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A Note on Normal Homogeneous Riemannian Spaces

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Introduction. Let M be a Riemannian space whose group of isometries admits the compact connected identity component G_0 which acts transitively on M. G_0 has an $\mathrm{Ad}(G_0)$-invariant Riemannian metric, and if the action $\mathrm{Ad}(H)$ of the isotropy group H of $o\in M$ is irreducible on $T_0(M)$, then the induced invariant metric on M coincides with the original one. More generally, let G be a connected Lie group and H be a closed subgroup of G. Denote the Lie algebras of G and H by $\mathfrak{G}$ and $\mathfrak{H}$. We suppose that G admits an $\mathrm{Ad}(G)$-invariant metric $\langle\,,\rangle$. Let $\mathfrak{M}$ be the orthogonal complement of $\mathfrak{H}$ in $\mathfrak{G}$. Since $\mathfrak{M}$ is invariant by $\mathrm{Ad}(H)$ we can extend the inner product $\langle\,,\rangle$ of $\mathfrak{M}$ to all of the coset space $M=G/H$ by the left translation of G on G/H. Thus the space M is Riemannian homogeneous and we call M with this metric a normal Riemannian homogeneous space. Our main purpose in this note is to study the nullity space of the affine curvature tensor of a normal Riemannian homogeneous space. We obtain a sufficient condition under which a normal Riemannian homogeneous space becomes the product space of a group space and a normal Riemannian homogeneous subspace. The result can be applied to the symmetric case.

1. Normal Riemannian homogeneous spaces. Let G, H and $M=G/H$ be as in the introduction. Then for all x, y of $\mathfrak{G}$ and $a\in G$, we have

$$\langle \mathrm{Ad}(a)x, \mathrm{Ad}(a)y\rangle=\langle x, y\rangle, \tag{1}$$

and hence

$$\langle [x, y], z\rangle+\langle y, [x, z]\rangle=0 \tag{2}$$

for all x, y and z of $\mathfrak{G}$. From (2) it follows that the space G/H is naturally reductive Riemannian homogeneous, i.e.,

$$[\mathfrak{M}, \mathfrak{H}]\subset\mathfrak{M} \tag{3}$$

holds good. There exists the so called canonical connection D on G/H. Denote the Riemannian connection on G/H by ∇ and the Riemannian curvature tensor by R. Then D satisfies

$$D_X Y = \nabla_X Y - \frac{1}{2} T(X, Y)$$

for vector fields X and Y on G/H. The torsion and curvature tensors T and B of D satisfy

$$T(x, y) = [x, y]_{\mathfrak{M}}, \tag{4}$$

$$B(x, y)z = [[x, y]_{\mathfrak{H}}, z] \tag{5}$$

for x, y, z of $\mathfrak{M}$. B and T are G-invariant and parallel with respect to the connection D. It can be shown easily that it holds for a vector fields X on G/H

$$R_X = B_X - \frac{1}{4} T_X^2 \tag{6}$$

where R_X, B_X and T_X are the (1, 1)-tensors given by

$$R_X(Y) = R(X, Y)X, \qquad B_X(Y) = B(X, Y)X,$$
$$T_X(Y) = T(X, Y).$$

Then by virtue of (2), (5) and (6), the Riemannian sectional curvature of the 2-plane spanned by orthonormal vectors x and y in $\mathfrak{M}$ is given by

$$k(x, y) = \|[x, y]_{\mathfrak{H}}\|^2 + \frac{1}{4} \|[x, y]_{\mathfrak{M}}\|^2. \tag{7}$$

It follows that the normal Riemannian homogeneous space has the nonnegative sectional curvature.

2. Nullity spaces. Let $\mathfrak{M}_0$ be the nullity space in $\mathfrak{M}$ of the curvature operator B_X, that is, $\mathfrak{M}_0 = \{x \in \mathfrak{M} ;\ B_x y = 0$ for all $y \in \mathfrak{M}\}$. If $x \in \mathfrak{M}_0$, then we have $\langle B_X y, y \rangle = \langle [[x, y]_{\mathfrak{H}}, x], y \rangle = \|[y, x]_{\mathfrak{H}}\|^2 = 0$ for all $y \in \mathfrak{M}$. Hence $\mathfrak{M}_0$ coincides with the space $\{x \in \mathfrak{M} ;\ [x, y]_{\mathfrak{H}} = 0$ for all $y \in \mathfrak{M}\}$ $= \{x \in \mathfrak{M} ;\ B_y x = 0$ for all $y \in \mathfrak{M}\}$. We put $\mathfrak{M}'$ = the subspace of $\mathfrak{M}$ spanned by $[\mathfrak{M}, \mathfrak{H}]$ which we write simply as $[\mathfrak{M}, \mathfrak{H}]$ and $\mathfrak{M}_0^{\perp}$ the orthogonal complement of $\mathfrak{M}_0$ in $\mathfrak{M}$.

LEMMA 1. *The subspaces $\mathfrak{M}_0, \mathfrak{M}'$ and $\mathfrak{M}_0^{\perp}$ are $Ad(H)$-invariant.*

PROOF. Since $\mathrm{Ad}(h)$ is an isomorphism of the Lie algebra $\mathfrak{G}$ for any $h \in H$, we have $\mathrm{Ad}(h)[\mathfrak{M}, \mathfrak{H}] = [\mathrm{Ad}(h)\mathfrak{M}, \mathrm{Ad}(h)\mathfrak{H}] = [\mathfrak{M}, \mathfrak{H}]$. Thus $\mathfrak{M}'$ is $\mathrm{Ad}(H)$-invariant. Next if $x \in \mathfrak{M}_0$, then $[\mathrm{Ad}(h)x, y]_{\mathfrak{H}} = (\mathrm{Ad}(h)[x, \mathrm{Ad}(h^{-1})y])_{\mathfrak{H}}$ $= \mathrm{Ad}(h)[x, \mathrm{Ad}(h^{-1})y]_{\mathfrak{H}} = 0$ for all $y \in \mathfrak{M}$, and hence $\mathrm{Ad}(h)x$ belongs to $\mathfrak{M}_0$. As $\mathrm{Ad}(h)$ is an orthogonal transformation on $\mathfrak{M}$, the orthogonal complement $\mathfrak{M}_0^{\perp}$ is $\mathrm{Ad}(H)$-invariant.

LEMMA 2. *$\mathfrak{M}'$ is contained in $\mathfrak{M}_0^{\perp}$ and especially* $[\mathfrak{M}_0, \mathfrak{H}] = (0)$.

PROOF. Let $x \in \mathfrak{M}$ and $\alpha \in \mathfrak{H}$. Then $\langle [x, \alpha], y \rangle = -\langle \alpha, [x, y] \rangle =$

$-\langle\alpha, [x, y]_{\mathfrak{H}}\rangle=0$ for all $y\in\mathfrak{M}_0$. It follows that $[x, \alpha]\in\mathfrak{M}_0^{\perp}$ and hence $\mathfrak{M}'\subset\mathfrak{M}_0$. As $\mathfrak{M}_0$ is invariant by $\mathrm{Ad}(H)$, we have $[\mathfrak{M}_0, \mathfrak{H}]\subset\mathfrak{M}_0$. On the other hand $[\mathfrak{M}_0, \mathfrak{H}]$ is a subspace of $\mathfrak{M}'$ and hence of $\mathfrak{M}_0^{\perp}$. Therefore we have $[\mathfrak{M}_0, \mathfrak{H}]=(0)$.

LEMMA 3. *If $y\in\mathfrak{M}_0^{\perp}$ satisfies $[y, \alpha]=0$ for all $\alpha\in\mathfrak{H}$, then $y=0$.*

PROOF. For any x in $\mathfrak{M}$, we have $\|[x, y]_{\mathfrak{H}}\|^2=\langle[x, y]_{\mathfrak{H}}, [x, y]\rangle=\langle[[x, y]_{\mathfrak{H}}, x], y\rangle=-\langle x, [[x, y]_{\mathfrak{H}}, y]\rangle=0$, and hence $[x, y]_{\mathfrak{H}}=0$ holds good. Thus y belongs to both of the spaces $\mathfrak{M}_0$ and $\mathfrak{M}_0^{\perp}$, from which $y=0$ follows.

THEOREM 1. *Let G/H be a normal Riemannian homogeneous space and $\mathfrak{G}=\mathfrak{M}+\mathfrak{H}$ be the natural decomposition. Then we have the following orthogonal decomposition*

$$\mathfrak{M}=\mathfrak{M}_0+[\mathfrak{M}, \mathfrak{H}]$$

where $\mathfrak{M}_0$ is the nullity space of the curvature tensor B.

PROOF. It is sufficient to show $\mathfrak{M}'=\mathfrak{M}_0^{\perp}$. For this purpose, we take the orthogonal complement $\mathfrak{M}''$ of $\mathfrak{M}'$ in $\mathfrak{M}$. Then $\mathfrak{M}''$ is invariant by $\mathrm{Ad}(H)$, and hence we have $[\mathfrak{M}'', \mathfrak{H}]\subset\mathfrak{M}''$. Clearly $[\mathfrak{M}'', \mathfrak{H}]$ is contained in $\mathfrak{M}'$. Thus $[\mathfrak{M}'', \mathfrak{H}]$ is zero space. By virtue of Lemma 3 if follows $\mathfrak{M}''=(0)$.

The following lemma is obtained by direct calculation:

LEMMA 4. *For x, y and z of $\mathfrak{M}$, we have*

$$R(x, y)z=B(x, y)z+\frac{1}{4}\{2[[x, y]_{\mathfrak{M}}, z]_{\mathfrak{M}}-[x, [y, z]_{\mathfrak{M}}]_{\mathfrak{M}}+[y, [x, z]_{\mathfrak{M}}]_{\mathfrak{M}}\}.$$

If one of the vectors x, y and z belongs to $\mathfrak{M}_0$, then

$$R(x, y)z=\frac{1}{4}[[x, y], z]=\frac{1}{4}[[x, y]_{\mathfrak{M}}, z]_{\mathfrak{M}}.$$

3. Geodesic spaces.

A subspace $\mathfrak{K}$ of $\mathfrak{M}$ is called geodesic if $T(x, y)$ and $B(x, y)z$ belong to $\mathfrak{K}$ for all x, y and z of $\mathfrak{K}$. Then it is known that $K=\mathrm{Exp}_0\mathfrak{K}$ is a complete totally geodesic subspace of $M=G/H$ and taking the smallest subalgebra $\mathfrak{G}'$ of $\mathfrak{G}$ containing $\mathfrak{K}$, K is the normal Riemannian homogeneous space G'/H' where G' is the associated connected subgroup to $\mathfrak{G}'$ of G and $H'=G'\cap H$.

LEMMA 5. *We have $[\mathfrak{M}_0, \mathfrak{M}']\subset\mathfrak{M}'$.*

PROOF. Let $x\in\mathfrak{M}_0$, $y\in\mathfrak{M}$, $\alpha\in\mathfrak{H}$. Then from Jacobi identity and Lemma 2 we have $[x, [y, \alpha]]=-[y, [\alpha, x]]-[\alpha, [x, y]]=-[\alpha, [x, y]]$. Since $[x, y]_{\mathfrak{H}}=0$, it follows that $[x, [y, \alpha]]=[[x, y]_{\mathfrak{M}}, \alpha]\in\mathfrak{M}'$. The lemma follows easily because any element of $\mathfrak{M}'$ is the union of $[y, \alpha]$, $y\in\mathfrak{M}$, $\alpha\in\mathfrak{H}$.

THEOREM 2. *$\mathfrak{M}_0$ is a subalgebra and a geodesic space.*

PROOF. Take $x, y \in \mathfrak{M}_0$. Then it is trivial that $B(x, y)$ vanishes. Let z be in $\mathfrak{M}'$. From Lemma 5 we have $[x, z] \in \mathfrak{M}'$. Hence we have $\langle T(x, y), z\rangle = \langle [x, y]_m, z\rangle = -\langle y, [x, z]_m\rangle = 0$. This means $T(x, y)$ is orthogonal to $\mathfrak{M}'$ and hence belongs to $\mathfrak{M}_0$. Therefore it follows that $[x, y] \in \mathfrak{M}_0$ which shows $\mathfrak{M}_0$ is a subalgebra.

THEOREM 3. *$\mathfrak{M}' = [\mathfrak{M}, \mathfrak{H}]$ is geodesic if and only if $[\mathfrak{M}_0, \mathfrak{M}'] = (0)$.*

PROOF. Let x, y and z be in $\mathfrak{M}'$. Then $B(x, y)z = [[x, y]_{\mathfrak{H}}, z] \in \mathfrak{M}'$, and hence $\mathfrak{M}'$ is geodesic if and only if $T(x, y) = [x, y]_{\mathfrak{M}} \in \mathfrak{M}'$, that is $[\mathfrak{M}', \mathfrak{M}']_{\mathfrak{M}} \subset \mathfrak{M}'$. On the other hand we have $\langle [x, y]_{\mathfrak{M}}, w\rangle = -\langle y, [x, w]\rangle$ for $w \in \mathfrak{M}_0$. Therefore $T(x, y) \in \mathfrak{M}'$ if and only if $[x, w]$ is orthogonal to $\mathfrak{M}'$. Since $[x, w]_{\mathfrak{H}} = 0$ and by virtue of Lemma 5, it follows that $[x, w] \in \mathfrak{M}_0 \cap \mathfrak{M}' = (0)$. The theorem is proved.

REMARK. Since the geodesic curves with respect to the canonical affine connection D are the same as those with respect to the Riemannian connection ∇, we see that $\mathfrak{M}_0$ is geodesic and $M_0 = \mathrm{Exp}_0\, \mathfrak{M}_0$ is a totally geodesic subspace of the Riemannian metric of M. M_0 is a subgroup of G.

4. Riemannian product structure. Taking consideration of $\mathrm{Ad}(H)$-invariance of $\mathfrak{M}_0$ and $\mathfrak{M}'$, there exist two orthogonal distributions V_0 and V' on G/H defined by $V_0(L_g o) = L_g \mathfrak{M}_0$, $V'(L_g o) = L_g \mathfrak{M}'$ where L_g denotes the left translation by $g \in G$ on M and o is the image of the identity by projection $G \to G/H$.

LEMMA 6. *The distribution V'(resp. V_0) is parallel along V_0(resp. V') with respect to the Riemannian connection provided that $[\mathfrak{M}_0, \mathfrak{M}'] = (0)$.*

PROOF. Take $x \in \mathfrak{M}_0$ and $y \in \mathfrak{M}'$. Then the geodesic curve $c_t = L_{\exp tx} o$ is in M_0. We show that V' is parallel along c_t. From the definition of V', it is sufficient to show that the vector field $Y = L_{\exp tx} y$ is parallel along c_t, i.e., $\nabla_X Y = 0$, $X = L_{\exp tx} x$. Since $[\mathfrak{M}_0, \mathfrak{M}'] = (0)$, we have $T(X, Y) = L_{\exp tx}[x, y]_{\mathfrak{M}} = 0$, and hence $\nabla_X Y = D_X Y = 0$ because Y is D-parallel along the geodesic c_t. On the other hand if $[\mathfrak{M}_0, \mathfrak{M}'] = (0)$, then by virtue of Theorem 3, the space $\mathfrak{M}'$ is geodesic and there is a totally geodesic subspace M' of M at o. Then the same argument as above shows that V_0 is parallel along V'.

From this lemma, it follows that the space M has the local Riemannian product structure of $M_0 \times M'$ if $[\mathfrak{M}_0, \mathfrak{M}'] = (0)$ is satisfied.

LEMMA 7. *If the sectional curvature of any 2-plane spanned by* *orthonormal vectors* $x \in \mathfrak{M}_0$ *and* $y \in \mathfrak{M}'$ *is zero, we have* $[\mathfrak{M}_0, \mathfrak{M}'] = (0)$.

PROOF. By virtue of Lemma 4, we have for $x \in \mathfrak{M}_0$, and $y \in \mathfrak{M}'$ $R(x, y)x = \frac{1}{4}[[x, y], x]$. It follows that the sectional curvature $k(x, y)$ of the 2-plane spanned by x and y is $\langle R(x, y)x, y\rangle = \frac{1}{4}\|[x, y]\|^2$. Thus $k(x, y) = 0$ if and only if $[x, y] = 0$.

Next we suppose that the group G is simply connected and $[\mathfrak{M}_0, \mathfrak{M}'] = (0)$. Taking the subspace $\mathfrak{G}' = \mathfrak{M}' + \mathfrak{H}$, we get the direct sum decomposition $\mathfrak{G} = \mathfrak{M}_0 + \mathfrak{G}'$. Since $\mathfrak{M}_0$ commutes with $\mathfrak{H}$, $[\mathfrak{M}_0, \mathfrak{G}'] = (0)$ holds good. By virtue of Theorem 3, $\mathfrak{G}'$ is a subalgebra, and $\mathfrak{M}_0$ is too. Now take the connected Lie groups G_0 and G' in G associated to subalgebras $\mathfrak{M}_0$ and $\mathfrak{G}'$. Then we have the direct product decomposition $G = G_0 \times G'$ on account of simply connectedness of G. Since H is contained in G', it follows that $G/H = G_0 \times G'/H$, and $G_0 = \exp \mathfrak{M}_0 = M_0$ and $G'/H = \mathrm{Exp}_0 \mathfrak{M}' = M'$. We see that locally the product is Riemannian, and hence we obtained the following theorem.

THEOREM 4. *Let* G/H *be a normal Riemannian homogeneous space of a simply connected Lie group* G *and* $\mathfrak{G} = \mathfrak{M} + \mathfrak{H}$ *be the natural decomposition. We suppose that the sectional curvature* $k(x, y)$ *of any 2-plane spanned by the orthonormal vectors* $x \in \mathfrak{M}_0$ *and* $y \in \mathfrak{M}'$ *is zero. Then we have the Riemannian product* $G/H = G_0 \times G'/H$. *In the group manifold* G_0, $k(x, y)$ *is given by* $\frac{1}{4}\|[x, y]\|^2$ *for orthonormal* $x, y \in \mathfrak{M}_0$ *and the submanifold* G'/H *is a normal Riemannian homogeneous space.*

REMARK. G_0 is flat if and only if $\mathfrak{M}_0$ is abelian. It is shown that if $\mathrm{Ad}(H)$ acts transitively on $\mathfrak{M}'$, then G'/H has positive sectional curvature.

COROLLARY. *Let* G *be a simply connected Lie group. If the normal Riemannian homogeneous space* G/H *is symmetric, then the nullity and its orthogonal distributions are parallel along each other, and we have the Riemannian product structure* $G/H = G_0 \times G'/H$ *where* G_0 *is flat and* G'/H *is a normal symmetric homogeneous space.*

Bibliography

[1] S. Kobayashi and K. Nomizu: Foundations of differential geometry, vol. II, New York, 1969.

[2] I. Chavel: Riemannian symmetric spaces of rank one, New York, 1972.