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A Method of Quantization and Its Application to the Theory of Action at a Distance

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§1. Introduction and summary

In dealing with a system of similar particles the method of composition has been found equivalent to the canonical quantization in non-relativistic quantum mechanics. The method of composition consists of two processes, that is, to regard each of a set of complete orthonormal vectors in a Hilbert space as a state of particle and to transform the Schroedinger equation into its number-representation form, taking account of the statistics of particles.

In relativistic quantum mechanics, however, the method of composition is overshadowed completely by the method of canonical quantization. An aim of this paper is to revive the method of composition in its relativistic form. Another aim is to apply it to the quantization of the theory of action at a distance in classical electrodynamics.[1,2]

The quantization of the theory of action at a distance has recently been carried out by Hoyle and Narlikar[3,4] by use of Feynman's path integral. Their result seems, however, to contradict our expectation for constructing a consistent theory free from divergence difficulties.

To achieve the preceding aims, we depart from conventional ideas in some points.

In the first place, we consider a state of particle extending over the entire space-time and represent it by a vector in a Hilbert space of 4 variables. According to the conventional idea, a state of particle is represented by a vector in a Hilbert space of 3 variables changing its position with the lapse of time. This special role of time seems to have marred simplicity and clarity of the formulation of relativistic quantum mechanics. In our formulation, time is deprived of its special role so that there is no state whose amplitude varies with the lapse of time. The variation in the course of time is fully decomposed into complete orthonormal components. We might say with Herakleitos "it is not possible to step twice into the same river".

In the second place, the concept of representation invariance and the Lagrangian formulation are used throughout this paper, which is a continuation to a previous paper[5], written from the same point of view.

In the third place, a multiple space-time of $4n$ dimensions, n being the number of particles, is introduced instead of conventional configuration space of $3n+1$ dimensions.

In § 2, the quantization of the action is carried out with the aid of an expression either symmetric or antisymmetric according to the statistics of particles and its hermite conjugate, along a general line parallel to Fock's method.

In § 3, non-hermitian mixed tensor fields are quantized along the line of § 2. But an hermitian mixed tensor field does not span a Hilbert space, so another method must be devised, by introducing another hermitian field.

§ 4 deals with a brief comparison between the theory of field and the theory of action at a distance.

In § 5, a reconciliation of the two theories is proposed which employs auxiliary fields, each of them generated by one of charged particles.

In §6, the total action is quantized. And the interaction of particles with fields is eliminated by a unitary transformation to reveal the existence of the direct interaction of particles as desired.

§ 2. System of similar particles

Let the action of one particle be denoted by

$$L=\langle\phi|x\rangle\langle x|H|x'\rangle\langle x'|\psi\rangle$$

where ϕ, ψ are wave functions conjugate to each other, x standing for a quadruplet of space-time coordinates x^0, x^1, x^2, x^3 together with spin variables if necessary. H takes a form $p^\mu p_\mu - m^2$ for a scalar particle of mass m, $\gamma^\mu p_\mu - m$ for a spinor particle, etc. L is called Lagrangian in the previous paper, and denotes the expectation value of H, but it may be called action more legitimately for the role of it corresponds to that of the action $\int m\,ds$ in classical dynamics, both quantities being required to be stationary by virtue of the variation principle for realizable states.

The total action of n particles is taken to be the expectation value of the sum of individual L's,

$$\text{Total action}=\langle\Phi|x_1\dots x_n\rangle\langle x_1\dots x_n|H(1)+\dots+H(n)|x_1'\dots x_n'\rangle\langle x_1'\dots x_n'|\Psi\rangle$$
$$=\langle\Phi|\boldsymbol{x}\rangle\langle\boldsymbol{x}|H(1)+\dots+H(n)|\boldsymbol{x}'\rangle\langle\boldsymbol{x}'|\Psi\rangle$$

where the bold letter $\boldsymbol{x}$ represents a set of $x_1, x_2, \dots, x_n$, each x referring to each particle. $\langle \Phi | \boldsymbol{x} \rangle$, $\langle \boldsymbol{x} | \Psi \rangle$ are required to be either symmetric or anti-symmetric in arguments $x_1, x_2, \dots, x_n$, according to the statistics of particles.

Our task is to transform the total action into its number representation form. One may follow the original method of Fock, with necessary alterations. In this paper we make a small change and introduce from the start boson or fermion operators, $\langle x | \psi \rangle$, $\langle \phi | x \rangle$, and vacuum state vectors 1, $1^\dagger$ hemite-conjugate to each other, that satisfy the following conditions

$$\left.\begin{aligned} &\langle x|\psi\rangle\langle x'|\psi\rangle-\rho\langle x'|\psi\rangle\langle x|\psi\rangle=0, \quad \langle\phi|x\rangle\langle\phi|x'\rangle-\rho\langle\phi|x'\rangle\langle\phi|x\rangle=0 \\ &\langle x|\psi\rangle\langle\phi|x'\rangle-\rho\langle\phi|x'\rangle\langle x|\psi\rangle=\langle x|x'\rangle, \quad \langle x|\psi\rangle\cdot 1=0, \\ &\qquad\qquad\qquad\qquad\qquad\qquad 1^\dagger\cdot\langle\phi|x\rangle=0, \quad 1^\dagger\cdot 1=1 \end{aligned}\right\} \tag{1}$$

$\rho=1$ for boson, $\rho=-1$ for fermion

and form an expression either symmetric or antisymmetric according to the statistics of particles

$$\langle \boldsymbol{x} | T \rangle = \frac{1}{\sqrt{n!}} \langle x_1|\psi\rangle\langle x_2|\psi\rangle\dots\langle x_n|\psi\rangle$$

and its hermite conjugate (2)

$$\langle T^\dagger | \boldsymbol{x} \rangle = \frac{1}{\sqrt{n!}} \langle\phi|x_n\rangle\langle\phi|x_{n-1}\rangle\dots\langle\phi|x_1\rangle .$$

We see then, by moving $\langle\phi|x\rangle$ to the left of $\langle x|\psi\rangle$,

$$\langle \boldsymbol{x}|T\rangle\langle T^\dagger|\boldsymbol{x}'\rangle\cdot 1=(n!)^{-1}\sum_P \varepsilon(P)\langle P\boldsymbol{x}|\boldsymbol{x}'\rangle\cdot 1$$

$$1^\dagger\cdot\langle \boldsymbol{x}|T\rangle\langle T^\dagger|\boldsymbol{x}'\rangle=(n!)^{-1}\,1^\dagger\cdot\sum_P \varepsilon(P)\langle P\boldsymbol{x}|\boldsymbol{x}'\rangle$$

P ranging over all permutations of n letters $x_1, x_2, \dots, x_n$, $\varepsilon(P)$ being equal to 1 for bosons and $\varepsilon(P)$ being equal to 1 or -1 for fermions according as P is even or odd.

Therefore, when $\langle \boldsymbol{x}|T\rangle\langle T^\dagger|\boldsymbol{x}'\rangle$ is multiplied by $\langle F|\boldsymbol{x}\rangle\cdot 1^\dagger$ from the left, $\langle F|\boldsymbol{x}\rangle$ being a function of $x_1, x_2, \dots, x_n$ of the same symmetric property as that of $\langle \boldsymbol{x}|T\rangle$, we see that

$$\begin{aligned} 1^\dagger\cdot\langle F|\boldsymbol{x}\rangle\langle \boldsymbol{x}|T\rangle\langle T^\dagger|\boldsymbol{x}'\rangle &=(n!)^{-1}\,1^\dagger\cdot\sum_P \langle F|\boldsymbol{x}\rangle\varepsilon(P)\langle P\boldsymbol{x}|\boldsymbol{x}'\rangle \\ &=1^\dagger\cdot\langle F|\boldsymbol{x}'\rangle . \end{aligned}$$

In short $\langle \boldsymbol{x}|T\rangle\langle T^\dagger|\boldsymbol{x}'\rangle$ is equivalent to $\langle \boldsymbol{x}|\boldsymbol{x}'\rangle=\langle x_1|x_1'\rangle\dots\langle x_n|x_n'\rangle$. In the same manner we see that

$$\langle \boldsymbol{x}|T\rangle\langle T^\dagger|\boldsymbol{x}'\rangle\langle \boldsymbol{x}'|G\rangle\cdot 1=\langle \boldsymbol{x}|G\rangle\cdot 1$$

$\langle \boldsymbol{x}|G\rangle$ being a function of $x_1, x_2, \dots, x_n$ of the same symmetric property as that of $\langle \boldsymbol{x}|T\rangle$.

Hence we have

$$\begin{aligned}&\text{Total action}\\&=1^\dagger\cdot\langle\Phi|\boldsymbol{x}\rangle\langle\boldsymbol{x}|T\rangle\langle T^\dagger|\boldsymbol{x}'\rangle\langle\boldsymbol{x}'|\Sigma H(i)|\boldsymbol{x}''\rangle\langle\boldsymbol{x}''|T\rangle\langle T^\dagger|\boldsymbol{x}'''\rangle\langle\boldsymbol{x}'''|\Psi\rangle\cdot 1\\&=\Phi\langle T^\dagger|\boldsymbol{x}\rangle\langle\boldsymbol{x}|\Sigma H(i)|\boldsymbol{x}'\rangle\langle\boldsymbol{x}'|T\rangle\Psi\end{aligned}$$

where we put for simplicity

$$\Phi=1^\dagger\cdot\langle\Phi|\boldsymbol{x}\rangle\langle\boldsymbol{x}|T\rangle$$
$$\Psi=\langle T^\dagger|\boldsymbol{x}\rangle\langle\boldsymbol{x}|\Psi\rangle\cdot 1$$

Φ, Ψ are representation invariants and denote a state where n particles are present.

Since $\Sigma H(i)$ is the sum of individual L's, we see

$$\begin{aligned}&\langle\boldsymbol{x}|\Sigma H(i)|\boldsymbol{x}'\rangle\\&=\langle x_1|H(1)|x_1'\rangle\langle x_2|x_2'\rangle\dots\langle x_n|x_n'\rangle+\dots+\langle x_1|x_1'\rangle\dots\langle x_{n-1}|x_{n-1}'\rangle\langle x_n|H(n)|x_n'\rangle.\end{aligned}$$

Hence

$$\begin{aligned}&\langle T^\dagger|\boldsymbol{x}\rangle\langle\boldsymbol{x}|\Sigma H(i)|\boldsymbol{x}'\rangle\langle\boldsymbol{x}'|T\rangle\\&=(n\,!)^{-1}\{\langle\phi|x_n\rangle\dots\langle\phi|x_2\rangle\langle\phi|x_1\rangle\langle x_1|H(1)|x_1'\rangle\langle x_1'|\phi\rangle\langle x_2|\phi\rangle\dots\langle x_n|\phi\rangle+\dots\}.\end{aligned}$$

If we define an operator $N=\langle\phi|x\rangle\langle x|\phi\rangle$, N satisfies the following relations

$$\left.\begin{aligned}&\langle\phi|x\rangle N=(N-1)\langle\phi|x\rangle,\quad \langle x|\phi\rangle N=(N+1)\langle x|\phi\rangle\\&N\cdot 1=0,\quad 1^\dagger\cdot N=0\\\text{and}\quad&\langle\phi|x_n\rangle\dots\langle\phi|x_2\rangle\langle x_2|\phi\rangle\dots\langle x_n|\phi\rangle=N(N-1)\dots(N-n+2).\end{aligned}\right\}\tag{3}$$

Therefore we have, summing up n similar terms,

$$\begin{aligned}&\langle T^\dagger|\boldsymbol{x}\rangle\langle\boldsymbol{x}|\Sigma H(i)|\boldsymbol{x}'\rangle\langle\boldsymbol{x}'|T\rangle\\&=\{(n-1)\,!\}^{-1}(N-1)\dots(N-n+1)\langle\phi|x\rangle\langle x|H|x'\rangle\langle x'|\phi\rangle\end{aligned}$$

where $\langle x|H|x'\rangle=\langle x_i|H(i)|x_i'\rangle$ for $i=1, 2, \dots, n$.

On the other hand, since Φ, Ψ are of the n-th degree in $\langle\phi|x\rangle$, $\langle x|\phi\rangle$ we see

$$\Phi N=n\Phi,\quad N\Psi=n\Psi.$$

In short the operator N is an operator representing the number of particles. Hence we have

$$\text{Total action}=\Phi L\Psi,\quad L=\langle\phi|x\rangle\langle x|H|x'\rangle\langle x'|\phi\rangle.\tag{4}$$

We invoke here the variation principle that asserts the action to be stationary for realizable states and get fundamental equations to be satisfied by Ψ and Φ,

$$L\Psi=0, \quad \Phi L=0. \tag{5}$$

But, in our previous formulation, Φ, Ψ are conditioned to be an eigenstate of the operator L, thus satisfying

$$L\Psi=l\Psi, \quad \Phi L=l\Phi.$$

In retrospect, we find that the previous formulation has imposed inadvertently an unnecessary normalization condition $\Phi\Psi=1$. The condition should be discarded. When $N=1$, we can put

$$\Psi=\langle\phi|x\rangle\langle x|\Psi\rangle\cdot 1, \quad \Phi=1^{\dagger}\cdot\langle\Phi|x\rangle\langle x|\phi\rangle$$

so that the action reduces to the action of one particle

$$L=\langle\Phi|x\rangle\langle x|H|x'\rangle\langle x'|\Psi\rangle$$

whence we derive the wave equation

$$\langle x|H|x'\rangle\langle x'|\Psi\rangle=0, \quad \langle\Phi|x\rangle\langle x|H|x'\rangle=0$$

by virtue of the variation principle, as is desired.

§ 3. Quantization of mixed tensor fields

We may remark here at first that a set of mixed tensors span a Hilbert space. In fact, 1) if $\langle x|U|x'\rangle$, $\langle x|V|x'\rangle$ are mixed tensors, $a\langle x|U|x'\rangle+b\langle x|V|x'\rangle$, a, b being any complex numbers, is again a mixed tensor. 2) The scalar product of U and V can be defined by

$$(U, V)=\langle x|V^{\dagger}|x'\rangle\langle x'|U|x\rangle=\mathrm{tr}\, V^{\dagger}U=\langle x'|V|x\rangle^{*}\langle x'|U|x\rangle.$$

3) There exists a complete orthonormal set, for example, $\langle x|p\rangle\langle p'|x'\rangle$, p, p' being eigenvalues of momentum ranging from $-\infty$ to ∞.

Therefore the method of composition is applicable.

Let the action of a particle be denoted for example by

$$\begin{aligned} L&=\mathrm{tr}\{[p_{\mu}, u^{\dagger}][u, p^{\mu}]-m^2u^{\dagger}u\} \\ &=\langle x''x'''|Q|xx'\rangle\langle x|u^{\dagger}|x''\rangle\langle x'|u|x'''\rangle \end{aligned}$$

$$\begin{aligned} \langle x''x'''|Q|xx'\rangle=&\langle x''|x'\rangle\langle x'''|p^{\mu}p_{\mu}|x\rangle-2\langle x''|p_{\mu}|x'\rangle\langle x'''|p^{\mu}|x\rangle \\ &+\langle x'''|x\rangle\langle x''|p_{\mu}p^{\mu}|x'\rangle-m^2\langle x''|x'\rangle\langle x'''|x\rangle \end{aligned}$$

and further the total action of n particles by

$$\begin{aligned} \text{Total action}=&\sum_i \mathrm{tr}\{[p_{\mu}^{(i)}, U^{\dagger}][U, p^{\mu(i)}]-m^{2(i)}U^{\dagger}U\} \\ =&\langle x_1\ldots x_n|U^{\dagger}|x_1''\ldots x_n''\rangle\langle x_1'\ldots x_n'|U|x_1'''\ldots x_n'''\rangle \\ &\cdot\{\langle x_1''x_1'''|Q(1)|x_1x_1'\rangle\langle x_2''x_2'''|x_2x_2'\rangle\ldots \\ &\ldots\langle x_n''x_n'''|x_nx_n'\rangle+\ldots\}. \end{aligned}$$

Introducing boson or fermion operators $\langle x|u|x'\rangle$, $\langle x|u^\dagger|x'\rangle$ obeying the following commutation relations

$$\left.\begin{aligned}&\langle x|u|x'\rangle\langle x''|u^\dagger|x'''\rangle-\rho\langle x''|u^\dagger|x'''\rangle\langle x|u|x'\rangle=\langle x|x'''\rangle\langle x''|x'\rangle\\&\langle x|u|x'\rangle\langle x''|u|x'''\rangle-\rho\langle x''|u|x'''\rangle\langle x|u|x'\rangle=0\\&\langle x|u^\dagger|x'\rangle\langle x''|u^\dagger|x'''\rangle-\rho\langle x''|u^\dagger|x'''\rangle\langle x|u^\dagger|x'\rangle=0,\end{aligned}\right\}\tag{6}$$

we put, as in the preceding section,

$$\left.\begin{aligned}&\langle \boldsymbol{x}|T|\boldsymbol{x}'\rangle=(n\,!)^{-1/2}\langle x_1|u|x_1'\rangle\ldots\langle x_n|u|x_n'\rangle\\&\langle \boldsymbol{x}|T^\dagger|\boldsymbol{x}'\rangle=(n\,!)^{-1/2}\langle x_n|u^\dagger|x_n'\rangle\ldots\langle x_1|u^\dagger|x_1'\rangle\end{aligned}\right\}\tag{7}$$

$$\langle x|u|x'\rangle\cdot 1=0,\quad 1^\dagger\langle x|u^\dagger|x'\rangle=0$$

$$\Phi=1^\dagger\cdot\langle \boldsymbol{x}|U^\dagger|\boldsymbol{x}'\rangle\langle \boldsymbol{x}'|T|\boldsymbol{x}\rangle$$

$$\Psi=\langle \boldsymbol{x}|T^\dagger|\boldsymbol{x}'\rangle\langle \boldsymbol{x}'|U|\boldsymbol{x}\rangle\cdot 1$$

$$N=\langle x|u^\dagger|x'\rangle\langle x'|u|x\rangle.$$

The resulting formula is quite similar to the preceding one,

$$\text{Total action}=\Phi L\Psi$$

$$L=\langle x''x'''|Q|xx'\rangle\langle x|u^\dagger|x''\rangle\langle x'|u|x'''\rangle.$$

When the field of a particle is represented by an Hermitian mixed tensor, the method of composition is not applicable, for Hermitian mixed tensors do not span a Hilbert space. In fact, if $\langle x|U|x'\rangle$ is Hermitian, that is, $\langle x|U|x'\rangle^*=\langle x'|U^\dagger|x\rangle$, $a\langle x|U|x'\rangle$, a being any complex number, is not Hermitian unless a is real.

In this case we decompose U into the sum of two terms

$$U=u+u^\dagger$$

and impose on u and $u^\dagger$ the condition that the action of a particle, which is assumed to be quadratic in U, do not contain terms quadratic in u or in $u^\dagger$.

For example, we take the action of a particle to be

$$L=\frac{1}{2}\operatorname{tr}[p^\mu,\,U][U,\,p_\mu]+\operatorname{tr}AU$$

where $\operatorname{tr}AU$ is the interaction of U and another field A. If U is decomposed as above, the action becomes

$$L=\operatorname{tr}[p_\mu,\,u^\dagger][u,\,p^\mu]+\operatorname{tr}A(u+u^\dagger)$$
$$+\frac{1}{2}\operatorname{tr}\{[p_\mu,\,u][u,\,p^\mu]+[p_\mu,\,u^\dagger][u^\dagger,\,p^\mu]\}.$$

The additional condition is taken to be

$$\mathrm{tr}\{[p_\mu, u][u, p^\mu]+[p_\mu, u^\dagger][u^\dagger, p^\mu]\}=0.$$

Then the action becomes

$$L=\mathrm{tr}\{[p_\mu, u^\dagger][u, p^\mu]+A(u+u^\dagger)\}.$$

Another method is to introduce an auxiliary Hermitian field V and to add the action L' of V to the action L of U.

$$L'=\frac{1}{2}\mathrm{tr}[p_\mu, V][V, p^\mu]+\mathrm{tr}\,AV.$$

The total action $L+L'$ gives respective equations separately. Instead of Hermitian fields U, V two fields u, $u^\dagger$ hermite-conjugate to each other, defined by

$$\left.\begin{aligned} U&=\sigma u+\sigma^* u^\dagger \\ V&=\sigma^* u+\sigma u^\dagger \end{aligned}\right\} \quad \text{or} \quad \begin{aligned} u&=\sigma^* U+\sigma V \\ u^\dagger&=\sigma U+\sigma^* V \end{aligned} \qquad \sigma=\frac{1+i}{2}$$

are introduced so that the total action takes the form

$$\text{Total action}=\mathrm{tr}[p_\mu, u^\dagger][u, p^\mu]+\mathrm{tr}\,A(u+u^\dagger) \tag{8}$$

which is identical to the preceding.

§4. Field versus Action at a Distance

The theory of electromagnetic field of Faraday-Maxwell has gained the authority over the theory of action at a distance in explaining electromagnetic phenomena since the end of last century. The theory of field suffers, however, a serious fault, that is, the appearance of self energy, which is denied at the outset in the theory of action at a distance. The fault has been inherited by quantum field theory.

We consider the motion of particles from the two points of view. Following the papers of Wheeler and Feynman, we denote coordinates of a particle a by $a^1, \ldots, a^4$, mass by m_a, charge by e_a. The theory of field, which employs the concept of a universal field, gives the total action

$$\left.\begin{aligned} &J=\sum_a m_a\int\sqrt{da_\mu da^\mu}+\sum_a e_a\int A_\mu(a)da^\mu \\ &\qquad +\frac{1}{4}\int F_{\mu\nu}(x)F^{\mu\nu}(x)\,dx=\text{stationary}, \\ &\frac{\partial A^\mu}{\partial x^\mu}=0, \end{aligned}\right\} \tag{9}$$

$$(F_{\mu\nu}=\partial A_\nu/\partial x^\mu-\partial A_\mu/\partial x^\nu, \quad a^0=a_0,\ a^1=-a_1,\ \text{etc.})$$

where the integration extends over a finite domain D in space-time.

The variation principle gives the equations of motion of particles together with the field equations

$$m_a \ddot{a}_\mu = e_a F_{\mu\nu} \dot{a}^\nu$$

$$\frac{\partial F^{\mu\nu}}{\partial x^\nu} = \sum_a e_a \int \delta(x-a) da^\mu, \quad \delta(x-a) = \prod_\mu \delta(x^\mu - a^\mu)$$

boundary conditions for a^μ, A_μ being to be given at the boundary of the domain D.

The theory of action at a distance, which does not use the concept of field directly, gives the total action

$$J = \sum_a m_a \int \sqrt{da_\mu da^\mu} + \frac{1}{4\pi} \sum_{a \neq b} e_a e_b \iint D(a-b) da_\mu db^\mu \tag{10}$$

$$D(a-b) = \delta((a-b)_\mu (a-b)^\mu)$$

where the integration should be extended over the entire paths of particles from $-\infty$ to ∞, as Wheeler and Feynman have explained.

The variational principle gives the equations of motion of particles

$$m_a \ddot{a}_\mu = e_a \sum_{b \neq a} F_{\mu\nu}{}^{(b)} \dot{a}^\nu \tag{11}$$

where $F_{\mu\nu}{}^{(b)}$ is short for

$$F_{\mu\nu}{}^{(b)}(x) = \partial A_\nu{}^b / \partial x^\mu - \partial A_\mu{}^b / \partial x^\nu, \quad A_\mu{}^b(x) = e_b \int D(x-b) db_\mu.$$

There is a difficulty that the boundary conditions cannot be given at a definite epoch, since (11) is not a set of ordinary differential equations.

There is another difficulty that the theory of action at a distance cannot derive the existence of photon upon quantization, since it reduces the field to a useful accessory, depriving of its importance attested by the Faraday-Maxwell theory.

But this theory does not conjure up the self-energy difficulty. It is a great merit.

The universal field $F_{\mu\nu}$ has its sources at the positions of all particles, and affects all particles, so that it gives rise to the self-energy difficulty. Therefore, the theory of field fails to describe the interaction of particles. The concept of universal field seems, however, to be an illogical misconception when the interaction of elementary particles is considered. The first reason is that the definition of the strength of an electric field at a position as the force acting on a test particle placed at the position divided by its charge is subjected to the condition that the charge of the test particle be enough small not to disturb the electrification of other bodies. The second reason is that since there is the least unit of electric charge, it is impossible to place a test particle at a position without disturbing the state of other particles. Therefore, the concept of universal field is not tenable, seeing that the condition imposed on the measurement of the strength of electric field is never realizable for a system of interacting particles.

§ 5. The Concept of Multiple Field

Some reconciliation of the two theories must be needed to overcome the difficulties said above. If we remember the definition of electric field, it is logical to define a field for each particle, which field represents the effect experienced by the particle due to other particles. We have then a set of individual fields, which we call multiple field. To define a field for a particle experienced by the particle individually is equivalent to define a field for a particle produced by the particle individually. So we employ the latter definition in the following.

For a system of particles in otherwise charge-free space, we put the total action

$$J=\sum_a m_a\int\sqrt{da_\mu da^\mu}+\sum_a e_a\sum_{b\neq a}\int A_\mu{}^b(a)\,da^\mu$$
$$+\frac{1}{4}\sum_a\sum_{b\neq a}\int F_{\mu\nu}{}^a(x)\,F^{\mu\nu b}(x)\,dx \tag{12}$$

where the integration extends over a finite domain D in space-time. The variation principle gives

$$\left.\begin{aligned} m_a\ddot{a}_\mu &= e_a\dot{a}^\nu\sum_{b\neq a}F_{\mu\nu}{}^b \\ \frac{\partial F^{\mu\nu a}}{\partial x^\nu} &= e_a\int\delta(x-a)\,da^\mu \end{aligned}\right\}\ \text{inside } D \tag{13}$$

boundary conditions being given at the boundary for a_μ, $A_\mu{}^a$. The self-energy never appears. The existence of photon will be derived upon quantization of the total action (12) in the succeeding section.

If we eliminate the field variables in (12) with the aid of (13), we get the total action (10) in the theory of action at a distance.

If we write

$$j_\mu{}^a(x)=e_a\int_{-\infty}^{\infty}\delta(x-a)\,da_\mu$$

and extend the integration over the entire space-time as in (10), the total action takes the form

$$J=\sum m_a\int\sqrt{da_\mu da^\mu}+\sum_a\sum_{b\neq a}\int A_\mu{}^b(x)\,j^{\mu a}(x)\,dx$$
$$+\frac{1}{4}\sum_a\sum_{b\neq a}\int F_{\mu\nu}{}^a(x)\,F^{\mu\nu b}(x)\,dx, \tag{14}$$

and the field equations become

$$\left.\begin{aligned} &\frac{\partial F^{\mu\nu a}}{\partial x^\nu}=j^{\mu a} \\ \text{whence} \quad & \\ &A_\mu{}^a(x)=\frac{e_a}{4\pi}\int D(x-a)\,da_\mu . \end{aligned}\right\} \quad (15)$$

The integration by parts gives

$$\frac{1}{4}\sum_a \sum_{b\neq a}\int F_{\mu\nu}{}^a(x)\,F^{\mu\nu b}(x)\,dx$$

$$=\frac{1}{2}\sum_a \sum_{b\neq a}\int A_\mu{}^a \frac{\partial F^{\mu\nu b}}{\partial x^\nu}\,dx=-\frac{1}{2}\sum_a \sum_{b\neq a}\int A_\mu{}^a j^{\mu b}\,dx$$

Therefore, the second and third terms of J in (14) become, in view of (15),

$$\frac{1}{2}\sum_a \sum_{b\neq a}\int A_\mu{}^b(x)\,j^{\mu a}(x)\,dx$$

$$=\frac{1}{4\pi}\sum_{a<b} e_a e_b\int D(a-b)\,da_\mu db^\mu .$$

§6. Quantization of the Total Action

Our next task is to quantize the total action (12). The first step is to derive the quantum mechanical action in terms of wave functions. The second step is to quantize the action by the method of composition, leaving the field variables unquantized. The third step is to quantize the field variables by the method of § 3 and the final step is to transform the total action into that of a simpler form, by eliminating the interaction of particles and fields by way of a unitary transformation.

The action of a particle is assumed to be the action of a Dirac particle

$$\phi(\gamma^\mu p_\mu - m)\psi=\langle\phi\,|\,x,\ \alpha\rangle\langle x\alpha\,|\,H\,|\,x'\alpha'\rangle\langle x'\alpha'\,|\,\psi\rangle$$

$$H=\gamma^\mu p_\mu - m$$

then, the action of n similar particles becomes

$$\langle\Phi\,|\,\boldsymbol{x},\boldsymbol{\alpha}\rangle\langle \boldsymbol{x},\boldsymbol{\alpha}\,|\,H(1)+\ldots+H(n)\,|\,\boldsymbol{x}',\boldsymbol{\alpha}'\rangle\langle \boldsymbol{x}',\boldsymbol{\alpha}'\,|\,\Psi\rangle,\quad H(a)=\gamma^{\mu(a)}p_\mu{}^{(a)}-m^{(a)}$$

$\boldsymbol{x}$, $\boldsymbol{\alpha}$ standing for a set of variables $x_1, \ldots, x_n$, $\alpha_1, \ldots, \alpha_n$ (α: spin variables).

We write next the field variables in their representation tensor form by putting

$$A_\mu(x)\,\delta(x-x')=\langle x\,|\,A_\mu\,|\,x'\rangle$$

$$F_{\mu\nu}(x)\delta(x-x')=\langle x|F_{\mu\nu}|x'\rangle$$

$$F_{\mu\nu}=i[p_\mu, A_\nu]-i[p_\nu, A_\mu].$$

The interaction of a particle with the field has been given by

$$e\phi\gamma^\mu A_\mu\phi=e\langle\phi|x\rangle\gamma^\mu\langle x'|\phi\rangle\langle x|A_\mu|x'\rangle$$

so that we assume in the present case the following interaction

$$\langle\Phi|\boldsymbol{x}, \boldsymbol{\alpha}\rangle\langle\boldsymbol{x\alpha}|G(1)+\ldots+G(n)|\boldsymbol{x'\alpha'}\rangle\langle\boldsymbol{x'\alpha'}|\Psi\rangle$$

$$G(a)=e\gamma^{\mu(a)}\sum_{b\neq a}A_\mu{}^b.$$

The action of the field, which is in its tensor form

$$\frac{1}{4}\sum_{a\neq b}\mathrm{tr}\,F_{\mu\nu}{}^aF^{\mu\nu b}=\frac{1}{4}\sum_{a\neq b}\langle x|F_{\mu\nu}{}^a|x'\rangle\langle x'|F^{\mu\nu b}|x\rangle,$$

is now to be enclosed by Φ and Ψ, since it depends on the states of particles through suffices a, b.

Therefore the total action becomes

$$\left.\begin{aligned}&\text{Total action}=\langle\Phi|\boldsymbol{x\alpha}\rangle\langle\boldsymbol{x\alpha}|L|\boldsymbol{x'\alpha'}\rangle\langle\boldsymbol{x'\alpha'}|\Psi\rangle\\&L=\sum_a(\gamma^{\mu(a)}p_\mu{}^a-m^a)+e\sum_{b\neq a}\gamma^{\mu a}A_\mu{}^b+\frac{1}{4}\sum_{a\neq b}\mathrm{tr}\,F_{\mu\nu}{}^aF^{\mu\nu b}.\end{aligned}\right\}\quad(16)$$

The method of composition in §3 gives the action of particles, transformation functions and the particle number,

$$\left.\begin{aligned}&\text{Action}=\Phi\langle\phi|x, \alpha\rangle\langle x\alpha|H|x'\alpha'\rangle\langle x'\alpha'|\phi\rangle\Psi\\&\langle\boldsymbol{x\alpha}|T\rangle=(n\,!)^{-1/2}\langle x_1\alpha_1|\phi\rangle\ldots\langle x_n\alpha_n|\phi\rangle\\&\langle T^\dagger|\boldsymbol{x\alpha}\rangle=(n\,!)^{-1/2}\langle\phi|x_n\alpha_n\rangle\ldots\langle\phi|x_1\alpha_1\rangle\\&N=\langle\phi|x\alpha\rangle\langle x\alpha|\phi\rangle.\end{aligned}\right\}\quad(17)$$

The interaction of particles with fields becomes

$$\{n(n-1)\}^{-1}\sum_{a\neq b}e\Phi\langle\phi|x\alpha^a\rangle\langle\phi|x'\alpha'^b\rangle\gamma^{\mu a}_{\alpha\alpha''}\langle x|A_\mu{}^b|x''\rangle\langle x'''\alpha'''^b|\phi\rangle\langle x''\alpha''^a|\phi\rangle\Psi$$

where $\langle\phi|x'\alpha'^b\rangle$, $\langle x'''\alpha'''^b|\phi\rangle$ are left standing because the field variable $\langle x|A_\mu{}^b|x''\rangle$ depends on the state of the particle b through the suffix b, representing the field due to the particle b.

Since the term enclosed by Φ and Ψ should be invariant for the representation transformation, $\langle x|A_\mu{}^b|x''\rangle$ should transform as a tensor with left suffices x, $x'\alpha'^b$ and right suffices x'', $x'''\alpha'''^b$. To make the dependance explicit, the field variable may be written

$$\langle x|A_\mu{}^b|x''\rangle=\langle x, x'\alpha'^b|A_\mu{}^b|x'''\alpha'''^b, x''\rangle$$

so that the above interaction becomes

$$\Phi e\langle\phi|x\alpha\rangle\langle\phi|x'\alpha'\rangle\gamma^{\mu}_{\alpha\alpha''}\langle x,\ x'\alpha'|A_\mu|x'''\alpha''',\ x''\rangle\langle x'''\alpha'''|\phi\rangle\langle x''\alpha''|\phi\rangle\Psi$$

dropping the suffices a, b. Because of the property of fermion operators, this interaction excludes the interaction of a particle with itself. By a similar procedure,the action of the field becomes

$$\frac{1}{4}\Phi\langle\phi|x\alpha\rangle\langle\phi|x'\alpha'\rangle\langle y,\ x\alpha|F_{\mu\nu}|x''\alpha'',\ y'\rangle$$
$$\langle y',\ x'\alpha'|F^{\mu\nu}|x'''\alpha''',\ y\rangle\langle x'''\alpha'''|\phi\rangle\langle x''\alpha''|\phi\rangle\Psi.$$

Therefore the quantized total action becomes

$$\begin{aligned}L=&\langle\phi|x\alpha\rangle\langle x\alpha|H|x'\alpha'\rangle\langle x'\alpha'|\phi\rangle\\&+e\langle\phi|x\alpha\rangle\langle\phi|x'\alpha'\rangle\gamma^{\mu}_{\alpha\alpha'}\langle x,\ x'\alpha'|A_\mu|x'''\alpha''',\ x''\rangle\langle x'''\alpha'''|\phi\rangle\langle x''\alpha''|\phi\rangle\\&+\frac{1}{4}\langle\phi|x\alpha\rangle\langle\phi|x'\alpha'\rangle\langle y,\ x\alpha|F_{\mu\nu}|x''\alpha'',\ y'\rangle\langle y',\ x'\alpha'|F^{\mu\nu}|x'''\alpha''',\ y\rangle\\&\qquad\qquad\langle x'''\alpha'''|\phi\rangle\langle x''\alpha''|\phi\rangle.\end{aligned}\tag{18}$$

We quantize then the field variables following the second method of § 3. Introducing auxiliary fields B_μ, $G_{\mu\nu}$ related by the relation

$$G_{\mu\nu}=i[p_\mu,\ B_\nu]-i[p_\nu,\ B_\mu].$$

We rewrite the part involving field variables out of the total action, adding the similar part involving auxiliary fields,

$$e\sum_{a\neq b}\gamma^{\mu a}(A_\mu{}^b+B_\mu{}^b)+\frac{1}{4}\sum_{a\neq b}\mathrm{tr}\,(F_{\mu\nu}{}^aF^{\mu\nu b}+G_{\mu\nu}{}^aG^{\mu\nu b})$$
$$=e\sum_{a\neq b}\gamma^{\mu a}(a_\mu{}^b+a_\mu{}^{\dagger b})+\frac{1}{4}\sum_{a\neq b}\mathrm{tr}\,(f_{\mu\nu}{}^{\dagger a}f^{\mu\nu b})$$

where

$$a_\mu=\sigma^*A_\mu+\sigma B_\mu,\quad a_\mu{}^\dagger=\sigma A_\mu+\sigma^*B_\mu$$
$$f_{\mu\nu}=i[p_\mu,\ a_\nu]-i[p_\nu,\ a_\mu],\quad f_{\mu\nu}{}^\dagger=i[p_\mu,\ a_\nu{}^\dagger]-i[p_\nu,\ a_\mu{}^\dagger].$$

The total action becomes then

$$\begin{aligned}L=&\langle\phi|x\alpha\rangle\langle x\alpha|\gamma^\mu p_\mu-m|x'\alpha'\rangle\langle x'\alpha'|\phi\rangle\\&+e\langle\phi|x\alpha\rangle\langle\phi|x'\alpha'\rangle\gamma^{\mu}_{\alpha\alpha''}\langle x,\ x'\alpha'|a_\mu+a_\mu{}^\dagger|x'''\alpha''',\ x''\rangle\langle x'''\alpha'''|\phi\rangle\langle x''\alpha''|\phi\rangle\\&+\frac{1}{4}\langle\phi|x\alpha\rangle\langle\phi|x'\alpha'\rangle\langle y,\ x\alpha|f_{\mu\nu}{}^\dagger|x''\alpha'',\ y'\rangle\langle y',\ x'\alpha'|f^{\mu\nu}|x'''\alpha''',\ y\rangle\\&\qquad\qquad\langle x'''\alpha'''|\phi\rangle\langle x''\alpha''|\phi\rangle.\end{aligned}$$

We quantize field variables following the method of § 3 by setting the following commutation relations among them,

$$\left.\begin{aligned}&[\langle y,\, x\alpha\,|\,a_\mu\,|\,x'\alpha',\, y'\rangle,\ \langle y'',\, x''\alpha''\,|\,a_\nu\,|\,x'''\alpha''',\, y'''\rangle]=0\\&[\langle y,\, x\alpha\,|\,a_\mu{}^\dagger\,|\,x'\alpha',\, y'\rangle,\ \langle y'',\, x''\alpha''\,|\,a_\nu{}^\dagger\,|\,x'''\alpha''',\, y'''\rangle]=0\\&[\langle y,\, x\alpha\,|\,a^\mu\,|\,x'\alpha',\, y'\rangle,\ \langle y'',\, x''\alpha''\,|\,a_\nu{}^\dagger\,|\,x'''\alpha''',\, y'''\rangle]\\=&\langle y\,|\,y'''\rangle\langle y''\,|\,y'\rangle\langle x\alpha\,|\,x'''\alpha'''\rangle\langle x''\alpha''\,|\,x'\alpha'\rangle\,\delta^\mu_\nu\,.\end{aligned}\right\}\tag{19}$$

In the momentum representation, the last term of (18) becomes

$$\langle\phi\,|\,p\alpha\rangle\langle\phi\,|\,p'\alpha'\rangle(q-q')^2\langle q,\, p\alpha\,|\,a_\mu{}^\dagger\,|\,p''\alpha'',\, q'\rangle$$
$$\langle q',\, p'\alpha'\,|\,a^\mu\,|\,p'''\alpha''',\, q\rangle\langle p'''\alpha'''\,|\,\phi\rangle\langle p''\alpha''\,|\,\phi\rangle,$$
$$(q-q')^2=(q-q')_\mu(q-q')^\mu$$

q standing for p to save the use of primes. Therefore the total action becomes simpler in the momentum reperesentation.

When no particle is present, L reduces to nothing.

The interaction of particles with fields can be eliminated by a unitary transformation

$$\begin{aligned}e^{-X}Le^{X}&=L+[L,\, X]+\frac{1}{2}[[L,\, X],\, X]+\dots\\&=L_0+(L_1+[L_0,\, X])+([L_1,\, X]+\frac{1}{2}[[L_0,\, X],\, X])+\dots\end{aligned}\tag{20}$$

where

$L_0=L_M$ (the action of particles) $+L_F$ (the action of fields)
$L_I=$ the interaction of particles with fields

and X is to be determined by the condition

$$L_I+[L_0,\, X]=0.\tag{21}$$

We assume

$$\begin{aligned}X=&\langle r,\, r'\beta'\,|\,a_\mu\,|\,r''\beta'',\, r'''\rangle\langle r''',\, r''\beta''\,|\,\xi^\mu\,|\,r'\beta',\, r\rangle\\&+\langle r,\, r'\beta'\,|\,\eta^\mu\,|\,r''\beta'',\, r'''\rangle\langle r''',\, r''\beta''\,|\,a_\mu{}^\dagger\,|\,r'\beta',\, r\rangle\end{aligned}\tag{22}$$

with c-number coefficients ξ^μ, η^μ and compute $[L_0,\, X]$. Since the action of particles L_M contains no field variables, X commutes with it. Now we compute $[L_F,\, X]$

$$\begin{aligned}[L_F,\, X]=&\langle\phi\,|\,p\alpha\rangle\langle\phi\,|\,p'\alpha'\rangle\langle p'''\alpha'''\,|\,\phi\rangle\langle p''\alpha''\,|\,\phi\rangle(q-q')^2\\&[\langle q,\, p\alpha\,|\,a_\nu{}^\dagger\,|\,p''\alpha'',\, q'\rangle\langle q',\, p'\alpha'\,|\,a^\nu\,|\,p'''\alpha''',\, q\rangle,\ X]\\=&\langle\phi\,|\,p\alpha\rangle\langle\phi\,|\,p'\alpha'\rangle\langle p'''\alpha'''\,|\,\phi\rangle\langle p''\alpha''\,|\,\phi\rangle(q-q')^2\\&\{-\langle q,\, p\alpha\,|\,\xi^\mu\,|\,p''\alpha'',\, q'\rangle\langle q',\, p'\alpha'\,|\,a_\mu\,|\,p'''\alpha''',\, q\rangle\\&+\langle q,\, p\alpha\,|\,a_\mu{}^\dagger\,|\,p''\alpha'',\, q'\rangle\langle q',\, p'\alpha'\,|\,\eta^\mu\,|\,p'''\alpha''',\, q\rangle\}.\end{aligned}\tag{23}$$

On the other hand, the interaction can be written

$$L_I=\langle\phi|p\alpha\rangle\langle\phi|p'\alpha'\rangle\langle p'''\alpha'''|\phi\rangle\langle p''\alpha''|\phi\rangle$$
$$\cdot\{e\gamma^{\mu}_{\alpha\alpha''}\langle q',\, p'\alpha'|a_\mu|p'''\alpha''',\, q\rangle\langle p|q'\rangle\langle q|p''\rangle$$
$$+e\gamma^{\mu}_{\alpha'\alpha'''}\langle q,\, p\alpha|a_\mu^{\dagger}|p''\alpha'',\, q'\rangle\langle p'|q\rangle\langle q'|p'''\rangle\}. \tag{24}$$

Comparing (23) with (24), we have the equations to ξ, η

$$-(q-q')^2\langle q,\, p\alpha|\xi^{\mu}|p''\alpha'',\, q'\rangle=e\gamma^{\mu}_{\alpha\alpha''}\langle p|q'\rangle\langle q|p''\rangle$$
$$(q-q')^2\langle q',\, p'\alpha'|\eta^{\mu}|p'''\alpha''',\, q\rangle=e\gamma^{\mu}_{\alpha'\alpha'''}\langle p'|q\rangle\langle q'|p'''\rangle.$$

Therefore a solution is

$$\langle q,\, p\alpha|\xi^{\mu}|p''\alpha'',\, q'\rangle=-\frac{e\gamma^{\mu}_{\alpha\alpha''}\langle p|q'\rangle\langle q|p''\rangle}{(q-q')^2}$$
$$\langle q',\, p'\alpha'|\eta^{\mu}|p'''\alpha''',\, q\rangle=\frac{e\gamma^{\mu}_{\alpha'\alpha'''}\langle p'|q\rangle\langle q'|p'''\rangle}{(q-q')^2}.$$

This solution gives

$$[L_I,\, X]=\langle\phi|p\alpha\rangle\langle\phi|p'\alpha'\rangle\langle p\alpha'''|\phi\rangle\langle p'\alpha''|\phi\rangle\frac{2e^2\gamma^{\mu}_{\alpha\alpha''}\gamma_{\mu,\alpha'\alpha'''}}{(p-p')^2}.$$

Hence $[L_I,\, X]$ involves no field variables, so that $[[L_I,\, X],\, X]=0$. Therefore we get an exact formula

$$e^{-X}Le^{X}=L_0+\frac{1}{2}[L_1,\, X]$$
$$=L_0+e^2\langle\phi|p\alpha\rangle\langle\phi|p'\alpha'\rangle\langle p\alpha'''|\phi\rangle\langle p'\alpha''|\phi\rangle\frac{\gamma^{\mu}_{\alpha\alpha''}\gamma_{\mu,\alpha'\alpha'''}}{(p-p')^2}. \tag{25}$$

The last term in the right member denotes the direct interaction of particles, which corresponds to the interaction of particles in the theory of action at a distance

$$\frac{e^2}{4\pi}\sum_{a<b}\iint D(a-b)da_\mu db^{\mu}$$

precluding obviously the self-interaction.

The formula (25) is not an approximate but an exact one.

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