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Notes on infinitesimal HP-transformations in Kähler manifolds with constant scalar curvature

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Introduction. K. Yamauchi has proved the non-existence of infinitesimal projective transformations which are not Killing in a compact Riemannian manifold with non- positive constant scalar curvature. It is natural to consider infinitesimal holomorphic projective transformations in a compact Kähler manifold with non-positive constant scalar curvature corresponding to the result. So, this paper is devoted to this problem.

§1. An infinitesimal HP-transformation in a compact Kähler manifold. Let (M^n, g) be an n dimensional Kähler manifold and φ_{ij}, $R_{ijk}{}^{l}$, $R_{ij}(=R_{rij}{}^{r})$, R be its complex structure, the Riemannian curvature tensor, the Ricci tensor and the scalar curvature, respectively. Then $\varphi_i{}^r R_{ri} = -\varphi_j{}^r R_{ri}$.

If v is an infinitesimal HP-transformation, then it holds

$$\underset{v}{\mathcal{L}}\left\{{h \atop ij}\right\} = \nabla_j \nabla_i v^h + R_{rji}{}^{h} v^r = \rho_j \delta_i^h + \rho_i \delta_j^h - {}_j\tilde{\rho}\varphi_i{}^h - \tilde{\rho}_i \varphi_j{}^h \tag{1}$$

where we put $\quad \rho = \dfrac{1}{n+2} \nabla_r v^r, \qquad \rho_i = \nabla_i \rho, \qquad \tilde{\rho}_i = \varphi_i{}^r \rho_r.$

It is known [1] that in a compact Kähler manifold, an infinitesimal HP-transformation v is contravariant analytic and consequently ρ^i is contravariant analytic, $\tilde{\rho}^i$ is a Killing vector. In this manifold, the following identities are satisfied:

$$\nabla^r \nabla_r \rho^i + R_r{}^i \rho^r = 0, \tag{2}$$

$$2R_r{}^i \rho^r = \nabla_i(\Delta\rho), \qquad (\Delta\rho = -\nabla^r \nabla_r \rho) \tag{2'}$$

$$\varphi_i{}^r \nabla_r \rho_j + \varphi_j{}^r \nabla_i \rho_r = \nabla_i \tilde{\rho}_j + \nabla_j \tilde{\rho}_i = 0, \tag{3}$$

$$\underset{v}{\mathcal{L}} R_{ji} = -(n+2) \nabla_j \rho_i. \tag{4}$$

Henceforth, M^n will be compact.

§2. An infinitesimal HP-transformation in a compact Kähler manifold with constant scalar curvature. In this section, we assume R is constant. This implies $\nabla_r R_i{}^r = 0$.

We operate ∇^j to (4). As

$$\nabla^j \underset{v}{\mathcal{L}} R_{ji} = \underset{v}{\mathcal{L}} \nabla^j R_{ji} + \left(\underset{v}{\mathcal{L}} \left\{{k \atop rj}\right\} R_{ki} + \underset{v}{\mathcal{L}} \left\{{k \atop ri}\right\} R_{jk}\right) g^{rj}$$

$$= 2\rho^j R_{ji} + \rho_i R,$$

we get

$$nR_{ri}\rho^r = \rho_i R, \tag{5}$$

where we use (1), (2) and (4). By virtue of (2′) and (5), it follows

$$\nabla_i = (\Delta\rho) = \frac{2R}{n} \nabla_i \rho.$$

Then we have

$$\Delta\rho = \frac{2R}{n}\rho \tag{6}$$

because of $\int \Delta\rho d\sigma = \int \rho d\sigma = 0.$

Since

$$\rho_i \rho^i = \nabla_i(\rho\rho_i) + \rho\Delta\rho$$

it follows

$$\int \rho_i \rho^i d\sigma = \frac{2R}{n} \int \rho^2 d\sigma$$

So, if R is non-positive, then $\rho^i = 0$. This means v is affine and consequently Killing by the compactness of M^n. Hence we have the following.

THEOREM. *In a compact Kähler manifold with non-positive constant scalar curvature, an infinitesimal HP-transformation is a Killing vector.*

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