Natural Science Report, Ochanomizu University, Vol. 24, No. 1, 1973 1
お茶の水女子大学 自然科学報告 第24巻 第1号

Note on Regular K-c ntact 3-structures

Mariko Konishi

Department of Mathematics, Ochanomizu University and
Tokyo Institute of Technology

(Received February, 28, 1973)

In a previous paper [4] we studied fibred Riemannian manifolds with Sasakian 3-structure. That is a fibred Riemannian manifold $(\tilde{M}, M, \pi; \tilde{g})$ such that $\tilde{M}$ has a Sasakian 3-structure and their associated 3 vector fields are tangent to each fibre $\pi^{-1}(p)$ $(p \in M)$. Then it has been shown in [2] that the base manifold M has a quaternion Kaehler structure.

More generally, if $\tilde{M}$ is a fibred Riemannian manifold with K-contact 3-structure, an almost quaternion structure can be induced in the base manifold. In this note, we show that if the structure induced in M is a quaternion Kaehler structure, then the K-contact 3-structure in $\tilde{M}$ is necessarily Sasakian 3-structure.

§1. Preliminaries

First we recall definitions and some fundamental properties of a fibred Riemannian manifold with K-contact 3-structure (cf. [3, 4]). Let $(\tilde{M}, \tilde{g})$ be a Riemannian manifold and ξ, η, ζ be three unit Killing vector fields which are mutually orthogonal and satisfy

$$\xi=\frac{1}{2}[\eta, \zeta], \quad \eta=\frac{1}{2}[\zeta, \xi], \quad \zeta=\frac{1}{2}[\xi, \eta]. \tag{1.1}$$

Such a set $\{\xi, \eta, \zeta\}$ is called a triple of Killing vectors, for simplicity.

We put

$$\phi=\tilde{\nabla}\xi, \quad \psi=\tilde{\nabla}\eta, \quad \theta=\tilde{\nabla}\zeta, \tag{1.2}$$

$\tilde{\nabla}$ being the Riemannian connection of $(\tilde{M}, \tilde{g})$. Assume that the sectional curvature of $(\tilde{M}, \tilde{g})$ with respect to any section containing at least one of ξ, η and ζ is equal to 1, then $(\phi, \xi, \tilde{g})$, $(\psi, \eta, \tilde{g})$ and $(\theta, \zeta, \tilde{g})$ are all K-contact structures. Assume further that

$$\theta\psi=\phi+\beta\otimes\zeta, \qquad \phi\theta=\psi+\gamma\otimes\xi, \qquad \psi\phi=\theta+\alpha\otimes\eta,$$
$$\psi\theta=-\phi+\gamma\otimes\eta, \quad \theta\phi=-\psi+\alpha\otimes\zeta, \quad \phi\psi=-\theta+\beta\otimes\xi,$$

α, β and γ being 1-forms associated with ξ, η and ζ, respectively. Then we call $\{\xi, \eta, \zeta\}$ a *K-contact 3-structure*. In particular, if ξ, η, ζ define Sasakian structures, $\{\xi, \eta, \zeta\}$ is called a *Sasakian 3-structure*.

Let $(\tilde{M}, M, \pi; \tilde{g})$ be a fibred Riemannian manifold with triple of Killing vectors $\{\xi, \eta, \zeta\}$. That is, a fibred Riemannian manifold such that $\tilde{M}$ admits a triple of Killing vectors $\{\xi, \eta, \zeta\}$ and each fibre is a maximal integral manifold of the distribution spanned by ξ, η and ζ. In the sequel we denote $(\tilde{M}, M, \pi; \tilde{g})$ by $(\tilde{M}, \tilde{g})$ for brevity.

If a vector field is tangent to each fibre, then it is called a *vertical vector field*. If a vector field is orthogonal to each fibre, then it is called a *horizontal vector field*. A 1-form ω is called *horizontal* if $\omega(\xi)=\omega(\eta)=\omega(\zeta)=0$. And a horizontal tensor field of any type is defined in the usual way (cf. [**3**, **4**]). We define the horizontal part $\tilde{T}^H$ of any kind of tensor field $\tilde{T}$ following [**3**, **4**].

A tensor field $\tilde{T}$ in $\tilde{M}$ is projectable, if it satisfies

$$(\pounds_{\tilde{X}}\tilde{T}^H)^H=0 \qquad \text{for any vertical vector field } \tilde{X},$$

where $\pounds_{\tilde{X}}$ denotes the Lie derivative with respect to $\tilde{X}$. Then $\tilde{T}$ is projectable if $\pounds_\xi\tilde{T}=\pounds_\eta\tilde{T}=\pounds_\zeta\tilde{T}=0$. If $\tilde{T}$ is projectable, we can define a tensor field in M, which is called the projection of $\tilde{T}$ and is denoted by $p\tilde{T}$. The Riemannian metric $\tilde{g}$ is projectable, since $\pounds_\xi\tilde{g}=\pounds_\eta\tilde{g}=\pounds_\zeta\tilde{g}=0$. So we adopt g $(=p\tilde{g})$ as a Riemannian metric in the base manifold M. We can see that the Riemannian connection ∇ in M is identical with the connection induced by π from $\tilde{\nabla}$ (cf. [**4**]).

Given a vector field X in M. Then there is a unique horizontal projectable vector field X^L, called the lift of X, such that $d\pi(X^L)=X$, $d\pi$ being the differential of π. Let there be given a tensor field $\tilde{F}$ (not necessarily projectable) of type (1,1) in $\tilde{M}$. We take a coordinate neighborhood U of M and a local cross section $\tau: U\to\pi^{-1}(U)$, where $\pi\circ\tau$ is the identity. We now define a local tensor field F_τ in U by

$$(F_\tau X)_p=d\pi(\tilde{F}X^L)_{\tau(p)}, \text{ for any } p\in U,$$

X being an arbitrary vector field in U. Then tensor field F_τ thus constructed in U is called the projection of $\tilde{F}$ in U with respect to the cross section τ.

§2. Quaternion Kaehler structure.

Let $(\tilde{M}, \tilde{g})$ be a fibred Riemannian manifold with triple of Killing vectors $\{\xi, \eta, \zeta\}$. We define ϕ, ψ and θ by (1.2). Then we have

LEMMA 1. *A triple of Killing vectors* $\{\xi, \eta, \zeta\}$ *is a K-contact 3-structure if and only if*

$$(\phi^H)^2=(\psi^H)^2=(\theta^H)^2=-I^H$$
$$\theta^H\psi^H=-\psi^H\theta^H=\phi^H, \quad \phi^H\theta^H=-\theta^H\phi^H=\psi^H \tag{2.1}$$
$$\psi^H\phi^H=-\phi^H\psi^H=\theta^H$$

hold, where I is the identity tensor field (see Proposition 3.3 and 3.5 in [**3**]).

In the sequel $\{\xi, \eta, \zeta\}$ is assumed to be a K-contact 3-structure. Then we have seen in [**3**]

$$\begin{array}{lll} (\pounds_\xi\phi^H)^H=0, & (\pounds_\eta\phi^H)^H=-2\theta^H, & (\pounds_\zeta\phi^H)^H=2\psi^H \\ (\pounds_\xi\psi^H)^H=2\theta^H, & (\pounds_\eta\psi^H)^H=0, & (\pounds_\zeta\psi^H)^H=-2\phi^H \\ (\pounds_\xi\theta^H)^H=-2\psi^H, & (\pounds_\eta\theta^H)^H=2\phi^H, & (\pounds_\zeta\theta^H)^H=0. \end{array} \tag{2.2}$$

We now consider local tensor fields F_τ, G_τ and H_τ which are the projections of ϕ, ψ and θ with respect to a local cross section τ in the sense of § **1**. That is, F_τ, G_τ and H_τ are defined as

$$(F_\tau X)_p=d\pi(\phi X^L)_{\tau(p)}, \quad (G_\tau X)_p=d\pi(\psi X^L)_{\tau(p)}, \quad (H_\tau X)_p=d\pi(\theta X^L)_{\tau(p)}.$$

Then we easily find

$$F_\tau^2=G_\tau^2=H_\tau^2=-I$$
$$G_\tau F_\tau=-F_\tau G_\tau=H_\tau, \quad H_\tau G_\tau=-G_\tau H_\tau=F_\tau$$
$$F_\tau H_\tau=-H_\tau F_\tau=G_\tau,$$

because of Lemma 1 (cf. [**2**]).

If we take various local cross sections, we can construct in M, an almost quaternion metric structure $\{F_\tau, G_\tau, H_\tau\}$ together with the metric g (cf. [**1**]). And by means of (2.2)

$$\tilde{\Lambda}=\phi\otimes\phi+\psi\otimes\psi+\theta\otimes\theta \tag{2.3}$$

is a projectable tensor field of type (2.2) in $\tilde{M}$, and its projection $p\tilde{\Lambda}$ coincides with

$$\Lambda=F_\tau\otimes F_\tau+G_\tau\otimes G_\tau+H_\tau\otimes H_\tau. \tag{2.4}$$

for any local cross section τ. An almost quaternion structure $\{(F_\tau, G_\tau, H_\tau), g\}$ is a quaternion Kaehler structure if and only if

$$\nabla\Lambda=0 \tag{2.5}$$

(cf. [**1**]).

§ 3. Condition to induce a quaternion Kaehler structure.

Let $(\tilde{M}, \tilde{g})$ be a fibred Riemannian manifold with K-contact 3-structure $\{\xi, \eta, \zeta\}$. We define ϕ, ψ and θ by (1.2). We are now finding a condition that the induced almost quaternion structure is a quaternion Kaehler structure.

First we have

LEMMA 2. *A necessary and sufficient condition that ϕ, ψ and θ induce a quaternion Kaehler structure in the base manifold M is that*

$$(\tilde{\nabla}\tilde{A}^H)^H=0 \tag{3.1}$$

holds in $\tilde{M}$.

This is clear from the definitions and (2.5).

We take cordinate neighborhoods $\{\tilde{U}, x^h\}$ of $\tilde{M}$ and $\{U, v^a\}$ of M such that $\pi(\tilde{U})=U$. Then π can be expressed by

$$v^a=v^a(x^h)$$

For a fibre F $(F\cap\tilde{U}\neq\phi)$, we introduce coordinates (u^α) such that (v^a, u^α) is a system of coordinates in $\tilde{U}$. In the usual way, we take a local coframe $\{E_b, C_\beta\}$ in $\tilde{U}$ and the coframe $\{E^a, C^\alpha\}$ dual to $\{E_b, C_\beta\}$. That is, E^a has components $E_i^a=\partial v^a/\partial x^i$ and $C_\beta=\partial/\partial u^\beta$.

Then the horizontal part of any tensor field in $\tilde{M}$, say $\tilde{T}$ of type (1.2), can be represented by

$$\tilde{T}^H=T_{cb}{}^aE^c\otimes E^b\otimes E_a,$$

where $T_{cb}{}^a$ are local functions in $\tilde{U}$. We put

$$\tilde{g}=g_{cb}E^c\otimes E^b$$

$$\phi^H=\phi_c{}^bE^c\otimes E_b,\quad \psi^H=\psi_c{}^bE^c\otimes E_b,\quad \theta^H=\theta_c{}^bE^c\otimes E_b,$$

$$(\tilde{\nabla}\phi^H)^H=(\nabla_d\phi_c{}^b)E^d\otimes E^c\otimes E_b,\quad (\tilde{\nabla}\psi^H)^H=(\nabla_d\psi_c{}^b)E^d\otimes E^c\otimes E_b,$$

$$(\tilde{\nabla}\theta^H)^H=(\nabla_d\theta_c{}^b)E^d\otimes E^c\otimes E_b,$$

then we have

$$(\tilde{\nabla}\tilde{A}^H)^H=\{\nabla_e(\phi_d{}^c\phi_b{}^a+\psi_d{}^c\psi_b{}^a+\theta_d{}^c\theta_b{}^a)\}E^e\otimes E^d\otimes E^b\otimes E_c\otimes E_a.$$

The condition (3.1) can be written by

$$(\nabla_e\phi_d{}^c)\phi_b{}^a+\phi_d{}^c\nabla_e\phi_b{}^a+(\nabla_e\psi_d{}^c)\psi_b{}^a+\psi_d{}^c\nabla_e\psi_b{}^a+(\nabla_e\theta_d{}^c)\theta_b{}^a+\theta_d{}^c\nabla_e\theta_b{}^a=0. \tag{3.2}$$

Transvecting (3.2) with $\phi_c{}^d$ and taking account of Lemma 1, we see that $(\tilde{\nabla}\phi^H)^H$ is linear combination of ψ^H and θ^H. Similary, we see that $(\tilde{\nabla}\psi^H)^H$ and $(\tilde{\nabla}\theta^H)^H$ are linear combinations of ϕ^H, ψ^H and θ^H. That is, we get

LEMMA 3. *The condition* (3.1) *is equivalent to that*

$$(\tilde{\nabla}_{\tilde{X}}\phi^H)^H=\quad c(\tilde{X})\psi^H-b(\tilde{X})\theta^H$$

$$\begin{aligned}(3.3) \qquad (\tilde{\nabla}_{\tilde{X}}\phi^H)^H &= -c(\tilde{X})\phi^H \qquad + a(\tilde{X})\theta^H \\ (\tilde{\nabla}_{\tilde{X}}\theta^H)^H &= b(\tilde{X})\phi^H - a(\tilde{X})\psi^H\end{aligned}$$

holds for certain horizontal 1-forms a, b, c.

If we define 2-forms Φ, Ψ and Θ by

$$\Phi(\tilde{X}, \tilde{Y}) = \tilde{g}(\phi\tilde{X}, \tilde{Y}), \quad \Psi(\tilde{X}, \tilde{Y}) = \tilde{g}(\psi\tilde{X}, \tilde{Y}), \quad \Theta(\tilde{X}, \tilde{Y}) = \tilde{g}(\theta\tilde{X}, \tilde{Y}),$$

then Φ, Ψ and Θ are all closed forms, i. e.,

$$d\Phi = 0, \quad d\Psi = 0, \quad d\Theta = 0,$$

and hence

$$(3.4) \qquad \begin{aligned}&\nabla_d\phi_{cb} + \nabla_c\phi_{bd} + \nabla_b\phi_{dc} = 0, \quad \nabla_d\psi_{cb} + \nabla_c\psi_{bd} + \nabla_b\psi_{dc} = 0 \\ &\nabla_d\theta_{cb} + \nabla_c\theta_{bd} + \nabla_b\theta_{dc} = 0.\end{aligned}$$

Substituting the relation (3.3), into (3.4), and contracting with $\psi^{cb} = g^{ca}\psi_a{}^b$, we get

$$(3.5) \qquad \phi_d{}^a b_a = (1-2m)c_d,$$

where $b = b_a E^a$, $c = c_a E^a$ and $4m = \dim M$. Next, contracting (3.4), with $\theta^{cb} = g^{ca}\theta_a{}^b$, then we have

$$(3.6) \qquad \phi_d{}^a c_a = -(1-2m)b_d.$$

Since $(\phi^H)^2 = -I^H$, (3.5) and (3.6) lead us to

$$c_d = 0.$$

In the similar way, we have $a \equiv 0$, $b \equiv 0$. That is, we conclude that

$$(\tilde{\nabla}\phi^H)^H = 0, \quad (\tilde{\nabla}\psi^H)^H = 0, \quad (\tilde{\nabla}\theta^H)^H = 0.$$

From Lemma 3.7 in [**3**], ϕ, ψ and θ are seen to be all Sasakian structures. Thus we get

THEOREM. *Let $(\tilde{M}, \tilde{g})$ be a fibred Riemannian manifold with triple of Killing vectors $\{\xi, \eta, \zeta\}$. In the base manifold M, a quaternion Kaehler structure can be induced from $\{\xi, \eta, \zeta\}$, when and only when $\{\xi, \eta, \zeta\}$ defines a Sasakian 3-structure.*

Bibliography

[1] S. Ishihara, Notes on Quaternion Kaehler manifolds, to appear in J. Diff. Geometry.

[2] S. Ishihara, Quaternion Kaehler manifolds and fibred Riemannian space with Sasakian 3-structure, to appear in Kōdai Math. Sem. Rep.

[3] S. Ishihara and M. Konishi, Fibred Riemannian space with triple of Killing vectors, to appear in Kōdai Math, Sem. Rep.

[4] S. Ishihara and M. Konishi, Fibred Riemannian space with Sasakian 3-structure. Differential Geometry, in honor of K. Yano, Kinokuniya, Tokyo, 1972, 179–194.