On Special Almost Kählerian Spaces

Dedicated to Professor S. Sasaki on his 60th birthday

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Introduction. A differentiable manifold of odd dimension $2n+1$ is called to be contact if it has a 1-form η such that the 2-form $d\eta$ has the maximal rank on the space. In relation with the almost contact metric structure, many results of contact spaces were known. When the dimension of the space M is $2n$, M is called an even dimensional contact space if it admits a 1-form η such that $d\eta$ has the maximal rank on M. Then there exists naturally an almost Kählerian structure (g, ϕ) such that $g\circ\phi=d\eta$. By the condition of linear isotropy groups of the holonomy groups, S. Sasaki [1] gave a sufficient condition that this almost Kählerian structure is integrable.

On the other hand, Y. Muto [4] studied the almost Kählerian space in which the fundamental 2-form is given by $d\eta$, η being a Killing 1-form. This structure is a special case of even dimensional contact spaces. In this paper, we call such a space to be e-K-contact in short. Then we show that there naturally exists an integrable distribution, and that each integral submanifold admits ordinary (odd dimensional) K-contact structure. Our main purpose is to give sufficient conditions for an e-K-contact space to be isometric with a Euclidean space.

1. Let M be an N-dimensional symplectic space with the fundamental 2-form φ and $N=2(n+1)$. Then we can take a positive definite Riemannian metric tensor g and an almost complex structure tensor ϕ which satisfy

$$g\circ\phi=\varphi, \tag{1.1}$$

since the rank of φ is maximal (Y. Hatakeyama [6]). We assume that M has a 1-form η which satisfies

$$\varphi=d\eta, \tag{1.2}$$

that is, M is an even dimensional contact space. The associated vector field of the 1-form η is denoted by $\eta^{\#}$, and we write σ the length of $\eta^{\#}$, which is a non-negative scalar function of M. Let M_0 be the set of zero points of $\eta^\sharp$, then it coincides with the set of zero points of σ. We define $\xi^\sharp=\phi\eta^\sharp$, and denote by ξ the associated 1-form of $\xi^\sharp$. Clearly the length of $\xi^\sharp$ is equal to σ on M. We set $M'=M-M_0$, then we can take unit vectors $e_\varDelta$ and $e_{\varDelta^*}$ on a neighbourhood in M' such that

$$e_\varDelta=\frac{1}{\sigma}\eta^\sharp, \qquad e_{\varDelta^*}=\frac{1}{\sigma}\xi^\sharp.$$

We take an orthonormal frame field $e_1, \dots, e_n$ and $e_{1^*}, \dots, e_{n^*}$ and $e_\varDelta$, $e_{\varDelta^*}$ where $e_{i^*}=\phi e_i$, and the frame $\{e_A\}$ is called an adapted one. Denote its dual basis by $\{\omega_A\}$. Then $\omega_\varDelta$ and $\omega_{\varDelta^*}$ satisfy

$$\omega_\varDelta=\frac{1}{\sigma}\eta \quad \text{and} \quad \omega_{\varDelta^*}=\frac{1}{\sigma}\xi.$$

We shall denote the Riemannian connection on M' by

$$(1.3) \qquad \begin{aligned} &dP=\omega_A e_A, \\ &de_A=\omega_{AB}e_B, \qquad \omega_{AB}+\omega_{BA}=0. \end{aligned}$$

Then the structure equations are

$$\begin{aligned} &d\omega_A=\omega_{AB}\wedge\omega_B, \\ &d\omega_{AB}=\omega_{AC}\wedge\omega_{CB}+\Omega_{AB}, \end{aligned}$$

where Ω_{AB} denotes the curvature form.

The almost complex structure ϕ is a vector valued 1-form, and can be written on M' as

$$(1.4) \qquad \phi=-\omega_{a^*}e_a+\omega_a e_{a^*}.$$

The fundamental 2-form ψ is also written as

$$(1.5) \qquad \psi=-\omega_a\wedge\omega_{a^*}.$$

Putting $\omega_{\varDelta A}=a_{AB}\omega_B$, we have

$$\begin{aligned} d\eta=d(\sigma\omega_\varDelta)&=d\sigma\wedge\omega_\varDelta+\sigma d\omega_\varDelta \\ &=\sigma a_{\alpha\beta}\omega_\beta\wedge\omega_\alpha+\sigma a_{\alpha\varDelta}\omega_\varDelta\wedge\omega_\alpha+\sigma(a_{\alpha\varDelta^*}-a_{\varDelta^*\alpha})\omega_{\varDelta^*}\wedge\omega_\alpha \\ &\quad-\sigma a_{\varDelta^*\varDelta}\omega_{\varDelta^*}\wedge\omega_\varDelta+d\sigma\wedge\omega_\varDelta. \end{aligned}$$

By virtue of (1.5), we get

The ranges of indices are as follows:

$A, B, C\ldots=1, \dots, n, 1^*, \dots, n^*, \varDelta, \varDelta^*,$

$\lambda, \mu, \nu\ldots=1, \dots, n, 1^*, \dots, n^*, \varDelta,$

$\alpha, \beta, \gamma\ldots=1, \dots, n, 1^*, \dots, n^*, \quad \alpha^*=\alpha+n \pmod{2n},$

$a, b, c\ldots=1, \dots, n, \varDelta,$

$i, j, k\ldots=1, \dots, n.$

We assume the summation convention for all indices.

$$a_{ij}-a_{ji}=0, \qquad a_{i^*j^*}-a_{j^*i^*}=0,$$
$$(1.6) \qquad a_{\alpha\Delta^*}-a_{\Delta^*\alpha}=0, \qquad \sigma a_{\alpha\Delta}-\sigma_\alpha=0,$$
$$\sigma(a_{ij^*}-a_{j^*i})=\delta_{ij}, \qquad -\sigma a_{\Delta^*\Delta}+\sigma_{\Delta^*}=1,$$

where we put $d\sigma=\sigma_A\omega_A$. Conversely, if there exist a scalar function σ and connection forms ω_A, ω_{AB} which satisfy the relation $\omega_{\Delta A}=a_{AB}\omega_B$ with the condition (1.6), then the 1-form $\eta=\sigma\omega_\Delta$ satisfies

$$d\eta=-\omega_a\wedge\omega_{a^*}.$$

Thus M' becomes an even dimensional contact space, where M' is an open submanifold of all points p such that $\sigma(p)\neq 0$.

In an even dimensional contact space M, we assume that $\eta^\#$ is a Killing vector field, and we call such a space to be Killing contact (e-K-contact, in short). The condition that $\eta^\#$ is Killing is that the Lie derivative of the metric tensor g with respect to $\eta^\#$ vanishes, and hence we have

$$\theta(\eta^\#)g(X, Y)=g(\theta(\eta^\#)X, Y)+g(X, \theta(\eta^\#)Y)$$

for vector fields X, Y, where $\theta(\eta^\#)$ denotes the Lie derivative. Using the equation $\theta(\eta^\#)X=\nabla_{\eta^\#}X-\nabla_X\eta^\#$, we get

$$g(\nabla_X\eta^\#, Y)+g(X, \nabla_Y\eta^\#)=0.$$

Then we have

$$2d\eta(X, Y)=X\cdot\eta(Y)-Y\cdot\eta(X)-\eta([X, Y])$$
$$=2g(X, \nabla_Y\eta^\#).$$

By assumption (1.2), we see that

$$(1.7) \qquad \phi=-d\eta^\#$$

is valid on M'. Moreover, from (1.3) we have

$$d(\sigma e_\Delta)=d\sigma e_\Delta+\sigma de_\Delta$$
$$=d\sigma e_\Delta+\sigma(\omega_{\Delta\alpha}e_\alpha+\omega_{\Delta\Delta^*}e_{\Delta^*}).$$

Comparing this equation with (1.4), we get

$$\omega_{\Delta i}=\frac{1}{\sigma}\omega_{i^*}, \qquad \omega_{\Delta i^*}=-\frac{1}{\sigma}\omega_i,$$
$$(1.8)$$
$$\omega_{\Delta\Delta^*}=-\frac{1}{\sigma}\omega_\Delta,$$

and

$$(1.9) \qquad \omega_{\Delta^*}=d\sigma.$$

Conversely if there exist a scalar function σ on M and the connection forms ω_A, ω_{AB} which satisfy the relations (1.8) and (1.9) with respect to an adapted basis $\{e_A\}$, then we have for the vector field $\eta^\#=\sigma e_\Delta$

$$d\eta^{\#}=\omega_{a^*}e_a-\omega_a e_{a^*}.$$

Thus the relation $\phi=-d\eta^{\#}$ holds good and $\eta^{\#}$ is Killing. This shows that M' is e-K-contact.

2. In this section we suppose that M is an N-dimensional complete e-K-contact space ($N=2n+2$). As for the notations we obey to the preceding section.

LEMMA 2.1. *In an e-K-contact space, the vector field* e_Δ *is parallel along the orbits of* e_{Δ^*}, *that is,* $\nabla_{e_{\Delta^*}}e_\Delta=0$.

PROOF. From (1.8), we have

$$\nabla_{e_{\Delta^*}}e_\Delta=\frac{1}{\sigma}\,\omega_{i^*}(e_{\Delta^*})e_i-\frac{1}{\sigma}\,\omega_i(e_{\Delta^*})e_{i^*}-\frac{1}{\sigma}\,\omega_\Delta(e_{\Delta^*})e_\Delta=0.$$

We consider a distribution defined by $\omega_{\Delta^*}=0$ on M'. By virtue of (1.9), this distribution is completely integrable, and there exists a maximal integral submanifold $B_\sigma(p)$ through any point p of M', where $\sigma=\sigma(p)$. Then $B_\sigma(p)$ is a $(2n+1)$-dimensional manifold, and the scalar function σ is constant on it. $B_\sigma(p)$ has the Riemannian metric and the connection which are induced from those of M'.

THEOREM 2.2. *Each of* $B_\sigma(p)$ *admits the induced K-contact structure, that is, there exists a contact form which is Killing and of constant length, and the sectional curvature for planes containing the associated vector is equal to* $1/\sigma^2$.

PROOF. We can take $\{e_\lambda\}$ and $\{\omega_\lambda\}$ as an orthonormal frame and its dual basis on a neighbourhood of p on $B_\sigma(p)$. Then the induced connection d' on $B_\sigma(p)$ is given by

$$d'P=\omega_\lambda e_\lambda,\qquad d'e_\lambda=\omega_{\lambda\mu}e_\mu \tag{2.1}$$

where $\omega_{\lambda\mu}$ is considered under the condition that σ is a constant value $\sigma(p)$. We have from (1.8)

$$d'e_\Delta=\frac{1}{\sigma}\,\omega_{i^*}e_i-\frac{1}{\sigma}\,\omega_i e_{i^*}.$$

Hence if we define the tensor field ϕ' of type (1.1) by

$$\phi'=-d'(\sigma e_\Delta)=-\omega_{i^*}e_i+\omega_i e_{i^*}, \tag{2.2}$$

then we obtain that

$$\phi' e_\Delta=0,\qquad \phi' e_i=e_{i^*},\qquad \phi' e_{i^*}=-e_i,\qquad \phi'^2(X)=-X+\omega_\Delta(X)e_\Delta. \tag{2.3}$$

Since the induced metric g' on $B_\sigma(p)$ is given by $d's^2=\sum_\lambda \omega_\lambda^2$, we have

$$g'(\phi'X, \phi'X)=g'(X, X)-\omega_\Delta(X)^2 \tag{2.4}$$

for the tangent vector X of $B_\sigma(p)$. Thus $(\phi', \sigma e_\Delta, g')$ defines the almost contact metric structure on $B_\sigma(p)$. Moreover the fundamental 2-form $\psi'=g'\circ\phi'=-\omega_i\wedge\omega_{i^*}$ is given by

$$\psi'=d'(\sigma\omega_\Delta).$$

By virtue of (2.2), σe_Δ is a Killing vector field on $B_\sigma(p)$, and therefore the almost contact metric structure is K-contact. Since the vector σe_Δ is of length σ on $B_\sigma(p)$, we see that the sectional curvature for a plane containing the vector e_Δ is equal to $1/\sigma^2$.

Now we express the forms $\omega_{\Delta^*\lambda}$ on M' as $\omega_{\Delta^*\lambda}=b_{\lambda A}\omega_A$. Differentiating (1.9), we have

$$\begin{aligned}0&=\omega_{\Delta^*\lambda}\wedge\omega_\lambda=b_{\lambda A}\omega_A\wedge\omega_\lambda\\ &=b_{\alpha\beta}\omega_\beta\wedge\omega_\alpha+b_{\alpha\Delta^*}\omega_{\Delta^*}\wedge\omega_\alpha+b_{\alpha\Delta}\omega_\Delta\wedge\omega_\alpha.\end{aligned}$$

It follows that

$$\begin{aligned}&b_{\alpha\beta}=b_{\beta\alpha},\\ &b_{\alpha\Delta}=b_{\alpha\Delta}=0.\end{aligned} \tag{2.5}$$

From $(1.8)_2$, we have $b_{\Delta\Delta}=1/\sigma$. The quantity $b_{\lambda\mu}$ corresponds to the second fundamental tensor of the hypersurface $B_\sigma(p)$ with respect to the normal vector e_Δ.

LEMMA 2.3. *On an e-K-contact space M, the vector field e_{Δ^*} is autoparallel, that is, the orbits of e_{Δ^*} are geodesics.*

PROOF. It is evident from (1.8) and (2.5).

Through any point p of M', there exist an integral submanifold $B_\sigma(p)$ and an orbit of e_{Δ^*} which intersect to each other orthogonally. We show the following

LEMMA 2.4. *A connected orbit of the vector field e_{Δ^*} through p of M' can intersect only once with the hypersurface $B_\sigma(p)$.*

PROOF. As the space is complete, the orbit $c(t)$ of e_{Δ^*} can be extended for infinitely large value of its arc length t. The length σ of the vector field $\eta^\#$ can be considered as a function of variable t along the orbit. Since $\omega_{\Delta^*}=d\sigma$, it holds that $\sigma'(t)\neq 0$ on $c(t)$, and hence the function σ is a monotonous function of t. From the fact that σ is constant on $B_\sigma(p)$, we conclude that the orbit $c(t)$ can not intersect with the hypersurface $B_\sigma(p)$ again.

THEOREM 2.5. *On a complete e-K-contact space, there exists at least one zero point of* $\eta^{\#}$.

PROOF. We take an orbit $c(t)$ through a point p $(=c(0))$ with arc length parameter t. Then along $c(t)$, we have

$$\omega_{\Delta^*}=d\sigma=\frac{d\sigma}{dt}dt,$$

and since the tangent vector of $c(t)$ is e_{Δ^*}, we get

$$\frac{d\sigma}{dt}=1$$

on $c(t)$. Therefore

$$\sigma(t)=t+\sigma(0) \tag{2.6}$$

is valid for any t. (2.6) shows that $\sigma(t)$ is an unbounded function of t going towards one direction. Putting $t_0=-\sigma(0)$, we see

$$\lim_{t\to t_0}\sigma(t)=0$$

on M'. The function $\sigma(t)$ being continuous along $c(t)$ on M, we obtain

$$\sigma(t_0)=0.$$

Thus the point $c(t_0)$ is a zero point of the vector field $\eta^{\#}$.

3. The line element of M' can be written as

$$\begin{aligned} ds^2&=\sum_A(\omega_A)^2\\ &=(\omega_{\Delta^*})^2+\sum_\lambda(\omega_\lambda)^2. \end{aligned}$$

Since $\omega_{\Delta^*}=d\sigma$, we can take the scalar function σ as the Δ^*-th coordinate function of M', and we have

$$ds^2=(d\sigma)^2+\sum_\lambda(\omega_\lambda)^2.$$

If we put $\omega_{\lambda\mu}=\Gamma_{\lambda\mu A}\omega_A$, then using the structure equations, we have

$$\begin{aligned} d\left(\frac{1}{\sigma}\omega_\lambda\right)&=-\frac{1}{\sigma^2}d\sigma\wedge\omega_\lambda+\frac{1}{\sigma}\omega_{\lambda\Delta^*}\wedge\omega_{\Delta^*}+\frac{1}{\sigma}\omega_{\lambda\mu}\wedge\omega_\mu\\ &=\frac{1}{\sigma}\left(\frac{1}{\sigma}\omega_\lambda-\omega_{\Delta^*\lambda}+\Gamma_{\lambda\mu\Delta^*}\omega_\mu\right)\wedge d\sigma+\frac{1}{\sigma}\Gamma_{\lambda\mu\nu}\omega_\nu\wedge\omega_\mu. \end{aligned}$$

Hence we can see that the 1-form $\dfrac{1}{\sigma}\omega_\lambda$ is independent from σ if the following two conditions are satisfied ;

$$\Gamma_{\lambda\mu\Delta^*}=0, \tag{3.1}$$

(3.2) $$\omega_{\Delta^*\lambda}-\frac{1}{\sigma}\omega_\lambda=0.$$

Then the line element on M' becomes

(3.3) $$ds^2=(d\sigma)^2+\sigma^2\sum_\lambda(\omega'_\lambda)^2$$

where $\{\omega'_\lambda\}$ is a basis of 1-forms on M' and independent from the function σ. We shall study the conditions which are equivalent to (3.1) and (3.2).

Let us suppose that

(3.4) $$\nabla_{e_{\Delta^*}}\phi=0$$

is valid. Then we can select an orthonormal frame $\{e_A\}$ which is parallel along the orbits of e_{Δ^*} taking account of Lemma 2.1 and 2.3. This fact is equivalent to the condition

$$\omega_{AB}(e_{\Delta^*})=0$$

and especially $\omega_{\lambda\mu}(e_{\Delta^*})=0$, which is the same condition as (3.1). Hence if (3.4) is valid, (3.1) holds good.

Now we define 1-forms L_{AB} on M' by

$$\nabla_X\phi\cdot e_A=L_{AB}(X)e_B$$

for any vector X. Then we have

$$\begin{aligned}\nabla_X\phi\cdot e_A&=\nabla_X e_{A^*}-\phi(\nabla_X e_A)\\&=(\omega_{A^*B}(X)+\omega_{AB^*}(X))e_B\end{aligned}$$

and hence

(3.5) $$L_{AB}=\omega_{A\ B}+\omega_{AB}\,.$$

It is easily seen that L_{AB} satisfy

(3.6) $$\begin{aligned}&L_{AB}=-L_{A^*B^*}, \qquad L_{A^*B}=L_{AB^*},\\&L_{AB}=-L_{BA}.\end{aligned}$$

Since (g,ϕ) is an almost Kählerian structure, we have $d\phi=0$, and hence

$$g(e_A,\nabla_{e_B}\phi\cdot e_C)+g(e_B,\nabla_{e_C}\phi\cdot e_A)+g(e_C,\nabla_{e_A}\phi\cdot e_B)=0.$$

Then we see that

$$g(e_C,d\phi(e_A,e_B))+g(e_B,L_{AD}(e_C)e_D)=0,$$

and consequently

$$\omega_C(d\phi(e_A,e_B))+L_{AB}(e_C)=0$$

holds good. Thus we obtain

(3.7) $$d\phi(e_A,e_B)=L_{BA}(e_C)e_C.$$

LEMMA 3.1. *On an e-K-contact space,* (3.4) *is equivalent to*

$$\Omega_{\varDelta\varDelta^*}=0.$$

PROOF. We assume that (3.4) is valid. Then we have

$$\nabla_{e_{\varDelta^*}}\phi\cdot e_B=L_{BC}(e_{\varDelta^*})e_C=0,$$

and therefore

$$L_{BC}(e_{\varDelta^*})=0$$

hold for any B and C. Hence we obtain by virtue of (3.7)

$$\omega_{\varDelta^*}(L_{BC}(e_D)e_D)=-\omega_{\varDelta^*}(d\phi(e_B,\ e_C))=0. \tag{3.8}$$

On the other hand, the equation (1.7) shows that

$$\phi=d\sigma e_{\varDelta}+\sigma de_{\varDelta}$$

and that

$$d\phi=\sigma d^2e_{\varDelta}=\sigma\Omega_{\varDelta D}e_D.$$

Therefore (3.8) becomes

$$\omega_{\varDelta^*}(\sigma\Omega_{\varDelta D}(e_B,\ e_C)e_D)=\sigma\Omega_{\varDelta\varDelta^*}(e_B,\ e_C)=0.$$

The converse statement can be given only pursueing reversely the above process.

LEMMA 3.2. *On an e-K-contact space,* (3.2) *implies* (3.1).

PROOF. By virtue of (1.8), we have

$$\begin{aligned}\Omega_{\varDelta\varDelta^*}&=d\omega_{\varDelta\varDelta^*}-\omega_{\varDelta\alpha}\wedge\omega_{\alpha\varDelta^*}\\&=\frac{1}{\sigma^2}d\sigma\wedge\omega_{\varDelta}-\frac{1}{\sigma}d\omega_{\varDelta}-\frac{1}{\sigma}\omega_{\alpha^*}\wedge\omega_{\alpha\varDelta^*}\\&=\frac{1}{\sigma}\left(\frac{1}{\sigma}\omega_{\alpha}-\omega_{\varDelta^*\alpha}\right)\wedge\omega_{\alpha}\ .\end{aligned}$$

Thus if (3.2) holds, then $\Omega_{\varDelta\varDelta^*}=0$ and from Lemma 3.1, (3.4) is proved.

It is evident that (3.2) is the condition that the integral manifolds $B(p)$ are all totally umbilical. Because of (1.8), we can easily see that the equation

$$\omega_{\lambda^*\varDelta}+\omega_{\lambda\varDelta^*}=0 \tag{3.9}$$

means (3.2). By virtue of (3.5) and (3.7), it follows that (3.9) is equivalent to

$$d\phi(e_{\lambda},\ e_{\varDelta^*})=0. \tag{3.10}$$

As is shown in the proof of Lemma 3.1, $d\phi=\sigma\Omega_{\varDelta A}e_A$ holds good and therefore (3.9) implies

$$\Omega_{\varDelta A}(e_{\lambda},\ e_{\varDelta^*})=0.$$

Using the fact that

$$\Omega_{\Delta\Delta^*}(e_\lambda, e_{\Delta^*}) = -\frac{1}{\sigma}\left(\frac{1}{\sigma}\omega_\alpha(e_{\Delta^*}) - \omega_{\Delta^*\alpha}(e_{\Delta^*})\right)\delta_{\lambda\alpha^*} = 0,$$

we have

$$\Omega_{\Delta\mu}(e_{\Delta^*}, e_\lambda) = 0 \tag{3.11}$$

for any λ, μ. Thus we have proved the following

PROPOSITION 3.3. *On an e-K-contact space, if*

$$\Omega_{\Delta\lambda}(e_{\Delta^*}, e_\mu) = 0$$

is valid, then all hypersurfaces $B_\sigma(p)$ are totally umbilical and the line element of M' has the form of (3.3). [*see* [4], *Corollary of Theorem 5.1.*]

If the structure is Kählerian, then it is evident that (3.9) is satisfied and therefore we have

COROLLARY 3.4. *If an e-K-contact space is Kählerian, then all hypersurfaces $B_\sigma(p)$ are totally umbilical and the line element has the form of* (3.3).

THEOREM 3.5. *All hypersurfaces $B_\sigma(p)$ in an e-K-contact space are totally umbilic if $\Omega_{\Delta\Delta^*} = 0$ and the sectional curvature in M' for the 2-plane containing the vector $\eta^\#$ is zero.*

PROOF. In the proof of Lemma 3.2 we get

$$\Omega_{\Delta\Delta^*} = \frac{1}{\sigma}\left(\frac{1}{\sigma}\omega_\alpha - \omega_{\Delta^*\alpha}\right) \wedge \omega_{\alpha^*}.$$

If $\Omega_{\Delta\Delta^*} = 0$, then we have

$$\begin{aligned} &b_{ij^*} = b_{i^*j} = 0, \\ &b_{ij} + b_{i^*j^*} = \frac{2}{\sigma}\delta_{ij}. \end{aligned} \tag{3.12}$$

Taking consideration of the symmetric $2n$-matrix $B = (b_{ij})$, let X be an eigenvector of B. Then (3.12) shows that ϕX is also an eigenvector of B. Hence we can take an adapted basis $\{e_i, e_{i^*}, e_\Delta, e_{\Delta^*}\}$ such that the matrix B is diagonal with respect to $\{e_\alpha\}$. Next we calculate the curvature forms $\Omega_{\Delta\alpha}$ on M' and $\Omega'_{\Delta\alpha}$ on $B_\sigma(p)$. Since the differentiation d on M' can be written as

$$d = d' + d\sigma\frac{\partial}{\partial\sigma},$$

we have

$$\begin{aligned} d\omega_\alpha &= d'\omega_\alpha + d\sigma \wedge \frac{\partial\omega_\alpha}{\partial\sigma} \\ &= \omega_{\alpha\lambda} \wedge \omega_\lambda + \omega_{\alpha\Delta^*} \wedge d\sigma, \end{aligned}$$

and hence wet get

$$\frac{\partial \omega_\alpha}{\partial \sigma} = \omega_{\Delta^\cdot \alpha}.$$

Making use of the structure equations on M', we have

$$\begin{aligned}\Omega_{\Delta\alpha} &= d\omega_{\Delta\alpha} - \omega_{\Delta\rho}\wedge\omega_{\rho\alpha} - \omega_{\Delta\Delta^*}\wedge\omega_{\Delta\alpha}\\ &= (d'\omega_{\Delta\alpha} - \omega_{\Delta\rho}\wedge\omega_{\rho\alpha}) + d\sigma\wedge\frac{\partial}{\partial\sigma}\left(\frac{1}{\sigma}\omega_{\alpha^*}\right) + \frac{1}{\sigma}\omega_\Delta\wedge\omega_{\Delta^*\alpha}\\ &= \Omega'_{\Delta\alpha} - \frac{1}{\sigma}b_{\alpha\beta}\omega_\beta\wedge\omega_\Delta + \frac{1}{\sigma}\left(\frac{1}{\sigma}\omega_{\alpha^*} - \omega_{\Delta^\cdot\alpha}\right)\wedge\omega_{\Delta^\cdot}.\end{aligned}$$

On the other hand, by virtue of Theorem 2.2 the sectional curvature in $B_\sigma(p)$ satisfies $-\Omega'_{\Delta\alpha}(e_\Delta, e_\alpha) = 1/\sigma^2$. Therefore the sectional curvature of the plane spanned by the vectors e_Δ, e_α in M' is given by

$$K(e_\Delta, e_\alpha) = \frac{1}{\sigma}\left(\frac{1}{\sigma} - b_{\alpha\alpha}\right).$$

From our assumption, we have $b_{\alpha\alpha} = 1/\sigma$, and hence $\omega_{\Delta^\cdot\alpha} = (1/\sigma)\omega_\alpha$ holds good. Taking account of $(1.8)_3$, we have (3.2). This shows the theorem.

4. In this section we suppose that the complete e-K-contact space M satisfies (3.2), for example, that all hypersurfaces $B_\sigma(p)$ are totally umbilical. Then the line element of M' has the form of (3.3).

LEMMA 4.1. *The set M_0 of vanishing points of $\eta^\#$ is a 0-dimensional submanifold of M.*

PROOF. By virtue of Theorem 2.5, M_0 is non-empty. Assuming that M_0 is not zero dimensional, we take a point O of M_0 and a non-zero vector X which is tangential to M_0 at O. The metric g and ds^2 on M are given by (3.3) at each point of M' and by continuity of (3.3) at each point of M_0, since the dimension of M_0 is less than that of M. As ω'_λ is independent of σ, $\sigma^2 \sum_\lambda \omega'^2_\lambda$ comes to zero when σ draws near zero, hence we have

$$g(X, X) = d\sigma(X)^2 = (X\cdot\sigma)^2 = 0.$$

Thus we get $X = 0$, which is a contradiction.

LEMMA 4.2. *M_0 consists of only one point O.*

PROOF. We consider a geodesic $c(t)$ through O ($c(0) = O$) with the unit tangent vector $X(t)$ and the arc length parameter t. Since M_0 is zero dimensional, we can assume that there exist no points of M_0 on the geodesic arc $c(t)$ $(0 < t < k)$. If we define the function f on $c(t)$ by

$$f(t)=g(\xi^{*}(c(t)),\ X(t)),$$

then we have $f(0)=O$. Differentiating $f(t)$ to the direction $X(t)$ it follows that

$$\begin{aligned}\nabla_X f &= g(\nabla_X \xi^{*},\ X)\\ &= g((X\cdot\sigma)e_{\Delta^{*}},\ X)+\sigma g(b_{\lambda\mu}\omega_{\mu}(X)e_{\lambda},\ X).\end{aligned}$$

Since $B_{\sigma}(p)$ are totally umbilical, we have $b_{\lambda\mu}=(1/\sigma)\delta_{\lambda\mu}$ and therefore

$$\begin{aligned}\nabla_X f &= \omega_{\Delta^{*}}(X)^2+\sum_{\lambda}\omega_{\lambda}(X)^2\\ &= g(X,\ X)=1\end{aligned}$$

holds good. Hence the function $f(t)$ is strictly monotonous increasing on $c(t)$. We assume that there exists a point Q of M_0 which differs from O. As the space is complete, we can take a geodesic $c(t)$ combining the points O and Q ($c(0)=O$, $c(k)=Q$). Applying the above arguement to $c(t)$, we reach to a contradiction because of $f(0)=f(k)=O$.

S. Sasaki has proved the following

THEOREM. *If a complete Riemannian space V admits the line element of the type* (3.3), *then V is locally flat.*

By virtue of this theorem, our e-K-contact space M with the metric (3.3) on $M'=M-\{O\}$ is seen that it is locally flat. However, we write for completeness the proof showing that M is isometric with the Euclidean space.

LEMMA 4.3. *Any point Q on the hypersurface $B_{\sigma}(p)$ has the constant distance $\sigma(p)$ from the point O, and the minimal geodesic joining the points O and Q is given by the orbit of $e_{\Delta^{*}}$.*

PROOF. As in the proof of Lemma 4.1, we define the function $f(t)$ on the minimal geodesic $c(t)$ joining O and Q ($c(0)=O$, $c(k)=Q$). Then we have

$$\frac{d}{dt}f(t)=1.$$

Taking account of the initial condition $f(0)=O$, we get

$$f(t)=t,$$

and this leads to the differential equation

$$\sigma d\sigma = t dt$$

on $c(t)$. Consequently we obtain under the initial condition $\sigma(0)=0$,

$$\sigma(t)=t$$

along $c(t)$. Therefore the length $k=k(Q)$ of the geodesic joining O and

Q is $\sigma(k)=\sigma(p)$ on $B_\sigma(p)$, thus it is constant on $B_\sigma(p)$. Conversely, if a point on the geodesic through O for any direction X has the length k from O, we see that it belongs to $B_\sigma(p)$, $\sigma=k$. By virtue of the proof of Theorem 2.5, we see that any orbit of e_{Δ^*} reaches at O, so it is concluded that the minimal geodesics through O are orbits of e_Δ, and that they intersect with $B_\sigma(p)$ orthogonally.

THEOREM 4.4. *If the complete e-K-contact space M with $\Omega_{\Delta\Delta^*}=0$ satisfies the condition that the sectional curvature of 2-plane containing $\eta^\sharp$ is zero, then M is isometric with the Euclidean space E^N of flat metric, where N is the dimension of M.*

PROOF. We define first the coordinate system of M as follows. Taking a fixed orthonormal basis $\{f_A\}$ of the tangent space at O, we draw a geodesic through O to the direction $a_A f_A$ $(\sum (a_A)^2=1)$. Then we correspond (a_A, σ) to the point on this geodesic which has the distance σ from the origin O. We define that the point O corresponds to $\sigma=0$. Since any geodesic starting from O is an orbit of e_{Δ^*}, and it intersects with each $B_\sigma(p)$ only once, it follows that the pair (a_A, σ) where $\sum (a_A)^2=1$ and $\sigma\geqq 0$ can be taken as the coordinate system of M. Thus M is globally diffeomorphic to the Euclidean space E^N, and hence M is simply connected. Since M is locally flat, we see that M is globally isometric with the flat Euclidean space E^N.

COROLLARY 4.5. *If a complete e-K-contact space M is Kählerian, then M is isometric with the flat Euclidean space E^N.*

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