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# On a Characterization of the Bochner Curvature Tensor=0

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**Introduction.** In this paper, we give a characterization of the Bochner curvature tensor=0. Definitions, lemmas and Theorem are given in § 1. We state in § 2 some preliminary facts about $J$-bases. In § 3 Theorem is proved.

## § 1. Preliminaries.[1]

We consider in this paper a Kählerian space $M^{2m}$ of complex dimension $m(>1)$. $M^{2m}$ is a $2m(=n)$ dimensional Riemannian space admitting a parallel tensor field $J=(\varphi_\lambda{}^\mu)$ such that

$$\varphi_\lambda{}^\alpha\varphi_\alpha{}^\mu=-\delta_\lambda{}^\mu, \qquad \varphi_{\lambda\mu}(=\varphi_\lambda{}^\alpha g_{\alpha\mu})=-\varphi_{\mu\lambda} \tag{1.1}$$

where $g=(g_{\lambda\mu})$ denotes the Riemannian metric tensor. We shall denote by $R_{\lambda\mu\nu}{}^k$, $R_{\mu\nu}=R_{\lambda\mu\nu}{}^\lambda$ and $R$ the Riemannian curvature tensor, the Ricci tensor and the scalar curvature of $M^{2m}$. Putting $R_{\lambda\mu\nu\omega}=g_{\omega\alpha}R_{\lambda\mu\nu}{}^\alpha$, we shall denote by $\hat{R}$ the tensor $(R_{\lambda\mu\nu\omega})$.

A tensor $\hat{U}=(U_{\lambda\mu\nu\omega})$ of type (0, 4) will be called curvature-like, if it satisfies

$$U_{\lambda\mu\nu\omega}=-U_{\mu\lambda\nu\omega}=-U_{\lambda\mu\omega\nu}, \tag{1.2}$$

$$U_{\lambda\mu\nu\omega}+U_{\mu\nu\lambda\omega}+U_{\nu\lambda\mu\omega}=0. \tag{1.3}$$

As is well known,

$$U_{\lambda\mu\nu\omega}=U_{\nu\omega\lambda\mu}$$

hold good.

For any curvature-like tensor $\hat{U}$, we shall denote $\rho_U(X, Y)$ the $U$-sectional curvature of a 2-plane spanned by tangent vectors $X$ and $Y$;

$$\rho_U(X, Y)=\frac{-\hat{U}(X, Y, X, Y)}{||X||^2\,||Y||^2-\langle X, Y\rangle^2},$$

where $\langle X, Y\rangle=g(X, Y)$ and $||X||^2=g(X, X)$

The following lemma is well known.

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1) In § 1, tensors are written in terms of their components with respect to a natural base and the Greek indices run from 1 to $n(=2m)$ unless otherwise stated.

LEMMA a. *For any curvature-like tensor $\hat{U}$, if any U-sectional curvature vanishes, then $\hat{U}$ vanishes identically.*

A 2-plane spanned by vectors $X$ and $JX$ is called a holomorphic 2-plane. An orthonormal pair of vectors $\{X, Y\}$ such that $g(X, JY)=0$ is called an anti-holomorphic orthonormal pair. The $U$-sectional curvature $\rho_U(X, JX)$ of the holomorphic plane spanned by $X$ and $JX$ is called a holomorphic $U$-sectional curvature. We shall denote it by $H_U(X)$. Especially, when $\hat{U}=\hat{R}$, we write $\rho_R=\rho$, $H_R=H$, and call them the sectional curvature, the holomorphic sectional curvature, respectively.

Furthermore a curvature-like tensor $\hat{U}$ will be called $K$-curvature-like, if it satisfies

$$U_{\lambda\mu\nu\alpha}\varphi_\omega{}^\alpha=-U_{\lambda\mu\alpha\omega}\varphi_\nu{}^\alpha. \tag{1.4}$$

This equation means that $\hat{U}$ is hybrid with respect to the last two indices. It is easily seen that a $K$-curvature-like tensor $\hat{U}$ satisfies

$$\varphi_\nu{}^\alpha U_{\alpha\mu\nu\omega}=-\varphi_\mu{}^\alpha U_{\lambda\alpha\nu\omega}\,.$$

LEMMA b. *For any K-curvature-like tensor $\hat{U}$, if any holomorphic U-sectional curvature vanishes, then $\hat{U}$ vanishes identically.*

PROOF. For any orthonormal vectors $X$ and $Y$, putting $g(X, JY)=\cos\theta$, we have (see [2] p 517)

$$\rho(X, Y)=\frac{1}{8}\Big[3(1+\cos\theta)^2H(X+JY)+3(1-\cos\theta)^2H(X-JY)$$
$$-H(X+Y)-H(X-Y)-H(X)-H(Y)\Big].$$

As the above equation is obtained from only the property of the $K$-curvature-likeness of $\hat{R}$, it is also true for any $K$-curvature-like tensor $\hat{U}$. Thus the proof of Lemma *b* is completed on taking account of Lemm *a*. Q. E. D.

The Riemannian curvature tensor $\hat{R}=(R_{\lambda\mu\nu\omega})$ of $M^{2m}$ is, of course, an example of $K$-curvature-like tensors, and other examples are given by the following $\hat{Q}=(Q_{\lambda\mu\nu\omega})$ and $\hat{T}=(T_{\lambda\mu\nu\omega})$:

$$Q_{\lambda\mu\nu\omega}=g_{\lambda\omega}R_{\mu\nu}-g_{\mu\omega}R_{\lambda\nu}+R_{\lambda\omega}g_{\mu\nu}-R_{\mu\omega}g_{\lambda\nu}+\varphi_{\lambda\omega}S_{\mu\nu}$$
$$-\varphi_{\mu\omega}S_{\lambda\nu}+S_{\lambda\omega}\varphi_{\mu\nu}-S_{\mu\omega}\varphi_{\lambda\nu}-2\varphi_{\lambda\mu}S_{\nu\omega}-2S_{\lambda\mu}\varphi_{\nu\omega}\,,$$

$$T_{\lambda\mu\nu\omega}=g_{\lambda\omega}g_{\mu\nu}-g_{\mu\omega}g_{\lambda\nu}+\varphi_{\lambda\omega}\varphi_{\mu\nu}-\varphi_{\mu\omega}\varphi_{\lambda\nu}-2\varphi_{\lambda\mu}\varphi_{\nu\omega}\,.$$

The tensor $S_{\lambda\mu}$ appeared above is a skew symmetric tensor defined by

$$S_{\lambda\mu}=\varphi_\lambda{}^\alpha R_{\alpha\mu}\,, \tag{1.5}$$

and satisfies $\varphi_\lambda{}^\alpha S_{\alpha\mu}=-R_{\lambda\mu}$.

Let $B_{\lambda\mu\nu}{}^{\omega}$ be the Bochner curvature tensor, then $\hat{B}=(B_{\lambda\mu\nu\omega})$ is a $K$-curvature-like tensor given by

$$\hat{B}=\hat{R}-\frac{1}{2(m+2)}\hat{Q}+\frac{R}{4(m+1)(m+2)}\hat{T}.$$

Now, the purpose of this paper is to prove the following.

THEOREM.[2)] *In a Kählerian space $M^{2m}(m>1)$, the following three propositions* **A, B** *and* **C** *are equivalent to one another.*

**A.** $$\rho(X, Y)=\rho(X, JY)$$

*holds good for any anti-holomorphic orthonormal pair* $\{X, Y\}$.

**B.** $$\rho(X, Y)=\frac{1}{8}\{H(X)+H(Y)\}$$

*holds good for any anti-holomorphic orthonormal pair* $\{X, Y\}$.

**C.** *The Bochner curvature tensor vanishes identically.*

§ 2. **J-base.** In the rest of this paper, we shall consider at $p$ which is any point of a Kählerian space $M^{2m}$. It is well known that there exists an orthonormal base $\{e_\lambda\}$ of tangent space $T_p(M^{2m})$ such as

$$e_i{}^*=Je_i, \qquad i=1, \dots, m; \quad i^*=i+m.$$

Such a base will be called a $J$-base.

Henceforth, all tensors are represented by their components with respect to $J$-base. Indices $\lambda$, $\mu$, $\nu$ and $\omega$ take from 1 to $n(=2m)$ and $i$, $j$ from 1 to $m$.

Taking account of $g_{\lambda\mu}=\delta_{\lambda\mu}$ and $e_\lambda=(\delta_{\lambda\mu})$, we have

(2.1) $$\varphi_{ii^*}=-\varphi_{i^*i}=1,$$
$$\varphi_{i\lambda}=0 \quad \text{for} \quad \lambda\neq i^*.$$

The Ricci tensor and $S_{\lambda\mu}$ in (1.5) satisfy

(2.2) $$R_{ij}=R_{i^*j^*}, \qquad R_{ij^*}=-R_{i^*j},$$

(2.3) $$S_{ij}=S_{i^*j^*}=R_{i^*j}, \quad S_{ij^*}=-S_{i^*j}=R_{ij}.$$

By virtue of (2.2) and (2.3), we can get

(2.4) $$T_{ijij}=T_{ij^*ij^*}, \quad Q_{ijij}=Q_{ij^*ij^*}, \quad (i\neq j).$$

From the hybrid property of a $K$-curvature-like tensoı $\hat{U}$, we know that its components $U_{\lambda\mu\nu\omega}=\hat{U}(e_\lambda, e_\mu, e_\nu, e_\omega)$ satisfy

(2.5) $$U_{ij\nu\omega}=U_{i^*j^*\nu\omega}, \quad U_{ij^*\nu\omega}=-U_{i^*j\nu\omega}.$$

Thus taking account of (2.5), we get

2) The analogous facts have been obtained by T. Kashiwada [3] independently.

$$\rho(e_i, e_j)=\rho(e_{i^*}, e_{j^*}), \quad \rho(e_i, e_{j^*})=\rho(e_{i^*}, e_j). \tag{2.6}$$

## § 3. The proof of Theorem.

**(I) A⇒B.** Let $\{X, Y\}$ be an anti-holomorphic orthonormal pair, then

$$\rho(X, Y)+\rho(X, JY)=\frac{1}{4}\Big\{H(X+JY)+H(X-JY)+H(X+Y)+H(X-Y)-H(X)-H(Y)\Big\}. \tag{3.1}$$

holds good [2]. If we put $X'=\frac{1}{\sqrt{2}}(X+Y)$, $Y'=\frac{1}{\sqrt{2}}(X-Y)$, then $\{X', Y'\}$ is anti-holomorphic orthonormal and hence by the assumption we get

$$\rho(X, Y)=\rho(X, JY), \quad \rho(X', Y')=\rho(X', JY') \tag{3.2}$$

On the other hand, the pairs $\{X, Y\}$ and $\{X', Y'\}$ span the same 2-plane, from which we have

$$\rho(X, Y)=\rho(X', Y'). \tag{3.3}$$

As the sectional curvature depends only on the plane, we obtain

$$\begin{aligned} &H(X'+JY')=H(X-JY), && H(X'-JY')=H(X+JY), \\ &H(X'+Y')=H(X), && H(X'-Y')=H(Y), \\ &H(X')=H(X+Y), && H(Y')=H(X-Y). \end{aligned} \tag{3.4}$$

By virtue of (3.1), (3.2), (3.3) and (3.4)

$$\begin{aligned} \rho(X, Y)+\rho(X, JY)&=\rho(X', Y')+\rho(X', JY') \\ &=\frac{1}{4}\Big\{H(X'+JY')+H(X'-JY')+H(X'+Y')+H(X'-Y') \\ &\quad -H(X')-H(Y')\Big\} \\ &=\frac{1}{4}\Big\{H(X-JY)+H(X+JY)+H(X)+H(Y)-H(X+Y) \\ &\quad -H(X-Y)\Big\} \end{aligned}$$

Comparing the right hand side of the last equation with that of (3.1), we can deduce

$$H(X+Y)+H(X-Y)=H(X)+H(Y). \tag{3.5}$$

Putting $X''=\frac{1}{\sqrt{2}}(X+JY)$, $Y''=\frac{1}{\sqrt{2}}(X-JY)$, we have similarly the following equation.

$$H(X+JY)+H(X-JY)=H(X)+H(Y). \tag{3.6}$$

Substituting (3.5) and (3.6) into (3.1), we have

$$\rho(X, Y)+\rho(X, JY)=\frac{1}{4}\Big\{H(X)+H(Y)\Big\}.$$

Using $\rho(X, Y)=\rho(X, JY)$, we complete the proof.

**(II) B⇒C.** We take any $J$-base $\{e_\lambda\}=\{e_i, e_{i^*}\}$ of $T_p(M^{2m})$. As the Bochner curvature tensor is $K$-curvature-like, it is sufficient to show $B_{ii^*ii^*}=0$ $(i=1, \dots, m)$ by virtue of Lemma $b$. By the assumption,

$$\rho(X, Y)=\frac{1}{8}\Big\{H(X)+H(Y)\Big\}$$

is valid for any anti-holomorphic orthonormal pair $\{X, Y\}$. Hence we have

$$\left.\begin{aligned}\rho(e_i, e_j)&=\frac{1}{8}\Big\{H(e_i)+H(e_j)\Big\},\\ \rho(e_i, e_{j^*})&=\frac{1}{8}\Big\{H(e_i)+H(e_{j^*})\Big\},\end{aligned}\right\}\quad (i\neq j) \tag{3.7}$$

and

$$\begin{aligned}R_{ii}&=H(e_i)+\sum_{q\neq i}^{m}\Big\{\rho(e_i, e_q)+\rho(e_i, e_{q^*})\Big\}\\ &=H(e_i)+\frac{1}{4}\Big\{(m-2)H(e_i)+\sum_{q=1}^{m}H(e_q)\Big\}\\ &=\frac{m+2}{4}H(e_i)+\frac{1}{4}\sum_{1=q}^{m}H(e_q).\end{aligned}$$

Consequently it follows that

$$R_{ii}=R_{i^*i^*}=\frac{m+2}{4}H(e_i)+\frac{1}{4}\sum_{q=1}^{m}H(e_q), \tag{3.8}$$

$$R=2\sum_{i=1}^{m}R_{ii}=(m+1)\sum_{i=1}^{m}H(e_i). \tag{3.9}$$

On the other hand, it holds from the definition of $\hat{B}$ that

$$\begin{aligned}B_{ii^*ii^*}&=R_{ii^*ii^*}-\frac{1}{2(m+2)}Q_{ii^*ii^*}+\frac{R}{4(m+1)(m+2)}T_{ii^*ii^*}\\ &=-H(e_i)+\frac{4}{m+2}R_{ii}-\frac{R}{(m+1)(m+2)}.\end{aligned}$$

Substituting (3.8), (3.9) into the above equation, we have

$$B_{ii^*ii^*}=0.$$

**(III) C⇒A.** We take any $J$-base $\{e_\lambda\}$ of $T_p(M^{2m})$. The definition of $\hat{B}$ gives us the following

$$B_{ijij}=R_{ijij}-\frac{1}{2(m+2)}Q_{ijij}+\frac{R}{4(m+1)(m+2)}T_{ijij}, \qquad (i\neq j),$$

$$B_{ij^*ij^*}=R_{ij^*ij^*}-\frac{1}{2(m+2)}Q_{ij^*ij^*}+\frac{R}{4(m+1)(m+2)}T_{ij^*ij^*}, \qquad (i\neq j).$$

By (2.4) $B_{ijij}=B_{ij^*ij^*}=0$ implies that

$$R_{ijij}=R_{ij^*ij^*} \qquad (i\neq j)$$

which means $\rho(e_i, e_j)=\rho(e_i, e_{j^*})$.

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