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Reduction of a Sub-Markov Operator to its Irreducible Components

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1. Introduction

We have proved in [15] the following theorem: Let T be a positive bounded linear operator in a Banach lattice E which is irreducible[1)] and has the spectral radius $r(T)=1$. Moreover let 1 be a pole of its resolvent $R(\alpha, T)$. Then the spectrum of T on the spectral circle consists of $\exp\left(\frac{2k\pi i}{n}\right)$ ($k=0, \dots, n-1$ for a certain natural number n), each of which is a pole of $R(\alpha, T)$.

The problem is how to reduce the properties of positive, not necessarily irreducible, operators to those of positive irreducible operators.

In the finite dimensional case, this is done by the classical theory of normal form of reducible matrices. In the following A), B) and C) we give a brief survey of this theory according to F.R. Gantmacher [3].

A)[2)] A nonnegative square matrix A can be reduced by a permutation of indices to the form

$$A=\begin{pmatrix} A_1 & O & \dots O & O & \dots O \\ O & A_2 & \dots O & O & \dots O \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ O & O & \dots A_g & O & \dots O \\ A_{g+1,1} & A_{g+1,2} & \dots A_{g+1,g} & A_{g+1} & \dots O \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ A_{s,1} & A_{s,2} & \dots A_{s,g} & A_{s,g+1} & \dots A_s \end{pmatrix},$$

1) An operator T in a Banach lattice E is called *positive* if $f \geqq 0$ implies $Tf \geqq 0$ and a positive operator T is called *irreducible* if there exists no nonzero T-invariant closed ideal of E.

2) For this, see (69) p. 75 *ibid.*

where $A_1, \dots, A_s$ are irreducible matrices and in each row $A_{f,1}, \dots, A_{f,f-1}$ $(f=g+1, \dots, s)$ at least one of them is not equal to zero matrix.

B)[3] In particular, if A is a stochastic matrix, then $A_1, \dots, A_g$ are also stochastic matrices and each of $A_{g+1}, \dots, A_s$ has the maximum eigenvalue less than 1. Therefore the spectrum of A on the unit circle is reduced to those of $A_1, \dots, A_g$.

C)[4] Let $\tilde{A}=\lim_n \frac{1}{n}(I+A+\dots+A^{n-1})$. Then $\tilde{A}$ has the form

$$\tilde{A}=\begin{pmatrix} \tilde{A}_1 & O & \dots\dots O & \\ O & \tilde{A}_2 & \dots\dots O & O \\ \dots & \dots & \dots\dots & \\ O & O & \dots\dots \tilde{A}_g & \\ & & \tilde{U} & O \end{pmatrix},$$

where $\tilde{A}_1, \dots, \tilde{A}_g$ are also stochastic.

The purpose of our paper is to extend these results A), B) and C) to sub-Markov ergodic operators in $C(X)$.

In §2, we are concerned with positive sub-Markov and strongly ergodic operators in $C(X)$ having spectral radius 1. In the first part of this section, we shall determine the structure of the eigenspace of such operators for the eigenvalue 1 by making use of Kakutani's representation theorem for (AM) spaces[5] and the characterization of a point measure by a lattice homomorphic linear functional on $C(\Omega)$ (Theorem 1 and its Corollary). Using this result, a decomposition of the space X will be made and an extension of A) to our case will be given (Theorems 2 and 3). Most of the results in Theorems 1, 2 and 3 have been obtained by many authors, namely K. Yosida [24], S. P. Lloyd [9], M. Rosenblatt [17], B. Jamison [4], H. H. Schaefer [21] and T. Ando [2]. However, we believe that our method gives a unified and at the same time a clearer formulation of these results. Theorem 4 in the final part of this section may be regarded as an extension of C).

Most of the results in §3 are contained implicitly or under more restricted assumptions in our previous papers. These results will be used in §4. Especially lemmas 2 and 5, together with Proposition 10 in §4 are fundamental in the investigation of §4. Among them lemma 2, although its proof is easy, makes clear the induction process in §3 and §4.

3) For this, see Theorem 6, p. 77 or the proof of Theorem 10, p. 84 *ibid.*
4) For this, see (112) p. 97 *ibid.*
5) "abstract (M)-space" in the paper by S. Kakutani [5] will be denoted by (AM) space.

§ 4 contains our main result, Theorem 6, which asserts that the spectral properties on the spectral circle of an arbitrary positive uniformly ergodic sub-Markov operator are determined by the spectral properties of its irreducible components. This theorem is an extension of B), whereas Theorem 5 is an extension of C). Finally as an application of Theorem 6 we shall give Theorem 7 stating the spectral properties on the spectral circie for positive uniformly ergodic sub-Markov operators.

Notations in this paper are as follows: For a Banach space E, $\mathfrak{L}(E)$ is the set of bounded linear operators on E and E' is the dual of E. For $T\in\mathfrak{L}(E)$, we shall denote by T', $r(T)$, $R(\alpha, T)$, $\rho(T)$, $\sigma(T)$, $P_\sigma(T)$, $R_\sigma(T)$ and $C_\sigma(T)$ the dual, the spectral radius, the resolvent operator, the resolvent set, the spectrum, the point spectrum, the residual spectrum and the continuous spectrum of T respectively. $\rho_\infty(T)$ is the unbounded component of $\rho(T)$. Γ stands for the unit circle, i. e., $\Gamma=\{\alpha: |\alpha|=1\}$. $C(X)$ is the Banach lattice consisting of continuous functions on a compact Hausdorff space X, and, for $x\in X$, ε_x is the point measure which corresponds to x. For a subset S of X, $\boldsymbol{1}_S$ is the characteristic function of S. In particular, $\boldsymbol{1}_X$ will be denoted simply by $\boldsymbol{1}$. A^c, A^- and A° is the complement, the closure and the open kernel of A respectively. Finally the usual notation of a vector lattice is used: $f\vee g$ is the join of f and g, and $f^+=f\vee 0$, $f^-=(-f)\vee 0$ and $|f|=f\vee(-f)$ is the positive part, the negative part and the absolute value of f respectively.

2. A decomposition theory.

Let X be a compact Hausdorff space and $E=C(X)$. We suppose that $T\in\mathfrak{L}(E)$ is a positive, sub-Markov and strongly ergodic operator with $r(T)=1$. Thus we assume that

I) $T\geqq 0$
II) $T\boldsymbol{1}\leqq\boldsymbol{1}$
III) $r(T)=1$
IV) $\dfrac{I+T+\dots+T^{n-1}}{n}=M_n$ converges strongly.

We denote by P the limit operator of M_n, then P is a nonzero[6], positive, sub-Markov projection operator with the spectral radius $r(P)=1$. Moreover following relations are clear:

V) $PT=TP=P$, $P'T'=T'P'=P'$
VI) For $f\in E$, $Tf=f$ is equivalent to $Pf=f$

6) By M.G. Krein and M.A. Rutman [8], 1 is an eigenvalue of T'.

VII) For $\varphi \in E'$, $T'\varphi = \varphi$ is equivalent to $P'\varphi = \varphi$.[7]

The eigenspace of T for eigenvalue 1 is the closed subspace PE. Inducing the order of E, PE becomes an ordered linear space. We can prove easily the following

PROPOSITION 1. *The order of PE induced by that of E is lattice ordered: for any $f, g \in PE$, $P(f \vee g)$ is the least upper bound of f and g which will be denoted hereafter by $f \curlyvee g$.*

REMARK 1. $f \curlyvee g \geqq f \vee g$. $f \curlyvee g = f \vee g$ for any $f, g \in PE$ if and only if PE is a sublattice of E.

REMARK 2. Let P be *strictly positive, i.e.*, $f \geqq 0$ and $Pf = 0$ imply $f = 0$, then PE is a sublattice.

PROPOSITION 2. *PE is an* (AM) *space with order unit $P\boldsymbol{1}$.*

PROOF. Since P is a positive sub-Markov projection, the following inequalities for $f \in PE$ are equivalent to each other:

$$\|f\| \leqq 1, \quad -\boldsymbol{1} \leqq f \leqq \boldsymbol{1}, \quad -P\boldsymbol{1} \leqq f \leqq P\boldsymbol{1}. \tag{1}$$

We shall prove $\|f \curlyvee (-f)\| = \|f\|$. Let c be an arbitrary positive number. Then, Proposition 1 implies the equivalence of $-cP\boldsymbol{1} \leqq f \leqq cP\boldsymbol{1}$ to $-cP\boldsymbol{1} \leqq f \curlyvee (-f) \leqq cP\boldsymbol{1}$. This together with the equivalence of (1) implies the equivalence of $\|f\| \leqq c$ to $\|f \curlyvee (-f)\| \leqq c$. Therefore $\|f \curlyvee (-f)\| = \|f\|$. Since the order and the norm of PE are induced by those of E, the property of the norm that $\|f\| \leqq \|g\|$ whenever $0 \leqq f \leqq g$ is inherited to PE. Therefore PE is a Banach lattice. The equivalence of (1) shows that $P\boldsymbol{1}$ is the order unit of PE. Then PE is an (AM) space with order unit $P\boldsymbol{1}$. //

PROPOSITION 3. *$(PE)'$ is isomorphic to $P'E'$ as a Banach lattice. In more detail, for any $\varphi \in (PE)'$, the mapping*

$$\varphi \rightarrow \phi = \varphi \circ P$$

is a bijective linear, order-preserving, isometric mapping between $(PE)'$ and $P'E'$, the inverse mapping being given by the restriction of ϕ on the space PE.

PROOF. Since P is a positive contractive projection, the two mappings in the proposition are positive contractions and are inverses to each other. This proves the proposition. //

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7) T. Ando pointed out to us the following fact: For operator T such that $\|T^n\|$ is bounded, the existence of a projection operator P satisfying properties VI) and VII) is equivalent to the strong ergodicity of T with the limit operator P.

Let Φ be the set of invariant probability measures on X, i. e., $\Phi=\{\varphi:\ \varphi\in E',\ T'\varphi=\varphi,\ \varphi\geqq 0,\ \varphi(1)=1\}$ and let Λ be the set of extreme points of Φ. Then we have the following

THEOREM 1. *PE is isomorphic to* $C(\Lambda)$ *as a Banach lattice.*

PROOF. Applying Kakutani's representation theorem for (AM) spaces to proposition 2, we see that PE is isomorphic to $C(\Omega)$ as a Banach lattice, where Ω is a compact Hausdorff space. This together with proposition 3 implies that $P'E'$ is isomorphic to $C(\Omega)'$ as a Banach lattice. Since it is well known that Ω is homeomorphic to the set of extreme points of probability measures on Ω, we see that Ω is homeomorphic to Λ. Therefore PE is isomorphic to $C(\Lambda)$ as a Banach lattice. //

It is known, for example in H. H. Schaefer [20] p. 213, that a probability measure φ on a compact Hausdorff space Ω is an extreme point of probability measures if and only if φ is a lattice homomorphic linear functional on $C(\Omega)$. By this fact and Theorem 1, we have

COROLLARY. *An element* $\varphi\in\Phi$ *belongs to* Λ *if and only if* φ *is lattice homorphic on PE, i. e.,*

$$\varphi(f \veebar g)=\mathrm{Max}(\varphi(f),\ \varphi(g)) \qquad \textit{for any } f,\ g\in PE.$$

For any $\lambda\in\Lambda$, let S_λ be the support of λ and let S be the closure of $\bigcup_{\lambda\in\Lambda} S_\lambda$. Then we have

PROPOSITION 4. *Let* f *be an element of* E. *Then*

$$P|f|=0 \qquad \textit{if and only if} \qquad f=0 \ \textit{on}\ S.$$

PROOF. This follows from the fact that

$$\lambda(P|f|)=\lambda(|f|) \quad \text{for} \quad \lambda\in\Lambda$$

and Theorem 1. //

REMARK. $\{f:\ f=0$ on $S\}$ is the T-radical in the sense of H. H. Schaefer [22]. See also Theorem 4 ibid.

PROPOSITION 5. *Let* $\lambda\in\Lambda$. *Then* $P'\varepsilon_x=\lambda$ *for any* $x\in S_\lambda$.

PROOF. To prove this proposition, we remark that $P\mathbf{1}=\mathbf{1}$ on S_λ and $f\veebar g=f\vee g$ on S_λ for $f,\ g\in PE$. These are clear from the following relations:

$$f\veebar g-f\vee g\geqq 0,$$

$$\lambda(f\veebar g-f\vee g)=P'\lambda(P(f\vee g)-f\vee g)=0,$$

$$\mathbf{1}-P\mathbf{1}\geqq 0,$$
$$\lambda(\mathbf{1}-P\mathbf{1})=P'\lambda(\mathbf{1}-P\mathbf{1})=0.$$

Therefore $P'\varepsilon_x$ for any $x\in S_\lambda$ is lattice homomorphic on PE and $P'\varepsilon_x(\mathbf{1})=1$ and hence $P'\varepsilon_x\in\Lambda$. Suppose that $P'\varepsilon_x\neq\lambda$, then there exists $f_0\in PE$ such that $f_0\geqq 0$, $\lambda(f_0)=0$ and $P'\varepsilon_x(f_0)>0$ by Theorem 1. The latter inequality means $f_0(x)>0$. Since x is in the support of λ, $\lambda(f_0)>0$ must hold which is a contradiction. //

For any $\lambda\in\Lambda$, let

$$X_\lambda=\{x\in X:\ P'\varepsilon_x=\lambda\},$$
$$X_0=\{x\in X:\ P\mathbf{1}(x)=1,\ (f\veebar g)(x)=(f\vee g)(x)\ \text{for}\ f,\ g\in PE\}$$

and

$$X_1=\{x\in X:\ P\mathbf{1}(x)=1\}.$$

Then the following theorem are obtained:

THEOREM 2. i) *For $\lambda\in\Lambda$, S_λ is a compact subset of X_λ.* ii) *S is a set of maximal measure for $\varphi\in\Phi$, i.e., for any $\varphi\in\Phi$, $\varphi(S)=1$.* iii) *X_λ for $\lambda\in\Lambda$ is a compact set and X_λ and X_μ for $\lambda,\mu\in\Lambda$, $\lambda\neq\mu$ are disjoint.* iv) *X_0 is compact and $X_0=\bigcup_{\lambda\in\Lambda}X_\lambda$.* v) *Let $f\in E$ and $\lambda\in\Lambda$. Then*

a) *$f=0$ on S_λ implies $Pf=0$ on X_λ,*
b) *$f=0$ on S_λ implies $Tf=0$ on S_λ and $Pf=0$ on S_λ,*
c) *$f=0$ on S implies $Tf=0$ on S and $Pf=0$ on S,*
d) *$f=0$ on X_λ implies $Tf=0$ on X_λ and $Pf=0$ on X_λ,*
e) *$f=0$ on X_0 implies $Tf=0$ on X_0 and $Pf=0$ on X_0,*
f) *$f=0$ on X_1 implies $Tf=0$ on X_1 and $Pf=0$ on X_1.*

PROOF. i) follows from Proposition 5. Since φ is a $\sigma(E',E)$-limit of convex combinations of elements of Λ, the proof of ii) is clear. Let τ be the mapping of X into $P'E'$ equipped with $\sigma(E',E)$-topology such that

$$\tau:\ x\rightarrow P'\varepsilon_x. \tag{2}$$

Then τ is clearly continuous. This implies that X_λ for $\lambda\in\Lambda$ and X_0 are compact. The remaining part of iii) is a direct consequence of the definition and iv) follows from Corollary of Theorem 1. Let us prove v). a) is clear by the definition of X_λ. Since P is the strong ergodic limit of T, the conclusions for P in b)$\sim$f) follow from those for T. In proving the remaining part, we may assume that $0\leqq f\leqq\mathbf{1}$. Then b) is proved easiiy. c) follows from b) and e) follows from d) and iv). Therefore, we have to prove d) and f). To prove d), let η be an arbitrary positive number and let $A_\eta=\{x:\ f(x)\geqq\eta\}$. Then A_η is a compact subset of X and disjoint from X_λ. Let $\tau(A_\eta)=B_\eta$. Then

B_η is a compact subset of E' disjoint from $\tau(X_\lambda)=\lambda$. For any $\varphi\in B_\eta$, there exists $f_\varphi\in C(X)$ such that $\lambda(f_\varphi)=0$, $\varphi(f_\varphi)>0$. Since $\lambda, \varphi\in P'E'$, we may assume $f_\varphi\in PE$. By compactness of B_η, there exists a finite number of elements $f_1, \dots, f_n\in PE$ such that $\lambda(f_i)=0$ and $\mathrm{Max}\{\varphi(f_1), \dots, \varphi(f_n)\}>0$ for every $\varphi\in B_\eta$. Let $f_0=f_1\vee\dots\vee f_n\vee 0$. Then $f_0\in PE$. It is easy to show that $\lambda(f_0)=0$ and $\varphi(f_0)>0$ for every $\varphi\in B_\eta$. Since B_η is compact, $\varphi(f_0)$ has the positive minimum c on B_η. Then $f_0(x)\geqq c$ on A_η and $f_0(x)=0$ on X_λ. These imply

$$0\leqq f\leqq \eta+\frac{1}{c}f_0$$

and hence

$$0\leqq Tf\leqq T\eta+\frac{1}{c}Tf_0\leqq \eta+\frac{1}{c}f_0.$$

Therefore

$$0\leqq Tf\leqq \eta \quad \text{on } X_\lambda.$$

This holds for any positive number η, and assertion d) is proved. To prove f), let $f_0=\boldsymbol{1}-P\boldsymbol{1}$. Then, $f_0(x)=0$ on X_1. For an arbitrary positive number η, there exists a positive number c such that $f_0(x)\geqq c>0$ on $A_\eta=\{x: f(x)\geqq\eta\}$. Since $Tf_0\leqq f_0$ is clear, the proof hereafter is similar to the corresponding part of the proof of d) //

REMARK. Let τ be the mapping defined in (2). Then the inverse image of Λ by τ is equal to X_0.

By decomposition $f=f^+-f^-$, we have

COROLLARY. *The equality sign* $=$ *in Theorem* 2, v) *can be replaced by the inequality sign* $\geqq$. (*For example,* $f\geqq 0$ *on* S_λ *implies* $Pf\geqq 0$ *on* X_λ.)

Let $f_\lambda\in C(S_\lambda)$, then there exists an element $f\in C(X)$ such that the restriction of f on S_λ is f_λ. By Theorem 2, v), b), the restriction $(Tf)_{|S_\lambda}$ of Tf on S_λ is uniqueiy determined by f_λ. We define this mapping by T_λ. Thus

$$T_\lambda : f_\lambda \rightarrow (Tf)_{|S_\lambda}.$$

Similarly we can define P_λ. For $C(X_\lambda)$ also we can go on in the same way as for $C(S_\lambda)$. The operators corresponding to T_λ and P_λ will be denoted by U_λ and Q_λ respectively. Then we have

THEOREM 3. *T_λ [resp. U_λ] is a positive Markox operator in $C(S_\lambda)$ [resp. $C(X_\lambda)$] with the spectral radius $=1$ and strongly ergodic, the limit operator being P_λ [resp. Q_λ]. The eigenspace for T_λ [resp. U_λ] is one dimensional with the base $\boldsymbol{1}_{S_\lambda}$ [resp. $\boldsymbol{1}_{X_\lambda}$] and the eigenspace for T_λ' [resp. U_λ'] is one dimensional with the base $\lambda_{|S_\lambda}$ [resp. $\lambda_{|X_\lambda}$]. Moreover T_λ*

is irreducible.

PROOF. Corollary of Theorem 2 implies that T_λ is positive. It is clear that T_λ is strongly ergodic with limit operator P_λ. $Pf(x) = P'\varepsilon_x(f) = \lambda(f)$ for any $x \in S_\lambda$ and $f \in E$ implies $P_\lambda f_\lambda = \lambda_{|S_\lambda}(f_\lambda)\mathbf{1}_{S_\lambda}$ for any $f_\lambda \in C(S_\lambda)$. Then T_λ is a Markov operator with $r(T_\lambda)=1$ and the eigenspace of T_λ is one dimensional with the base $\mathbf{1}_{S_\lambda}$. And also $P_\lambda'\varphi_\lambda(f_\lambda) = \lambda_{|S_\lambda}(f_\lambda)\varphi_\lambda(\mathbf{1}_{S_\lambda})$ for any $\varphi_\lambda \in C(S_\lambda)'$ and $f_\lambda \in C(S_\lambda)$. Therefore the eigenspace of T_λ' is one dimensional with the base $\lambda_{|S_\lambda}$. Replacing T_λ, S_λ and P_λ by U_λ, X_λ and Q_λ respectively, we have the proof of all the assertions for U_λ in the theorem. To prove the irreducibility of T_λ, let J be a nonzero closed T_λ-invariant ideal of $C(S_\lambda)$. Then J is P_λ-invariant since T_λ is strongly ergodic. For nonzero f_λ in J, $P_\lambda(|f_\lambda|) = \lambda_{|S_\lambda}(|f_\lambda|)\mathbf{1}_{S_\lambda}$ is in J. Since $\lambda_{|S_\lambda}$ is strictly positive, $\mathbf{1}_{S_\lambda}$ is in J which implies $\mathrm{J} = C(S_\lambda)$. //

For strict positivity of P we have

PROPOSITION 6. *The following five conditions are equivalent to each other:*

i) *P is strictly positive,*

ii) *$\bigcup_{\lambda \in \Lambda} S_\lambda$ is dense in X,*

iii) *there exists a strictly positive set of positive T-invariant functionals φ_α for $\alpha \in A$,*

iv) *Φ is strictly positive,*

v) *Λ is strictly positive,*

where the strict positivity of a set Ψ of positive functionals means that $\phi(|f|)=0$ for all $\phi \in \Psi$ implies $f=0$.

PROOF. Equivalence of i) and ii) follows from Proposition 4. Equivalence of ii) and v) is clear by the definition of S_λ. Equivalence of iii), iv) and v) is also clear //

REMARK 1. Let one of the conditions in Proposition 6 is satisfied, then $S=X_0=X_1=X$, in other words, $P\mathbf{1}=\mathbf{1}$ and $X=\bigcup_{\lambda \in \Lambda} X_\lambda = (\bigcup_{\lambda \in \Lambda} S_\lambda)^-$.

REMARK 2. If there exists a strictiy positive T-invariant functional on X, then condition v) in the proposition is satisfied and P is strictly positive. This result was given in H. H. Schaefer [22] Proposition 12.

THEOREM 4. *Let X be metrizable and P be strictly positive. Then $\{\lambda : X_\lambda \neq S_\lambda\}$ is a set of the first category in Λ.*

To prove Theorem 4, we give the following

PROPOSITION 7. *Let P be strictly positive and let O be an open subset of X. Then the set*

$$A=\{\lambda:\ \lambda\in\Lambda,\quad S_\lambda\cap O=\phi,\quad X_\lambda\cap O\neq\phi\}$$

is a nowhere dense subset of Λ.

PROOF. Let $A^{-\circ}=B$ be nonempty. Then, since the mapping τ defined by (2) in the proof of Theorem 2 is a continuous mapping from X onto Λ, the set $C=\{x:\ x\in X,\ P'\varepsilon_x\in B\}$ is a nonempty open subset of X and the set $D=C\cap O$ is also a nonempty open subset of X. There exists an element $f_0\in C(X)$ such that $f_0\geqq 0$, $f_0\neq 0$ and f_0 is identically 0 on D^c. Then, for $\lambda\in A$, $S_\lambda\subset O^c\subset D^c$. This implies $\lambda(f_0)=0$ for every λ in A and also $\lambda(f_0)=0$ for every λ in A^- and therefore

$$\lambda(f_0)=0\quad\text{for every}\quad \lambda\in B. \tag{3}$$

On the other hand, let $\lambda\in B^c$. Then $S_\lambda\subset C^c\subset D^c$. This implies

$$\lambda(f_0)=0\quad\text{for every}\quad \lambda\in B^c. \tag{4}$$

(3) and (4) imply $\lambda(f_0)=0$ for every $\lambda\in\Lambda$. This contradicts the strict positivity assumption for P by the equivalence of i) and v) in Proposition 6. //

PROOF OF THEOREM 4. Since X is a metrizable compact Hausdorff space, there exists a countable open base $\{O_n\}$ of X. Let

$$A_n=\{\lambda:\ S_\lambda\cap O_n=\phi,\ O_n\cap X_\lambda\neq\phi\}.$$

Then, by Proposition 7, A_n is a nowhere dense subset of Λ. Let $X_\lambda\neq S_\lambda$. Then $X_\lambda\backslash S_\lambda$ includes a nonempty set $O_n\cap X_\lambda$, since $X_\lambda\backslash S_\lambda$ is open in X_λ. Therefore $\{\lambda:\ S_\lambda\neq X_\lambda\}$ is included in the set $\bigcup_{n=1}^{\infty} A_n$. This shows that the set $\{\lambda:\ S_\lambda\neq X_\lambda\}$ is a set of the first category. //

REMARK. The conclusion of Theorem 4 can not be replaced by the assertion that the set $\{\lambda:\ X_\lambda\neq S_\lambda\}$ is empty.

Counter-example[8]: Let $X=[-1, 1]$ and let

$$Tf(x)=\frac{1}{2}(1-|x|)f(-|x|)+\frac{1}{2}(1+|x|)f(|x|).$$

Then $T=P$, $PE=\{f\in C(X):\ f(x)=f(-x)\}$ and P is strictly positive. Therefore $X=X_0$. For $x\in X$,

8) This example is due to S. Miyajima.

$$P'\varepsilon_x=\frac{1}{2}(1-|x|)\varepsilon_{-|x|}+\frac{1}{2}(1+|x|)\varepsilon_{|x|}.$$

Thus Λ is homeomorphis to $[0, 1]$ by the correspondence

$$\lambda_y=\frac{1}{2}(1-y)\varepsilon_{-y}+\frac{1}{2}(1+y)\varepsilon_y$$

for any $y\in[0, 1]$. Then $S_{\lambda_y}=X_{\lambda_y}=\{y, -y\}$ for $y\in[0, 1)$, but

$$S_{\lambda_1}=\{1\}\neq X_{\lambda_1}=\{-1, 1\}.$$

3. Some lemmas.

LEMMA 1[9]. *Let E be a Banach space, T be in $\mathfrak{L}(E)$, F be a T-invariant closed subspace of E and α be in the unbounded component $\rho_\infty(T)$ of $\rho(T)$. Then F is $R(\alpha, T)$-invariant.*

PROOF. Let $|\alpha|>\|T\|$. Then the expansion $R(\alpha, T)=\sum_{n=0}^{\infty}\frac{T^n}{\alpha^{n+1}}$ shows that F is $R(\alpha, T)$-invariant. Since $R(\alpha, T)$ is holomorphic in $\rho_\infty(T)$, the conclusion follows for all elements α in $\rho_\infty(T)$. //

REMARK. The conclusion of the lemma does not hold necessarilly if we replace the condition $\alpha\in\rho_\infty(T)$ by the one $\alpha\in\rho(T)$.

Counter-example: Let T be the bilateral shift, *i.e.*, let $n=0, \pm 1, \pm 2, \dots$, $E=\{\{x_n\} : x_n\in \boldsymbol{R}, -\infty<\lim_{n\to+\infty} x_n=\lim_{n\to-\infty} x_n<+\infty\}$ and T be the operator defined by $T\{x_n\}=\{x_{n+1}\}$. Then, $0\in\rho(T)$ and $R(0, T)=T^{-1}$ and the subspace $\{\{x_n\} : x_n=0 \text{ for } n\geqq 0\}$ is T-invariant but not $R(0, T)$-invariant.

For an arbitrary operator T in a Banach space E, and for an arbitrary T-invariant closed subspace F of E, we define the operator T/F by $(T/F)\pi(f)=\pi(Tf)$, where π is the canonical mapping of E onto E/F. This operator T/F will be called the operator induced on E/F by T.

LEMMA 2[10]. *Let $\mathfrak{R}=\{\alpha : \alpha\in\rho(T), R(\alpha, T)F\subset F\}$. Then the following relations hold:*

$$\rho_\infty(T)\subset\mathfrak{R}=\rho(T_{|F})\cap\rho(T/F)=\rho(T)\cap\rho(T/F)=\rho(T)\cap\rho(T_{|F}). \tag{5}$$

Further, for $\alpha\in\mathfrak{R}$,

9) For this lemma see I. Sawashima [18], Lemma 1.

10) For this lemma, see A.E. Taylor [23], p. 270, I. Sawashima [18], Lemma 1, 2, F. Niiro and Sawashima [15], footnote 10), H.P. Lotz [10], Lemma 4.8 and H.P. Lotz and H.H. Schaefer [11], Lemma.

$$R(\alpha, T_{|F}) = R(\alpha, T)_{|F} \tag{6}$$

$$R(\alpha, T/F) = R(\alpha, T)/F^{11)} \tag{7}$$

$$R(\alpha, T)f = g - R(\alpha, T_{|F})\{(\alpha - T)g - f\} \tag{8}$$

where g is any element such that $R(\alpha, T/F)\pi(f) = \pi(g)$.

PROOF. Lemma 1 implies $\rho_\infty(T) \subset \mathcal{R}$. We shall prove the following four inclusion relations from which (5) is proved easily:

$$\mathcal{R} \subset \rho(T_{|F}) \cap \rho(T/F) \tag{9}$$

$$\rho(T_{|F}) \cap \rho(T/F) \subset \rho(T) \tag{10}$$

$$\rho(T) \cap \rho(T_{|F}) \subset \mathcal{R} \tag{11}$$

$$\rho(T) \cap \rho(T/F) \subset \mathcal{R}. \tag{12}$$

By definition of $\mathcal{R}$, it is clear that (6) and (7) hold. Consequently (9) is proved. (11) and (12) are also clear. To prove (10), let us assume that α is an element of $\rho(T_{|F}) \cap \rho(T/F)$. For any $f \in E$, let g satisfy $\pi(g) = R(\alpha, T/F)\pi(f)$. Then $\alpha\pi(g) - (T/F)\pi(g) = \pi(f)$. This means $(\alpha g - Tg) - f \in F$. Let

$$h = g - R(\alpha, T_{|F})\{(\alpha g - Tg) - f\}.$$

Then h is uniquely determined by f, since $f = 0$ implies $g \in F$ and consequently $h = 0$. We define $Rf = h$. Then

$$\begin{aligned}(\alpha I - T)Rf &= (\alpha I - T)h \\ &= (\alpha g - Tg) - (\alpha I - T)R(\alpha, T_{|F})\{(\alpha g - Tg) - f\} \\ &= \alpha g - Tg - \{(\alpha g - Tg) - f\} \\ &= f.\end{aligned}$$

Thus $(\alpha I - T)R = I$.

Let f_0 be an arbitrary element of E and let $f = (\alpha I - T)f_0$. Then $\pi(f) = \alpha\pi(f_0) - (T/F)\pi(f_0)$ and consequently $\pi(f_0) = R(\alpha, T/F)\pi(f)$. Thus, by definition of R,

$$Rf = f_0 - R(\alpha, T_{|F})\{(\alpha f_0 - Tf_0) - f\} = f_0.$$

Therefore $R(\alpha I - T) = I$. Then R is the resolvent operator of T and so $\alpha \in \rho(T)$. Thus (10) is proved. The definition of $R = R(\alpha, T)$ implies (8). //

COROLLARY 1. *Let $\alpha \in \mathcal{R}$. Then*

$$\|R(\alpha, T/F)\|,\ \|R(\alpha, T_{|F})\| \leqq \|R(\alpha, T)\|$$

and

11) $R(\alpha, T)/F$ is the operator induced on E/F by $R(\alpha, T)$.

$$\|R(\alpha, T)\| \leqq \|R(\alpha, T/F)\| + \|R(\alpha, T_{|F})\| \{(|\alpha| + \|T\|) \|R(\alpha, T/F)\| + 1\}.$$

PROOF. The inequalities are clear by (6), (7) and (8) in Lemma 2.//

COROLLARY 2. *Let α_0 be an isolated point of $\mathcal{R}^c$. Then the following implications hold.*

i) *If α_0 is a pole of $R(\alpha, T)$ of order k, then α_0 is a pole of both $R(\alpha, T_{|F})$ and $R(\alpha, T/F)$ of order at most k.*

ii) *If α_0 is a pole of $R(\alpha, T_{|F})$ of order k_1 and of $R(\alpha, T/F)$ of order k_2, then α_0 is a pole of $R(\alpha, T)$ of order at most k_1+k_2.*

PROOF. Let r be a positive number such that

$$A=\{\alpha:\ 0<|\alpha-\alpha_0|\leqq r\}\subset\mathcal{R}.$$

Then α_0 is a pole of $R(\alpha, T)$ of order at most k if and only if $\|(\alpha-\alpha_0)^k R(\alpha, T)\|$ is bounded on A. This together with inequalities in Corollary 1, proves (i) and (ii). //

LEMMA 3. *Let E be a Banach space, $T\in\mathfrak{L}(E)$ and $\alpha_0\in\rho(T)$. If c and d are positive numbers such that $\|R(\alpha_0, T)\|\leqq c$ and $cd<1$, then $|\alpha-\alpha_0|<d$ implies $\alpha\in\rho(T)$ and $\|R(\alpha, T)\|<\dfrac{c}{1-cd}$.*

PROOF. It is clear by the expansion

$$R(\alpha, T)=\sum_{n=1}^{\infty} R(\alpha_0, T)^n(\alpha-\alpha_0)^{n-1}. \qquad //$$

LEMMA 4. *Let E be a Banach space, $T\in\mathfrak{L}(E)$ be a contraction. If α_0 is a complex number such that $|\alpha_0|=1$ and α_0 is a pole of $R(\alpha, T)$ of order k, then $k=0$ or $k=1$.*

PROOF. It is enough to prove the lemma in the case $\alpha_0=1$. Let $\alpha>1$, then the expansion

$$R(\alpha, T)=\sum_{n=0}^{\infty} T^n/\alpha^{n+1}$$

implies

$$\|R(\alpha, T)\|\leqq\frac{1}{\alpha-1}.$$

Let A_{-k} be the leading coefficient of $R(\alpha, T)$ at $\alpha=1$. Then

$$\|A_{-k}\|=\lim_{\alpha\downarrow 1}\|(\alpha-1)^k R(\alpha, T)\|\leqq\lim_{\alpha\downarrow 1}(\alpha-1)^{k-1}.$$

Therefore k must be less than or equal to 1. //

LEMMA 5.[12] *Let E be a Banach lattice and $T\in\mathfrak{L}(E)$ be a positive*

12) For this lemma see F. Niiro [12], Lemma 2, also F. Niiro [13], Theorem 2, I. Sawashima [18], Lemma 6 and F. Niiro and I. Sawashima [15], Proposition 7.4.

operator with $\|T\|=r(T)=1$ *and* 1 *be a pole of* $R(\alpha, T)$ *and let* P *be the residual operator of* $R(\alpha, T)$ *at* 1. *If* b *is a positive number such that*

$$\sup_{\alpha>1} \|R(\alpha, T)(I-P)\| \leqq b^{13)},$$

then the relations

$$f \in E, \quad |\alpha_0|=1 \quad and \quad \|Tf-\alpha_0 f\| < \frac{1}{16b}\|f\|$$

imply

$$\|P|f|\| \geqq \frac{1}{2}\|f\|.$$

PROOF. Let $Tf-\alpha_0 f=g$. Then

$$T|f| \geqq |f|-|g|.$$

Let $\alpha=1+\frac{1}{4b}$. Then

$$(\alpha I-T)|f| \leqq (\alpha-1)|f|+|g|.$$

Since $R(\alpha, T)$ is a positive operator, we get

$$|f| \leqq (\alpha-1)R(\alpha, T)|f|+R(\alpha, T)|g|.$$

Let $Q=I-P$. Then

$$\begin{aligned} |f| &\leqq (\alpha-1)R(\alpha, T)(P+Q)|f|+R(\alpha, T)|g| \\ &=P|f|+(\alpha-1)R(\alpha, T)Q|f|+R(\alpha, T)|g|. \end{aligned}$$

Therefore

$$\|f\| \leqq (\alpha-1)\|R(\alpha, T)(I-P)\|\,\|f\|+\|R(\alpha, T)\|\,\|g\|+\|P|f|\|.$$

This, together with $\|R(\alpha, T)\| \leqq 1/(\alpha-1)$ and $\|g\| < \frac{1}{16b}\|f\|$, implies

$$\|f\| \leqq \left(\frac{1}{4}+\frac{1}{4}\right)\|f\|+\|P|f|\|.$$

Therefore we get

$$\frac{1}{2}\|f\| \leqq \|P|f|\|. \qquad //$$

LEMMA 6. *Let* E *be a Banach lattice and* $T \in \mathfrak{L}(E)$ *be a positive irreducible operator such that* $r(T)=1$ *and* 1 *is a pole of* $R(\alpha, T)$. *Let* r *be a positive number such that*

$$\{\alpha:\ 0<|\alpha-1|<r\} \subset \rho(T)$$

and α_0 *be in* $\sigma(T) \cap \Gamma$. *Then there exists an operator* $D \in \mathfrak{L}(E)$ *such that*

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13) This supremum is finite and P is the projection to the eigenspace of T for 1 because 1 is a pole of $R(\alpha, T)$ of order 1 by Lemma 4.

$$D^{-1} \in \mathfrak{L}(E),$$

$$|Df| = |D^{-1}f| = |f| \quad \textit{for any} \quad f \in E,$$

$$\|D\| = \|D^{-1}\| = 1,$$

$$T = \alpha_0^{-1} D^{-1} T D$$

and

$$R(\alpha, T) = \alpha_0^{-1} D R(\alpha/\alpha_0, T) D^{-1} \quad \textit{for} \quad 0 < |\alpha - \alpha_0| < r.$$

PROOF. By Main Theorem in [15], α_0 is an eigenvalue of T. Then, by Theorem 5.2 *ibid.*, the assertions in the lemma except the last one are obtained. The last assertion follows in the same way as in the proof of Corollary 3 in [12]. //

LEMMA 7. *Let E be a Banach lattice and $T \in \mathfrak{L}(E)$ be a positive operator such that $r(T)=1$ and 1 is a pole of $R(\alpha, T)$ of order 1. Let $\mathrm{J} = \{f \in E;\ P|f| = 0\}$ where P is the residual operator of $R(\alpha, T)$ at 1. Then J is a T-invariant closed ideal and the operator T induces the operator T/J on E/J. The operator T/J has the same spectral properties on Γ as those T has, i.e.,*

$$\sigma(T/\mathrm{J}) \cap \Gamma = \sigma(T) \cap \Gamma,$$

$$P_\sigma(T/\mathrm{J}) \cap \Gamma = P_\sigma(T) \cap \Gamma,$$

$$R_\sigma(T/\mathrm{J}) \cap \Gamma = R_\sigma(T) \cap \Gamma,$$

$$C_\sigma(T/\mathrm{J}) \cap \Gamma = C_\sigma(T) \cap \Gamma,$$

for $\alpha_0 \in \Gamma$, α_0 is a pole of $R(\alpha, T/\mathrm{J})$ if and only if α_0 is a pole of $R(\alpha, T)$ and the dimension of the eigenspace of T/J [resp. $(T/\mathrm{J})'$] for α_0 is identical with that of T [resp. T'].

PROOF. It is clear that J is a T-invariant closed ideal and $r(T_{|\mathrm{J}}) \leqq 1$. By Corollary 2 of Lemma 2, 1 is a pole of $R(\alpha, T_{|\mathrm{J}})$ of order at most 1 with the residual operator $P_{|\mathrm{J}}$ which is the identically zero operator. Therefore $1 \in \rho(T_{|\mathrm{J}})$ which implies, by the well known theorem of positive operator, $r(T_{|\mathrm{J}}) < 1$. Consequently we get

$$\rho(T_{|\mathrm{J}}) \supset \Gamma.$$

From this result, together with Lemma 2 and its Corollary 2, the conclusions of the lemma follow without difficulty. We have only to notice the following fact: $(E/\mathrm{J})'$ is isomorphic to the set $\{\varphi : \varphi \in E',\ \varphi = 0 \text{ on } \mathrm{J}\}$ as a Banach lattice. //

LEMMA 8[14]. *Let E be a Banach lattice and $T \in \mathfrak{L}(E)$ be a positive operator such that $r(T)=1$ and 1 is a pole of $R(\alpha, T)$ of order 1. Let J be a closed T-invariant ideal including the eigenspace of T for 1. Then*

14) For this lemma, see I. Sawashima [18], Lemma 4.

the operator $T_{|J}$ *has the same spectral properties on* Γ *as those* T *has.*

PROOF. We can prove this lemma in the same way as Lemma 7. Notice only the following fact: Any eigenfunctional φ of $(T_{|J})'$ for $\alpha_0 \in \Gamma$ can be extended uniquely to the eigenfunctional $\tilde{\varphi}$ of T' for $\alpha_0 \in \Gamma$ by

$$\tilde{\varphi} = \phi - (R(\alpha_0, (T/J)')((\alpha_0\phi - T'\phi)/J)) \circ \pi$$

where ϕ is any extension of φ to an element of E', $(\alpha_0\phi - T'\phi)/J$ is defined by $(\alpha_0\phi - T'\phi)/J) \circ \pi = \alpha_0\phi - T'\phi$ and π is the canonical mapping of E onto E/J. //

4. Spectral properties on the spectral circle.

In this section we continue the notations of § 2. Thus $E = C(X)$ and $T \in \mathfrak{L}(E)$ satisfy conditions I), II), III) and IV) in the first paragraph of § 2 and so on.

Let Z be an arbitrary compact subset of an arbitrary compact Hausdorff space Y. Then $C(Z)$ is isomorphic to $C(Y)/J$ as a Banach lattice, where $J = \{f \in C(Y) : f(y) = 0 \text{ on } Z\}$, and also $\mathfrak{L}(C(Z))$ is isomorphic to $\mathfrak{L}(C(Y)/J)$ as an ordered Banach space under the above isomorphism. Therefore, we shall hereafter identify $C(Z)$ with $C(Y)/J$ and also identify $\mathfrak{L}(C(Z))$ with $\mathfrak{L}(C(Y)/J)$. For example, the operator T_λ will be identified with the operator T/J_{S_λ} and also with the operator U_λ/I_{S_λ}, where T_λ and U_λ are the operators defined in § 2 and $J_{S_\lambda} = \{f \in C(X) : f(x) = 0 \text{ on } S_\lambda\}$ and $I_{S_\lambda} = \{f \in C(X_\lambda) : f(x) = 0 \text{ on } S_\lambda\}$. By this identification, parts of Lemma 2 will be stated in the following form.

PROPOSITION 8. i) *For* $\lambda \in \Gamma$, $\rho_\infty(T_\lambda) \supset \rho_\infty(U_\lambda) \supset \rho_\infty(T)$. ii) *For* $\alpha \in \rho_\infty(T)$ *and* $\lambda \in \Lambda$,

$$\| R(\alpha, T_\lambda) \| \leqq \| R(\alpha, U_\lambda) \| \leqq \| R(\alpha, T) \|.$$

From the equivalence of (i) and (ii) in Proposition 6 we get

COROLLARY. *If* $\alpha \in \rho_\infty(T)$ *and* P *is strictly positive, then* $\sup_{\lambda \in \Lambda} \| R(\alpha, T_\lambda) \| = \| R(\alpha, T) \|$.

We also have

PROPOSITION 9. *The following inclusion relation holds:*

$$\sigma(T) \cap \Gamma \supset \Big(\bigcup_{\lambda \in \Lambda} \sigma(U_\lambda)\Big)^- \cap \Gamma \supset \Big(\bigcup_{\lambda \in \Lambda} \sigma(T_\lambda)\Big)^- \cap \Gamma.$$

PROOF. Since $\rho_\infty(T)$ is open, we get from i) of Proposition 8

$$\rho_\infty(T)\cap\Gamma\subset(\bigcap_{\lambda\in\Lambda}\rho_\infty(U_\lambda))^\circ\cap\Gamma\subset(\bigcap_{\lambda\in\Lambda}\rho_\infty(T_\lambda))^\circ\cap\Gamma.$$

$$\rho_\infty(T)\cap\Gamma=\rho(T)\cap\Gamma$$

and

$$(\bigcap_{\lambda\in\Lambda}\rho_\infty(U_\lambda))^\circ\cap\Gamma=(\bigcap_{\lambda\in\Lambda}\rho(U_\lambda))^\circ\cap\Gamma$$

are proved easily. Therefore

$$\rho(T)\cap\Gamma\subset(\bigcap_{\lambda\in\Lambda}\rho(U_\lambda))^\circ\cap\Gamma\subset(\bigcap_{\lambda\in\Lambda}\rho(T_\lambda))^\circ\cap\Gamma. \qquad //$$

Hereafter we assume, in place of conditions III) and IV) for T, the following condition:

IV′) 1 is a pole of $R(\alpha, T)$.

Then, since T is a contraction, Lemma 4 implies that 1 is a pole of $R(\alpha, T)$ of order 1. S. Karlin has shown in Theorems 4 and 5 in [6] that, for a positive operator $T\in\mathfrak{L}(E)$ with $r(T)=1$, T is uniformly ergodic with nonzero limit operator P if and only if 1 is a pole of $R(\alpha, T)$ of order 1 with the residual operator P. Owing to this fact, we can make use of the results obtained in §2.

THEOREM 5. *Let T be a positive, sub-Markov operator of $\mathfrak{L}(E)$ such that 1 is a pole of $R(\alpha, T)$. Then the following relations hold:*

$$\sigma(T)\cap\Gamma=\sigma(T_S)\cap\Gamma,$$

$$P_\sigma(T)\cap\Gamma=P_\sigma(T_S)\cap\Gamma,$$

$$R_\sigma(T)\cap\Gamma=R_\sigma(T_S)\cap\Gamma,$$

$$C_\sigma(T)\cap\Gamma=C_\sigma(T_S)\cap\Gamma,$$

where S is the set defined in p. 39, and T_S is the operator induced on $C(S)$ by T.

PROOF. Let P be the residual operator of $R(\alpha, T)$ at $\alpha=1$ and let $J=\{f\colon P|f|=0\}$. Then, by Proposition 4, $J=\{f\colon f(x)=0$ on $S\}$ and hence all the assertions in the theorem follow from Lemma 7. //

REMARK. If we replace condition IV′) by conditions III) and IV), then the conclusions of the theorem do not hold.

Counter-example: Let $X=\{1, 2, \dots, \infty\}$ be the one point compactification of the discrete set N. Then $f=\{f(1), \dots, f(n), \dots, f(\infty)\}\in C(X)$ means that $f(n)$ converges to $f(\infty)$. Let Tf be the shift, i.e., $Tf(n)=f(n+1)$ for $n\in N$ and $Tf(\infty)=f(\infty)$. Then Pf is the constant function $f(\infty)$. Therefore Λ consists of only one element $\varepsilon_\infty=\lambda_0$ and hence $S=S_{\lambda_0}=\{\infty\}$ and $X_{\lambda_0}=X$. It is easy to see that $\sigma(T)=\{\alpha: |\alpha|\leqq 1\}$ and

$\sigma(T_S)=\{1\}$.

Let λ_n be an arbitrarily chosen sequence of elements of Λ. Denote T_{λ_n}, P_{λ_n} and S_{λ_n} simply by T_n, P_n and S_n respectively. Let $\boldsymbol{m}=\{\{f_n\}: f_n\in C(S_n), \sup_n \|f_n\|<\infty\}$. With linear structure and order defined coordinatewise and norm defined by $\|\{f_n\}\|=\sup_n \|f_n\|$, $\boldsymbol{m}$ is a Banach lattice. Operators $\hat{T}$ and $\hat{P}$ will be defined by $\hat{T}\{f_n\}=\{T_n f_n\}$ and $\hat{P}\{f_n\}=\{P_n f_n\}$. Let $\mathcal{U}$ be an arbitrary fixed ultrafilter on $\boldsymbol{N}$ containing no finite set. Put

$$\mathrm{J}_{\mathcal{U}}=\{\{f_n\}\in \boldsymbol{m}: \mathcal{U}-\lim \|P_n|f_n|\|=0\}$$

where $\mathcal{U}-\lim$ is the ultrafilter limit of $\mathcal{U}$. Let $\tilde{E}$ be the factor space $\boldsymbol{m}/\mathrm{J}_{\mathcal{U}}$. Since $\mathrm{J}_{\mathcal{U}}$ is easily seen to be $\hat{T}$-invariant, the operator $\hat{T}$ induces an operator on $\tilde{E}$ which is denoted by $\tilde{T}$.

PROPOSITION 10.[15] *$\tilde{T}$ is a positive irreducible operator of $\mathfrak{L}(\tilde{E})$ with $r(\tilde{T})=1$ and 1 is a pole of $R(\alpha, \tilde{T})$ of order 1. $\hat{P}$ induces on $\tilde{E}$ an operator, denoted by $\tilde{P}$, which is the residual operator of $R(\alpha, \tilde{T})$ at 1.*

PROOF. The positivity of $\tilde{T}$ is clear. Let $\hat{\boldsymbol{1}}=\{\boldsymbol{1}_{S_n}\}$ and $\tilde{\boldsymbol{1}}$ be the element of $\tilde{E}$ corresponding to $\hat{\boldsymbol{1}}$. Then $\tilde{T}\tilde{\boldsymbol{1}}=\tilde{P}\tilde{\boldsymbol{1}}=\tilde{\boldsymbol{1}}$ is clear from $T_n\boldsymbol{1}_{S_n}=P_n\boldsymbol{1}_{S_n}=\boldsymbol{1}_{S_n}$. This implies $r(\tilde{T})=1$ since $\|\tilde{T}\|\leqq 1$ is also clear.

Let r be a positive number such that

$$\rho(T)\supset\{\alpha: 0<|\alpha-1|<r\},$$

then, by Proposition 8, the definition of $\hat{T}$ and Lemma 2, we get

$$\rho_\infty(\hat{T}),\ \rho_\infty(\tilde{T})\supset\{\alpha: 0<|\alpha-1|<r\}$$

and

$$\begin{aligned}\sup_{\alpha\in A}\|(\alpha-1)R(\alpha, T)\| &\geqq \sup_{\alpha\in A}\sup_n \|(\alpha-1)R(\alpha, T_n)\|\\ &=\sup_{\alpha\in A}\|(\alpha-1)R(\alpha, \hat{T})\|\\ &\geqq \sup_{\alpha\in A}\|(\alpha-1)R(\alpha, \tilde{T})\|\end{aligned}$$

where $A=\{\alpha: 0<|\alpha-1|<r/2\}$. Since 1 is a pole of $R(\alpha, T)$ of order 1, $\sup_{\alpha\in A}\|(\alpha-1)R(\alpha, \tilde{T})\|$ is finite. Therefore 1 is a pole of $R(\alpha, \tilde{T})$ of order 1.

It is clear that $\hat{P}$ induces the operator $\tilde{P}$ on $\tilde{E}$ which is the residual operator of $R(\alpha, \tilde{T})$ and that $\tilde{T}\tilde{f}=\tilde{f}$ if and only if $\tilde{P}\tilde{f}=\tilde{f}$. Let $\{f_n\}\in \boldsymbol{m}$ be mapped to an eigenvector $\tilde{f}$ of $\tilde{T}$ for 1 by the canonical

15) For this, see T. Ando [1] and H.P. Lotz und H.H. Schaefer [11], Theorem 1.

mapping π of $\boldsymbol{m}$ onto $\boldsymbol{m}/\mathrm{J}_{\mathcal{U}}$. Then $\tilde{f}=\pi\{P_nf_n\}$. The irreducibility of T_n implies $P_nf_n=\gamma_n\boldsymbol{1}_{S_n}$ where $\gamma_n=\lambda_{|S_n}(f_n)$. Put $\gamma=\mathcal{U}-\lim\gamma_n$. Then

$$\mathcal{U}-\lim\|P_n(|\gamma_n\boldsymbol{1}_{S_n}-\gamma\boldsymbol{1}_{S_n}|)\|=\mathcal{U}-\lim|\gamma_n-\gamma|=0,$$

since $\mathcal{U}-\lim$ is a lattice homomorphism. Thus $\tilde{f}=\gamma\tilde{\boldsymbol{1}}$ which proves that the eigenspace of $\tilde{T}$ for 1 is one dimensional with the base $\tilde{\boldsymbol{1}}$.

Let J be a nonzero $\tilde{T}$-invariant closed ideal of $\tilde{E}$ and $\tilde{f}$ be a nonzero positive element of J. Then $\tilde{P}\tilde{f}=\gamma\tilde{\boldsymbol{1}}$ is a nonzero element of J. This implies $\tilde{\boldsymbol{1}}\in\mathrm{J}$. For any element $\tilde{g}$ of $\tilde{E}$ we can show easily that there exists a positive number c such that $|\tilde{g}|\leqq c\tilde{\boldsymbol{1}}$[16] and hence $\tilde{g}\in\mathrm{J}$. Therefore $\mathrm{J}=\tilde{E}$. Thus $\tilde{T}$ is irreducible. //

COROLLARY. *If r is a positive number such that*

$$\{\alpha:\ 0<|\alpha-1|<r\}\subset\rho(T),$$

then

$$\{\alpha:\ 0<|\alpha-1|<r\}\subset\rho(\tilde{T}).$$

THEOREM 6. *Let E be the space $C(X)$, and $T\in\mathfrak{L}(E)$ be a positive sub-Markov operator and* 1 *be a pole of $R(\alpha, T)$. If Λ is the set of extreme points of the set of invariant probability measures on X and S_λ is the support of $\lambda\in\Lambda$, then*

$$\sigma(T)\cap\Gamma=(\bigcup_{\lambda\in\Lambda}\sigma(T_\lambda))^-\cap\Gamma$$

where T_λ is the operator induced on $C(S_\lambda)$ by T.

PROOF. By Theorem 5 and the relation $S_\lambda\subset S$, we can assume $X=S$, i.e., P is strictly positive. By Proposition 9, the inclusion

$$\sigma(T)\cap\Gamma\supset(\bigcup_{\lambda\in\Lambda}\sigma(T_\lambda))^-\cap\Gamma$$

is clear. To prove the inverse inclusion which is equivalent to

$$\rho(T)\supset(\bigcap_{\lambda\in\Lambda}\rho(T_\lambda))^\circ\cap\Gamma,$$

let α_0 be in $(\bigcap_{\lambda\in\Lambda}\rho(T_\lambda))^\circ\cap\Gamma$.

First we notice that it is enough to show that $\{\|R(\alpha_0, T_\lambda)\|:\lambda\in\Lambda\}$ is bounded: Lemma 3 shows that $\sup_{\lambda\in\Lambda}\|R(\alpha, T_\lambda)\|$ is bounded on a set $\{\alpha:\ |\alpha-\alpha_0|<d\}$ for a positive number d. Corollary of Proposition 8 shows that $\|R(\alpha, T)\|$ is bounded by the same upper bound on the set $\{\alpha:\ |\alpha-\alpha_0|<d,\ |\alpha|>1\}$. This implies $\alpha_0\in\rho(T)$.

16) Moreover, $\tilde{E}$ is isomorphic as a Banach lattice to $C(\Omega)$ for some compact Hausdorff space Ω and $\tilde{\boldsymbol{1}}$ is mapped under this isomorphism to the identity element $\boldsymbol{1}$ of $C(\Omega)$.

We shall show in the following four steps that the assumption of unboundedness of the set $\{\|R(\alpha_0, T_\lambda)\| : \lambda \in \Lambda\}$ yields a contradiction.

The first step: Let r_1, r_2 and b be positive numbers satisfying

$$\{\alpha : |\alpha-\alpha_0|<r_1\} \subset \bigcap_{\lambda\in\Lambda} \rho(T_\lambda),$$

$$\{\alpha : 0<|\alpha-1|<r_2\} \subset \rho(T)$$

and

$$\sup_{\alpha>1} \|R(\alpha, T)(I-P)\| \leqq b.$$

Let r be a positive number less than r_1, r_2 and $1/32b$. By assumption, there exists a sequence λ_n of Λ such that $\|R(\alpha_0, T_n)\|>n$ where T_{λ_n} is denoted by T_n. We shall show that the circle $\{\alpha : |\alpha-\alpha_0|=r\}$ contains at least one point α_1 such that $\{\|R(\alpha_1, T_n)\| : n \in \boldsymbol{N}\}$ is unbounded. If not, then $\sup_n \|R(\alpha, T_n)\|$ is finite for any α on the circle. This, together with Lemma 3 and the compactness of the circle, implies that $\sup_n \|R(\alpha, T_n)\|$ is bounded on the circle. Since

$$\{\alpha : |\alpha-\alpha_0| \leqq r\} \subset \bigcap_n \rho(T_n),$$

and $R(\alpha, T_n)$ is holomorphic in this disk, $\sup_n \|R(\alpha_0, T_n)\|$ is finite which is a contradiction. Therefore we can assume without loss of generality that the following relations hold.

$$\|R(\alpha_0, T_n)\|>n \quad \text{for any } n, \tag{13}$$

$$\|R(\alpha_1, T_n)\|>n \quad \text{for any } n. \tag{14}$$

The second step: Let, for $E_n=C(S_{\lambda_n})$ and T_n chosen above and for an arbitrary ultrafilter $\mathcal{U}$ containing no finite set, $J_{\mathcal{U}}$, $\tilde{E}$, $\tilde{T}$ and $\tilde{P}$ be those defined before Proposition 10. Then this proposition, together with its corollary, shows that $\tilde{T}$ is a positive and irreducible operator of $\mathfrak{L}(\tilde{E})$ and 1 is a pole of $R(\alpha, \tilde{T})$ and

$$\{\alpha : 0<|\alpha-1|<r_2\} \subset \rho(\tilde{T}).$$

The third step: We shall show that α_1 and α_0 belong to $P_c(\tilde{T})$. Relation (14) in the second step yields the existence of f_n and g_n in E_n satisfying

$$\|f_n\|=1, \quad T_n f_n=\alpha_1 f_n+g_n$$

and

$$\|g_n\|<\frac{1}{n}. \tag{15}$$

These imply

$$T_n f_n=\alpha_0 f_n+(\alpha_1-\alpha_0) f_n+g_n$$

and

$$\|(\alpha_1-\alpha_0)f_n+g_n\| \leqq |\alpha_1-\alpha_0|+\|g_n\|<r+\frac{1}{n}.$$

Therefore, for $n>\frac{1}{r}$,

$$\|T_n f_n-\alpha_0 f_n\|<2r<\frac{b}{16},$$

where r and b are those defined in the first step. Appealing to Lemma 5, we see

$$\|P_n|f_n|\| \geqq \frac{1}{2} \quad \text{for} \quad n>\frac{1}{r}.$$

Since $\mathcal{U}$ is a ultrafilter containing no finite set, the above inequality implies $\{f_n\} \notin \mathrm{J}_{\mathcal{U}}$. Relation (15) implies $\{g_n\} \in \mathrm{J}_{\mathcal{U}}$. These imply, by definition of $\tilde{T}$, $\alpha_1 \in P_\sigma(\tilde{T})$. $\alpha_0 \in P_c(\tilde{T})$ can be proved from (13) in a similar way as above.

The fourth step: Applying Lemma 6 to the results obtained in the second step and the fact $\alpha_0 \in P_\sigma(\tilde{T})$, we see that $\rho(\tilde{T})$ includes the set $\{\alpha: 0<|\alpha-\alpha_0|<r_2\}$. This contradicts $\alpha_1 \in P_\sigma(\tilde{T})$, since

$$|\alpha_1-\alpha_0|=r<r_2. \qquad //$$

As a direct consequence of Theorem 6 and Proposition 9, we get

THEOREM 6′. *Let the assumptions for T be as in Theorem 6 and let, for $\lambda \in \Lambda$, $X_\lambda=\{x \in X: P'\varepsilon_X=\lambda\}$ and U_λ be the operator induced on $C(X_\lambda)$ by T. Then*

$$\sigma(T) \cap \Gamma=(\bigcup_{\lambda \in \Lambda} \sigma(U_\lambda))^- \cap \Gamma.$$

REMARK. The conclusion of Theorem 6 can not be replaced by

$$\sigma(T) \cap \Gamma=((\bigcup_{\lambda \in \Lambda} \sigma(T_\lambda)) \cap \Gamma)^-.$$

Counter-example: Let $X=\left[-\frac{1}{2}, \frac{1}{2}\right]$ and

$$Tf(x)=|x|f(x)+(1-|x|)f(-x).$$

Then we can prove

$$(R(\alpha, T)f)(x)=\frac{(\alpha-|x|)f(x)+(1-|x|)f(-x)}{(\alpha+1-2|x|)(\alpha-1)}$$

for $\alpha \in \rho(T)$.

From this we see that $\sigma(T)=\{1\} \cup [-1, 0]$ and 1 is a pole of $R(\alpha, T)$ of order 1 whose residual operator P is given by

$$(Pf)(x)=\frac{1}{2}(f(x)+f(-x)).$$

Then

$$PE=\{f\in C(X):\ f(x)=f(-x)\},$$

P is strictly positive and $X=X_0$.

$$P'\varepsilon_x=\frac{1}{2}(\varepsilon_x+\varepsilon_{-x})\quad\text{for}\quad x\in X.$$

Thus Λ is homeomorphic to $\left[0,\ \frac{1}{2}\right]$ by the correspondence

$$\lambda_y=\frac{1}{2}(\varepsilon_y+\varepsilon_{-y})\quad\text{for}\quad y\in\left[0,\ \frac{1}{2}\right].$$

It is easy to see

$$\sigma(T_{\lambda_y})=\{1\}\cup\{2y-1\}\quad\text{for}\quad 0<y\leqq\frac{1}{2}$$

and

$$\sigma(T_{\lambda_y})=\{1\}\quad\text{for}\quad y=0.$$

Therefore

$$\sigma(T)\cap\Gamma=\{1\}\cup\{-1\}$$

and

$$(\bigcup_{\lambda\in\Lambda}\sigma(T_\lambda)\cap\Gamma)^-=\{1\}.$$

From the proof of Theorem 6 we get the following propositions of which Proposition 11 is a special case of Theorem 4.10 in H. P. Lotz [10]. In the proof of these propositions we may assume, by Theorem 5, that P is strictly positive.

PROPOSITION 11. *Let T satisfy the assumptions of Theorem* 6 *and let r_2 be a positive number such that*

$$\{\alpha:\ 0<|\alpha-1|<r_2\}\subset\rho(T).$$

Then $\sigma(T)\cap\Gamma$ is a set consisting of a finite number of elements. More precisely we have

$$\sigma(T)\cap\Gamma\subset\bigcup_{n=1}^{m}\bigcup_{k=0}^{n-1}\exp\left(\frac{2k\pi i}{n}\right),$$

where m is the smallest natural number satisfying

$$\left|1-\exp\left(\frac{2\pi i}{m+1}\right)\right|<r_2.$$

PROOF. Let $\alpha_0\in\sigma(T)\cap\Gamma$ and $\alpha_0\neq 1$. Then there are two cases:

$$1)\quad \alpha_0\in\bigcap_{\lambda\in\Lambda}\rho(T_\lambda)\quad\text{and}\quad 2)\quad \alpha_0\in\bigcup_{\lambda\in\Lambda}\sigma(T_\lambda).$$

In the first case 1) we see, along the proof of Theorem 6 concerning

α_0, that $\sup_{\lambda\in\Lambda}\|R(\alpha_0, T_\lambda)\|$ is not finite and that there exists a positive irreducible operator $\tilde{T}$ such that $r(\tilde{T})=1$ is a pole of $R(\alpha, \tilde{T})$, $\{\alpha: |\alpha-1|<r_2\}\subset\rho(\tilde{T})$ and $\alpha_0\in\sigma(\tilde{T})$. Then, by Main Theorem in [15], we get

$$\alpha_0=\exp\left(\frac{2k\pi i}{n}\right),$$

where $k\leqq n-1$, $k, n\in \boldsymbol{N}$ and k is relatively prime to n and hence

$$\exp\left(\frac{2\pi i}{n}\right)\in\sigma(\tilde{T}).$$

Therefore $\left|1-\exp\left(\frac{2\pi i}{n}\right)\right|\geqq r_2$ and consequently $n\leqq m$.
In the second case 2) there exists $\lambda\in\Lambda$ such that $\alpha_0\in\sigma(T_\lambda)$ and $\{\alpha: |\alpha-1|<r_2\}\subset\rho(T_\lambda)$ where the latter relation is obtained from Proposition 8. Hereafter the proof is the same as in case 1) and we get

$$\alpha_0=\exp\left(\frac{2k\pi i}{n}\right),\quad k\leqq n-1 \quad\text{and}\quad n\leqq m.$$

Thus the proposition is proved. //

COROLLARY. *Let T satisfy the assumptions of Theorem 6. Then there exists a compact set in the complex plane with the following properties:*

i) $C\subset\{\alpha: |\alpha|\leqq 1\}$,
ii) *$C\cap\Gamma$ consists of a finite number of elements,*
iii) $\sigma(T)\subset C$,
iv) *for any $\lambda\in\Lambda$, $\sigma(T_\lambda)\subset C$ and $\sigma(U_\lambda)\subset C$.*

PROOF. Let $C=\rho_\infty(T)^c$. Then Proposition 11 implies that C satisfies i), ii) and iii) and Proposition 8 implies that C satisfies iv). //

REMARK. An extension of this corollary will appear in F. Niiro and I. Sawashima [16]. See also Karpelevich [7].

Counter-example in the remark of Theorem 6 shows that $\sigma(T)\cap\Gamma$ does not consist of poles. However we have

PROPOSITION 12. *Let T satisfy the assumptions of Theorem 6 and let $\alpha_0\in\Gamma$ be an isolated point of $\sigma(T)$[17]. Then α_0 is a pole of $R(\alpha, T)$ of order 1 and α_0 is in $\bigcup_{\lambda\in\Lambda}\sigma(T_\lambda)$.*

17) Compare the following fact: An element α_0 in $\sigma(T)\cap\Gamma$ is, by Proposition 11, an isolated point of $\sigma(T)\cap\Gamma$.

PROOF. By assumption there exist positive numbers r_1 and r_2 such that

$$\{\alpha:\ 0<|\alpha-\alpha_0|<r_1\}\subset\rho(T)$$

and

$$\{\alpha:\ 0<|\alpha-1|<r_2\}\subset\rho(T).$$

Let

$$\Lambda_1=\{\lambda\in\Lambda:\ \alpha_0\in\rho(T_\lambda)\}$$

and

$$\Lambda_2=\Lambda\backslash\Lambda_1=\{\lambda\in\Lambda:\ \alpha_0\in\sigma(T_\lambda)\}.$$

Then, Theorem 6 implies $\Lambda_2\neq\phi$, namely $\alpha_0\in\bigcup_{\lambda\in\Lambda}\sigma(T_\lambda)$. By the same reasoning as in the proof of Theorem 6, we see that $\sup_{\lambda\in\Lambda_1}\|R(\alpha_0, T_\lambda)\|$ is a finite number which will be denoted by c. Let r be a positive number less than r_1, r_2 and $\frac{1}{c}$. Then, by Corollary of Proposition 8, we get

$$\sup_{0<|\alpha-\alpha_0|<r}|\alpha-\alpha_0|\,\|R(\alpha, T)\|$$

$$=\sup_{0<|\alpha-\alpha_0|<r}\ \sup_{\lambda\in\Lambda}|\alpha-\alpha_0|\,\|R(\alpha, T_\lambda)\|$$

$$=\mathrm{Max}\{\sup_{0<|\alpha-\alpha_0|<r}\ \sup_{\lambda\in\Lambda_1}|\alpha-\alpha_0|\,\|R(\alpha, T_\lambda)\|,\ \sup_{0<|\alpha-\alpha_0|<r}\ \sup_{\lambda\in\Lambda_2}|\alpha-\alpha_0|\,\|R(\alpha, T_\lambda)\|\}.$$

By Lemma 3, we get

$$\sup_{0<|\alpha-\alpha_0|<r}\ \sup_{\lambda\in\Lambda_1}|\alpha-\alpha_0|\,\|R(\alpha, T_\lambda)\|<\frac{cr}{1-cr}.$$

By Lemma 6, we get

$$\sup_{0<|\alpha-\alpha_0|<r}\ \sup_{\lambda\in\Lambda_2}|\alpha-\alpha_0|\,\|R(\alpha, T_\lambda)\|$$

$$=\sup_{0<\left|\frac{\alpha}{\alpha_0}-1\right|<r}\ \sup_{\lambda\in\Lambda_2}\left|\frac{\alpha}{\alpha_0}-1\right|\left\|R\left(\frac{\alpha}{\alpha_0}, T_\lambda\right)\right\|$$

$$\leqq\sup_{0<\left|\frac{\alpha}{\alpha_0}-1\right|<r}\left|\frac{\alpha}{\alpha_0}-1\right|\left\|R\left(\frac{\alpha}{\alpha_0}, T\right)\right\|,$$

where the last ineqnality is obtained from Proposition 8. Since 1 is a pole of $R(\alpha, T)$ of order 1 and $\frac{cr}{1-cr}$ is finite, $\sup_{0<|\alpha-\alpha_0|<r}|\alpha-\alpha_0|\,\|R(\alpha, T)\|$ is finite and the proposition is proved. //

Combining Propositions 11 and 12, we get

THEOREM 7. *Let E be the space $C(X)$, T be a positive sub-Markov operator of $\mathfrak{L}(E)$ and 1 be a pole of $R(\alpha, T)$. Then any element α_0 of*

$\sigma(T)\cap\Gamma$ is an isolated point of $\sigma(T)\cap\Gamma$. If α_0 is an isolated point of $\sigma(T)$, then α_0 is a pole of $R(\alpha, T)$ of order 1. *If α_0 is not an isolated point of $\sigma(T)$, then α_0 is a limit point of $\sigma(T)\cap\{\alpha: |\alpha|<1\}$.*

REMARK. H. H. Schaefer raised in [19] the problem: under what conditions does $\sigma(T)\cap\Gamma$ consist of poles of $R(\alpha, T)$ provided 1 is a pole of $R(\alpha, T)$. Theorem 7 may be regarded as an answer in some sense to his problem. See also F. Niiro and I. Sawashima [15] p. 182–183.

References

[1] T. Ando: Technical note (unpublished).
[2] ———: Invariante Masse positiver Kontractionen in $C(X)$. Studia Math., **31** (1968), 173–187.
[3] F.G. Gantmacher: The theory of matrices. vol. 2, Chelsea, New York (1959).
[4] B. Jamison: Ergodic decompositions induced by certain Markov operators. Trans. Amer. Math. Soc., **117** (1965), 451–468.
[5] S. Kakutani: Concrete representation of abstract (M)-spaces. Ann. Math., **42** (1941), 994–1024.
[6] S. Karlin: Positive operators. J. Math. Mech., **8** (1959), 907–937.
[7] F.I. Karpelevich: On the eigenvalues of a matrix with non-negative elements. Izv. Akad. Nauk SSSR Ser. Mat., **15** (1951), 361–383 (Russian).
[8] M.G. Krein and M.A. Rutman: Linear operators leaving invariant a cone in a Banach space. Uspehi Mat. Nauk, **3** no. 1 (23), (1948) 3–95; Amer. Math. Soc. Transl., ser. **1**, 10 (1962), 199–325.
[9] S.P. Lloyd: On certain projections in spaces of continuous functions. Pacific J. Math., **13** (1963), 171–175.
[10] H.P. Lotz: Über das Spectrum positiver Operatoren. Math. Z., **108** (1968), 15–32.
[11] H.P. Lotz und H.H. Schaefer: Über einen Satz von F. Niiro und I. Sawashima. Math. Z., **108** (1968), 33–36.
[12] F. Niiro: On indecomposable operators in l_p $(1<p<\infty)$ and a problem of H.H. Schaefer. Sci. Pap. Coll. Gen. Educ., Univ. Tokyo, **14** (1964), 165–179.
[13] ———: On indecomposable operators in L_p $(1<p<\infty)$ and a problem of H.H. Schaefer. Sci. Pap. Coll. Gen. Educ., Univ. Tokyo, **16** (1966), 1–24.
[14] ———: On positive operator, Reports of sixth functional analysis symposium. Ibaragi Univ., (1968) 29–38 (Japanese).
[15] F. Niiro and I. Sawashima: On the spectral properties of positive irreducible operators in an arbitrary Banach lattice and problems of H.H. Schaefer. Sci. Pap. Coll. Gen. Educ., Univ. Tokyo, **16** (1966), 145–183.
[16] ———: On a set including the spectra of positive operators. Sci. Pap. Coll. Gen. Educ., Univ. Tokyo, **23** (1973), 00–00.
[17] M. Rosenblatt: Equicontinuous Markov operators. Teor. Verojatnost. i Primenen. **9** (1964), 205–222.
[18] I. Sawashima: On spectral properties of positive irreducible operators in $C(S)$ and a problem of H.H. Schaefer. Nat. Sci. Rep. Ochanomizu Univ., **17** (1966), 1–15.
[19] H.H. Schaefer: Some spectral properties of positive linear operators, Pacific J. Math., **10** (1960), 1009–1019.
[20] ———: Topological vector spaces. Macmillan, New York (1966).

[21] ———: Invariant ideals of positive operators in $C(X)$, I. Illinois J. Math., 11 (1967), 703-715.
[22] ———: Invariant ideals of positive operators in $C(X)$, II. Illinois J. Math., 12 (1968), 525-538.
[23] A.E. Taylor: Introduction to functional analysis. Wiley, New York (1958).
[24] K. Yosida: Simple Markov process with a locally compact phase space. Math. Japon., 1 (1948), 99-103.