 2 and $q>2$.)

LEMMA 12. *If $C_G(P)=P$, then the possibilities of K are as follows:*
(1) *2 group*
(2) *a 2-Sylow group of K is cyclic and $|K|=2m_i$, $4m_i$, $8m_i$, $2r$, or $4r$ (m_i divides n_i $1 \leqq i \leqq 3$.)*
But if the elements of order p form a single conjugate class, the case (2) does not occur and $|P|=3^2$.

PROOF. We shall examine all possibilities of K.

(1) K is a 2 group.

Assume the elements of order p form a single class. Then from the theorem of Burnside (p. 240 [5]) it follows that K acts on $P^{\#}$ transitively. Hence $|P^{\#}|$ divides $|K|$. Since K is a 2 group $|P^{\#}|$ is a power of 2. Hence we get an equality

$$|P^{\#}|=p^{\alpha}-1=2^{\beta} \qquad (\alpha, \beta : \text{positive integer}).$$

If α is even, then we get $p^{\alpha}=9$ from this. If α is odd, then we get $\alpha=1$ and $|P|=3$ or 5 from this and by lemma 8. But if $|P|=3$ or 5, the assumption $C_G(P)=P$ contradicts the assumption that G has an element of order $2p$.

(2) K is not a 2 group and 2-Sylow group of K is cyclic or generalized quaternion.

First assume 2-Sylow group is a generalized quaternion group. By the classification of groups whose 2-Sylow groups are generalized quaternion ([3]) it follows $K/O_{2'}(K)$ has an unique involution. Hence $K/O_{2'}(K)$ is a 2 group. Now $O_{2'}(K)$ must be noncyclic r group. On the other hand $O_{2'}(K)P$ is a Frobenius group, where Frobenius complement is $O_{2'}(K)$. Hence a r-Sylow group of $O_{2'}(K)$ must be cyclic. This is a contradiction.

Next assume 2-Sylow group is cyclic. Let S_0 be a 2-Sylow group of K. Then we can write $K=O_{2'}(K)S_0$ by the theorem of Burnside (p. 252 [5]). Since S_0 is cyclic, $S_0=2$ or 4 or 8. Since $O_{2'}(K)$ induces a regular automorphism group of P, $O_{2'}(K)P$ is a Frobenius group. Hence using the property of a Frobenius complement (Theorem p. 39 [9]) we get

$$|O_{2'}(K)|=r \text{ or } m_i \qquad (m_i \text{ divides } n_i \ i=1 \text{ or } 2 \text{ or } 3).$$

Hence

$$|K|=2m_i,\ 4m_i,\ 8m_i,\ 2r \text{ or } 4r.$$

Especially if we consider the case that the elements of oder p form a single class, we have that $|P^\sharp|$ divides $|K|$ by the theorem of Burnside (p. 240 [5]), and in this case we see by the property of a Frobenius complement (Theorem p. 39 [9]), that K can not induce a regular automorphism of P. Hence in this case $|K|\neq 2m_i$, $2r$ and if $|K|=4m_i$ or $4r$, order of a stabilizer of an element of $P^\sharp$ is 2 in K and if $|K|=8m_i$, it is 2 or 4. If $|K|=4m_i$ or $4r$, we can write

$$P=C_P(\tau)\times[P,\tau] \qquad \text{for any fixed involution } \tau \text{ in } K.$$

Since $C_G(P)=P$, $[P,\tau]\neq 1$. We shall consider length of orbits of $P^\sharp$ under the action of $O_{2'}(K)$ and $\langle\tau\rangle O_{2'}(K)$. Under the action of $O_{2'}(K)$, $P^\sharp$ is distributed into two orbits of same length and one orbit has the inverse elements of the elements of another orbit. But τ inverts an element $x\neq 1$ in $[P,\tau]$. If x is contained one orbit, x^{-1} is contained another orbit. Therefore $\langle\tau\rangle O_{2'}(K)$ acts on $P^\sharp$ transitively. Hence $|\langle\tau\rangle O_{2'}(K)|$ $=|P^\sharp|$ and fixes no element in $P^\sharp$. This is a contradiction. If $|K|=8m_i$, then $|P^\sharp|=2m_i$ or $4m_i$. By the same way as above $1\neq[P,\tau]$ for any fixed involution τ in K. Here also we shall consider length of orbits of $P^\sharp$ under the action of $O_{2'}(K)$ and $O_2(K)\langle\tau\rangle$. Under the action of $O_{2'}(K)$, $P^\sharp$ is distributed into 2 or 4 orbits of same length m_i and if x is contained in an orbit, x^{-1} is not contained in the same orbit. Since τ inverts nonidentity elements of $[P,\tau]$, under the action of $\langle\tau\rangle O_{2'}(K)$, $P^\sharp$ is distributed into one orbit of length $2m_i$ if $|P^\sharp|=2m$, and two orbits of same length $2m_i$ or three orbits of length $2m_i$, m_i, and m_i if $|P^\sharp|$ $=4m_i$. If $|P^\sharp|=4m_i$, $[P,\tau]\geqq 2m_i+1>P/2$ so $[P,\tau]=P$. Then the last case of three orbits can not occur. Hence in either case $|P^\sharp|=2m_i$ or $4m_i$, and fixes no element of P. This is a contradiction.

(3) $K\cong PSL(2,p')$, $PSL(2,9)$, $PSL(2,q)$, $Sz(q)$, $PSL(3,4)$, M_9 (p': Fermat or Merssene prime q: a power of $2>2$).

If $K\cong PSL(2,9)$ or $PSL(3,4)$ or M_9, $|K|$ is divisible by $3\cdot 5$, which is a contradiction.

Suppose $K\cong PSL(2,q)$ or $Sz(q)$. Then an element of odd prime order in $N_G(S_0)$ induces a fixed-point-free automorphism of $S_0\cdot P$, so by the theorem of Thompson on the nilpotency of Frobenius kernal (p. 83–92 [9]), $S_0P=S_0\times P$, which contradicts the assumption $C_G(P)=P$.

Thus we have only to consider the case $K\cong PSL(2,p')$. Then $|S_0|$ $=p'\pm 1$ and S_0 is a dihedral group of order 16 at most. Then we have only three possibilities:

(a) $p'=17$, $|K|=8\cdot 17\cdot 18$

(b) $p'=7$, $|K|=3\cdot 7\cdot 8$

(c) $p'=5$, $|K|=4\cdot 5\cdot 3$.

But (c) does not occur, since $|K|$ is not divisible by $3\cdot 5$. In case (a) since the normalizer of a 17-Sylow group is a non-abelian subgroup of order $17 \cdot 8$, we obtain $r=17$, which is a contradiction by lemma 9. In case (b), since the normalizer of 7-Sylow group is a non-abelian subgroup of order $7 \cdot 3$, we obtain $r=7$. Hence $p=3$ (c. f. lemma 9), which is a contradiction since $|K|$ is relatively prime to p. The lemma is proved.

§ 4. Order of G.

By lemma 3 we can write

$$G=n_1(kn_1+2\varepsilon)(kn_1+\varepsilon)(ln_1+1) \qquad (l: \text{non-negative integer}).$$

And by the argument in section 2 we can write

$$kn_1+2\varepsilon=Dg(X_{1i})=an_2n_3 \qquad (a: \text{positive integer}).$$

Using the bound of $|G|$ and the simplicity of G, we have the following lemma.

LEMMA 13. *We have three possibilities:*
(a) $1=0$ $kn_1+2\varepsilon \neq n_2n_3$ $kn_1+\varepsilon \neq$ *a power of* 2
(b) $l=1$ $kn_1+2\varepsilon=n_2n_3$ $kn_1+\varepsilon \neq$ *a power of* 2
(c) $l=3$ $kn_1+2\varepsilon=n_2n_3$ $kn_1+\varepsilon \neq$ *a power of* 2.
Moreover if G has five classes other than the classes containing at least one element of A_1 or A_2 or A_3, then (b) *does not occur and if* (c) *occurs in this case,*

$$|G|=n_1(n_1+2)(n_1+1)(3n_1+1).$$

PROOF. If $l \geqq 4$, then

$$|G| \geqq n_1an_2n_3(an_2n_3-\varepsilon)(4n_1+1) \geqq n_1n_2n_3(n_2n_3-1)(4n_1+1) \geqq 3n_1^2n_2^2n_3^2.$$

This is a contradiction.

If $l=2$, then $m^2-1=2n_1$. Since the left side is divisible by 4, this is a contradiction.

Assume $l=0$. Suppose $kn_1+2\varepsilon=n_2n_3$. Then

$$|G|=n_1n_2n_3(n_2n_3-\varepsilon) \leqq n_1^2n_2^2n_3^2.$$

Hence 1_G, A_1-characters, A_2-characters, A_3-characters, Y_1, Y_2 and Y_3 are all the irreducible characters of G. Since the number of irreducible characters coincides with the number of the conjugate classes, G has no element of order p, which is a contradiction. If $kn_1+\varepsilon=a$ power of 2, then

$$p \cdot (kn_1+\varepsilon) \leqq |C_G(\tau)|=kn_1+\varepsilon-Y_1(\tau) \leqq 2 \cdot (kn_1+\varepsilon),$$

which is a contradiction.

Assume $l=1$ or 3. Suppose $kn_1+\varepsilon=an_2n_3$ $(a\geqq 2)$. Then

$$3n_1^2n_2^2n_3^2\geqq|G|\geqq n_1(kn_1+2\varepsilon)(kn_1+\varepsilon)\cdot m^2\geqq n_1 2n_2n_3(2n_2n_3-1)(n_1+1).$$

We can easily see this is a contradiction. Suppose $kn_1+\varepsilon$ is a power of 2. Since the order of a 2-Sylow group of G divides $|C_G(\tau)|=m(kn_1+\varepsilon-Y_1(\tau))$, it follows $Y_1(\tau)=-(kn_1+\varepsilon)=-Dg(Y_1)$, which contradicts the simplicity of G.

Assume G satisfies the assumption of the last part of lemma. Then

$$2n_1^2n_2^2n_3^2\leqq|G|\leqq 3n_1^2n_2^2n_3^2.$$

If $l=1$ we can easily obtain a contradiction from this. If $l=3$, then we obtain from this

$$\varepsilon=1 \quad \text{and} \quad |G|=3n_1^2n_2^2n_3^2+n_1n_2n_3(n_2n_3-3n_1-1).$$

Hence we have $kn_1+2=n_2n_3\leqq 3n_1+1$. Therefore $k=1$ and obtain the assertion. The lemma is proved.

§ 5. The case in which $|G|$ has no prime factor relatively prime to $2n_1n_2n_3p$.

CASE I. $C_G(P)=P$.

(A) $l=0$ (i.e. $m=1$).

In this case $|G|=(kn_1+2\varepsilon)n_1(kn_1+\varepsilon)$. Let S be a 2-Sylow group of G. By lemma 13 (a) we have

$$kn_1+2\varepsilon=|S|n_2n_3 \quad \text{and} \quad kn_1+\varepsilon=|P|.$$

Hence using the theorem of Brauer (p. 168 [5]) we obtain $X_{1i}(\tau)=0$. From an equality

$$\varphi_{1i}=1_G-\varepsilon X_{1i}+\varepsilon Y_1,$$

we obtain $Y_1(\tau)=-\varepsilon$. Then

$$|C_G(\tau)|=kn_1+2\varepsilon=|S|n_2n_3,$$

which is a contradiction.

(B) $l=1$ or 3 and the elements of order p form a single class.

In this case by lemma 12 $|P|=3^2$ and by lemma 13 (b), (c)

$$|G|=n_1(kn_1+2\varepsilon)(kn_1+\varepsilon)\cdot m^2, \quad m^2=2^{2d}, \quad kn_1+\varepsilon=2^b\cdot 3^2$$

$$(b,\ d:\ \text{positive integers}).$$

On the other hand we have

$$2^{2d+b}\cdot 3=|C_G(\tau)|=2^d(2^b\cdot 3^2-Y_1(\tau)),$$

and we can write

$$Y_1(\tau)=3\cdot 2^b\cdot f \qquad (f:\ \text{odd integer}).$$

Then $f=1$ or -1 since $Dg(Y_1)=3^2\cdot 2^b$ and since G is simple. Hence

$$m^2=2^{2d}=(3-f)^2=16 \text{ or } 4 .$$

Since $m^2=n_1+1$ or $3n_1+1$ and $n_1\geqq 5$, we obtain $n_1=15$ or 5. If $n_1=15$, this is a contradiction since 3 does not divide n_1. If $n_1=5$, we get $G>3n_1^2n_2^2n_3^2$, which is a contradiction.

(C) $l=1$ or 3 and the elements of order p form two classes.

In this case the assumption of the last part of lemma 13 holds. Then $l=3$ and $|G|=n_1(n_1+2)(n_1+1)(3n_1+1)$. By lemma 13 (c) we have

$$n_1+2=n_2n_3\,,\quad n_1+1=2p^e\,,\quad 3n_1+1=2^{2d} \qquad (e,\, d:\text{ positive integer})\,.$$

In case I. $p=3$ or 5 by lemma 11 (i), and clearly 3 divides $n_1(n_1+2)(n_1+1)$. Then one of the following two cases occurs:

(a) $p=5$ and 3 divides $n_1n_2n_3$.

(b) $p=3$.

In case (a) putting $n_1=2\cdot 5^e-1$ in the equality $3n_1+1=2^{2d}$ we obtain an equality $3(2\cdot 5^e-1)+1=2^{2d}$. But this is impossible since $-2\not\equiv 2^{2d}$ (mod 5). In case (b), we obtain the equality

$$3^{e+1}-1=2^{2d-1}$$

from two equalities $3n_1+1=2^{2d}$ and $n_1+1=2\cdot 3^e$. On the other hand $e\geqq 2$, since $C_G(P)=P$. But the solution of the equality contradicts $e\geqq 2$.

CASE II. $C_G(P)\neq P$.

(A) $l=0$ (i. e. $m=1$).

In this case $|G|=n_1(kn_1+2\varepsilon)(kn_1+\varepsilon)$. By lemma 13 (a) we obtain

$$kn_1+2\varepsilon=|S|n_2n_3 \quad \text{and} \quad kn_1+\varepsilon=|P|\,.$$

on the other hand

$$|C_G(\tau)|=|P|\cdot|S|=kn_1+\varepsilon-Y_1(\tau) \quad \text{for an involution } \tau \text{ in } C_G(P)\,.$$

Hence

$$Y_1(\tau)=-(kn_1+\varepsilon)=-Dg(Y_1)\,,$$

which contradicts the simplicity of G.

(B) $l=1$ or 3.

In this case $|G|=n_1(kn_1+2\varepsilon)(kn_1+\varepsilon)m^2$. By lemma 13 (b), (c) we obtain

$$kn_1+2\varepsilon=n_2n_3\,,\qquad kn_1+\varepsilon=2^b\cdot p^e\,,\qquad m^2=2^{2d}p^{2h}$$

$$(b, d, e:\text{ positive integers, } h:\text{ non-negative integer})\,.$$

Hence

$$2^{2d+b}p^{2h+e}=|P|\cdot|S|=|C_G(\tau)|=2^dp^h(2^bp^e-Y_1(\tau))\,.$$

Then $Y_1(\tau)=-Dg(Y_1)$, which contradicts the simplicity of G.

§ 6. The case $|G|$ has an odd prime divisor relatively prime to $pn_1n_2n_3$.

Clearly the elements of order p form a single class and $p=3$ or 5 in this case (c.f. lemma 6). And by the last part of lemma 13 we can assume $l=0$ or

$$|G|=n_1(n_1+2)(n_1+1)(3n_1+1)\,.$$

Moreover if $|C_G(P)|=|P|$, then $|P|=3^2$ (c.f. lemma 12).

CASE I. $C_G(P)=P$.

(A) $l=0$.

In this case $|G|=n_1(kn_1+2\varepsilon)(kn_1+\varepsilon)$. By lemma 13 (a) we obtain five possibilities:

(i) $kn_1+2\varepsilon=|S|n_2n_3$ and $kn_1+\varepsilon=|R|\cdot 9$,

(ii) $kn_1+2\varepsilon=|S|\cdot|R|n_2n_3$ and $kn_1+\varepsilon=9$,

(iii) $kn_1+2\varepsilon=|S|\cdot 9n_2n_3$ and $kn_1+\varepsilon=|R|$,

(iv) $kn_1+2\varepsilon=|R|\cdot n_2n_3$ and $kn_1+\varepsilon=|S|\cdot 9$,

(v) $kn_1+2\varepsilon=9n_2n_3$ and $kn_1+\varepsilon=|S|\cdot|R|$.

In case (i), (ii), (iii) $|S|$ divides $kn_1+2\varepsilon$. Hence by the theorem of Brauer (p. 168 [5]) we obtain $X_{1i}(\tau)=0$, hence $Y_1(\tau)=-\varepsilon$. Hence $|C_G(\tau)|=kn_1+2\varepsilon$, which is a contradiction.

In case (iv), (v) $|S|$ divides $kn_1+\varepsilon$. Hence by the theorem of Brauer we obtain $Y_1(\tau)=0$. Therefore $|C_G(\tau)|=kn_1+\varepsilon$. This implies a contradiction, since $|C_G(\tau)|=3\cdot|S|$.

(B) $l=3$.

In this case $|G|=n_1(n_1+2)(n_1+1)(3n_1+1)$. By lemma 13 (c) we obtain two possibilities:

(i) $n_1+1=2\cdot|R|3^2$, $3n_1+1=$ a power of 2,

(ii) $n_1+1=2\cdot|R|$, $3n_1+1=3^2\cdot$(a power of 2).

Since now $p=3$, $|R|$ must be 5 or 7 or 5^2 by lemma 9. If we put these values of $|R|$ in (i) and (ii), we can easily see either case dosen't occur.

CASE II. $C_G(P)\neq P$.

(A) $l=0$.

In this case $|G|=n_1(kn_1+2\varepsilon)(kn_1+\varepsilon)$. By lemma 13 (a) we have five possibilities:

(i) $kn_1+2\varepsilon=|S|n_2n_3$ $kn_1+\varepsilon=|R|\cdot|P|$,

(ii) $kn_1+2\varepsilon=|S|\cdot|R|n_2n_3$, $kn_1+\varepsilon=|P|$,

(iii) $kn_1+2\varepsilon=|S|\cdot|P|n_2n_3$, $kn_1+\varepsilon=|R|$,

(iv) $kn_1+2\varepsilon=|R|\cdot n_2n_3$, $kn_1+\varepsilon=|S|\cdot|P|$,

(v) $kn_1+2\varepsilon=|P|\cdot n_2n_3$, $kn_1+\varepsilon=|S|\cdot|R|$.

We can see (i), (ii), (iii) and (v) do not occur by the same way as case I. It remains to prove case (iv) does not occur. In case (iv) using the orthogonality relation of characters we get an inequality

$$(Dg(Y_1))^2+\sum_{i=1}^{\frac{n_1-1}{2}}(Dg(X_{1i}))^2+2n_1^2n_2^2n_3^2<n_1|R|n_2n_3(|R|n_2n_3-\varepsilon)\,.$$

Now

$$\begin{aligned}\text{left side } &=2n_1^2n_2^2n_3^2+|R|^2n_2^2n_3^2n_1/2+\{(|R|n_2n_3-2\varepsilon)^2-2\}/2\\ &>2n_1^2n_2^2n_3^2+|R|^2n_2^2n_3^2n_1/2\end{aligned}$$

$$\text{right side}\leqq|R|^2n_1n_2^2n_3^2+|R|n_1n_2n_3\,.$$

Hence we obtain

$$2n_1^2n_2^2n_3^2+|R|n_2^2n_3^2n_1/2<|R|^2n_1n_2^2n_3^2+|R|n_1n_2n_3\,.$$

Dividing above inequality by $n_1n_2n_3$ we get

$$2n_1n_2n_3<|R|^2n_2n_3/2+|R|<n_2n_3|R|^2/2+n_2n_3/2\,.$$

Here the last inequality is obtained from an inequality $|R|<n_2n_3/2$. Hence

$$2n_1n_2n_3>n_2n_3(|R|^2+1)/2\,.$$

Hence

(1) $$n_1<|R|^2+1/4\,.$$

On the other hand we have an inequality

$$3n_1^2n_2^2n_3^2\geqq|G|\geqq|R|n_1n_2n_3(|R|n_2n_3-1)\,.$$

Dividing above inequality by $n_1n_2n_3$ we obtain

$$3n_1n_2n_3\geqq|R|^2n_2n_3-|R|>|R|^2n_2n_3-n_2n_3\,.$$

Here the last inequality is obtained from an inequality $|R|<n_2n_3$. Hence

$$3n_1n_2n_3>n_2n_3(|R|^2-1)\,.$$

Hence

(2) $$n_1>|R|^2-1/3\,.$$

From (1) and (2) we obtain

$$|R|^2-1/3<|R|^2+1/4\,,$$

which implies $|R|^2<7$. This is a contradiction.

(B) $l=3$.

In this case $|G|=n_1(n_1+2)(n_1+1)(3n_1+1)$. By lemma 13 (b) and (c) we obtain two possibilities:

(i) $n_1+1=2|R|\cdot|P|$, $m^2=3n_1+1=2^{2d}$ (d: positive integer),

(ii) $n_1+1=2|R|$, $m^2=3n_1+1=2^{2d}\cdot|P|$ (d: as above).

In case (i)

$$|P|2^{2d+1}=|P|\cdot|S|=|C_G(\tau)|=m(kn_1+\varepsilon-Y_1(\tau))=2^d(2|R|\cdot|P|-Y_1(\tau))\,.$$

Since $Y_1(\tau)$ is an integer we can write

$$Y_1(\tau)=2|P|f \qquad (f\text{: odd integer}).$$

By the simplicity of G we have

$$-2|R|\cdot|P|+1\leqq Y_1(\tau)\leqq 2|R|\cdot|P|+1.$$

Then there are finite possibilities for f since there are only finite known possibilities for $|R|$. Since

$$|C_G(\tau)|=2^{d+1}|P|(|R|-f),$$

$|R|-f$ must be 2^d. Moreover since $n_1\geqq 5$, $2^{2d}=3n_1+1\geqq 16$. We can pick up finite possibilities satisfying above conditions and then easily see that all have contradictions.

In case (ii) we have

$$3n_1+3=3\cdot 2\cdot|R|.$$

Now since the possibilities of $|R|$ is finite and known, we can compute the possible values of $2^{2d}\cdot|P|=3n_1+1$. We can easily find contradictions in all possibilities.

Therefore no case can occur. The theorem is proved.

Bibliography

[1] Harada, A characterization of groups $LF(2, q)$, Illinois Journal of Math. vol. 11, 1967, p. 647–659.

[2] Stewart, In preparation. (c. f. G. Higman, Odd characterization of finite simple group)

[3] Brauer & Suzuki, On finite groups of even order whose 2-Sylow group is a quaternion group, Proc. Nat. Acad. Sci., 45, 1959, p. 1757–1759.

[4] Dornhoff, Group representation theory, Dekker.

[5] Gorenstain, Finite groups, Harper & Row.

[6] R. Brauer & K. A. Fowler, On groups of even order, Ann. of Math. vol. 62, 1955, p. 565–583.

[7] Suzuki, Finite groups with nilpotent centralizers, Trans. Amer. Math. Soc. 99, 1961, p. 425–470.

[8] Suzuki, Finite groups in which the centralizer of any element of order 2 is 2 closed, Ann. Math., 82, 1965, p. 191–212.

[6] Ito, Finite groups, Kyoritsu.