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On Balayaged Measures and Simplexes in Harmonic Spaces

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§1. **Introductoin.** Let Ω be a strict harmonic space in the sense of Bauer [1], and U be an open set in Ω, we shall consider the Dirichlet's problem on U. Using that the set P of all continuous potentials is adapted, we can define, simularly to the case of relatively compact U, the balayaged measure $(\varepsilon_x)^{U^c}$ and develop the argument on resolutivity and regular points. We shall denote by C the set of all continuous functions on $\bar{U}$ which are superharmonic in U. By regarding a balayage as a dilation, we shall prove that $(\bar{U}, C)$ is a simplex if Ω satisfies Axiom D in Brelot [2], and that for any $x \in \bar{U}$ the extremal measure μ_x with $\varepsilon_x \ll \mu_x$ coincides with the balayaged measure $(\varepsilon_x)^{U^c}$.

Further we shall prove that for any two $x, y \in U$, the balayaged measures $(\varepsilon_x)^{U^c}$ and $(\varepsilon_y)^{U^c}$ are mutually absolutely continuous when U is a connected open set in an elliptic harmonic space Ω.

§2. **Preliminaries.** Let Ω be a locally compact, σ-compact Hausdorff space and f, g be non-negative functions on Ω. We denote $f \uparrow \prec g$ if for any $\varepsilon > 0$ there exists a compact set K such that

$$x \in K^{c(*)} \Rightarrow f(x) \leqq \varepsilon g(x).$$

PROPOSITION 1. *Let μ be a positive Borel measure, f a Borel measurable function and $\{f_\alpha\}$ a decreasing net of upper semicontinuous μ-integrable functions converging to f. Suppose that there exist an index β and a non-negative μ-integrable function g satisfying $(f_\beta - f) \uparrow \prec g$. Then we have*

$$\int f d\mu = \inf \int f_\alpha d\mu.$$

PROOF. For any $\varepsilon > 0$ there exists a compact set K such that

$$x \in K^c \Rightarrow f_\beta(x) - f(x) \leqq g(x).$$

(*) We denote by K^c the complement of K in Ω.

Since on compact set K $\inf\int_K (f_\alpha - f)d\mu = 0$ holds, we have for sufficiently large index α

$$\int (f_\alpha - f)d\mu = \int_K (f_\alpha - f)d\mu + \int_{K^c} (f_\alpha - f)d\mu < \varepsilon + \varepsilon \int g d\mu.$$

Hence $\inf\int (f_\alpha - f)d\mu = 0$.

We shall call a convex cone P of $C^+(\Omega)$ adapted, if P satisfies the following two conditions (i) and (ii);

(i) for any $x \in \Omega$, there exists $u \in P$ such that $u(x) > 0$;
(ii) for any $u \in P$, there exists $v \in P$ satisfying $u \uparrow \prec v$.

Let us put for $u \in P$,

$$H_u = \{f \in C(\Omega)\ ;\ \exists\lambda > 0,\ |f| \leqq \lambda u\}.$$

Then H_u is a Banach space with norm

$$\|f\|_u = \inf\{\lambda\ ;\ |f| \leqq \lambda u\}.$$

We shall assign to the vector space $H_p = \bigcup_{g \in p} H_g$ the topology of inductive limits of Banach spaces $\{H_g\}_{g \in p}$.

PROPOSITION 2. *Any positive linear form L defined on dense subspace N of H_p is uniquely extended to a positive linear form on H_p.*

PROOF. Since N is dense in H_p, for any $u \in P$ we can find $u_1 \in N$ with $\|u - u_1\| < 1/2$, and holds $u_1 > u - \frac{1}{2}u \geqq \frac{1}{2}u \geqq 0$. For any $g \in H_u \cup N$, we get $-\|g\|_u u \leqq g \leqq \|g\|_u u$, whence $-2\|g\|_u u_1 \leqq g \leqq 2\|g\| u_1$. Since L is positive on N, we have

$$|L(g)| \leqq 2\|g\|_u L(u_1)$$

for any $g \in H_u \cap N$. Therefore, L can be uniquely extended to a positive linear form on H_u for any $u \in P$ and it can be uniquely to a positive linear form on H_p.

§ 3. Choquet boundaries and simplexes.

Let P be an adapted cone of $C^+(\Omega)$. We denote by $\mathfrak{M}_p^+$ the set of all P-integrable positive measures on Ω. Let C be an min-stable, linearly separating convex cone with $P \subset C \subset H_p$. A cone C is called linearly separating if for every two different x, y of Ω and any real $\lambda \geqq 0$ there exists $u \in C$ such that $u(x) \neq \lambda u(y)$.

For any two measures $\mu, \nu \in \mathfrak{M}_p^+$ we denote by $\mu \ll \nu$ if $\nu(v) \leqq \mu(v)$ for any $v \in C$. Then we see that $\ll$ is an order relation in $\mathfrak{M}_p^+$. A maximal measure $\mu \in \mathfrak{M}_p^+$, according to the order $\ll$, is called C-extremal (or simply extremal). We call the Choquet boundary with respect to C, denoted by $\delta(C)$, the set of all points x of Ω such that, if $\mathfrak{M}_p^+$, $\varepsilon_x \ll \mu$ then $\mu = \varepsilon_x$. If Ω has a countable base, then $\delta(C)$ is a G_δ-set. Further we shall denote by $\Omega^-(C) = \Omega^-$ the set of all points x such that there exists $u \in C$ with $u(x) < 0$. Then under the condition $\Omega^- = \Omega$, a measure $\mu \in \mathfrak{M}_p^+$ is extremal if and only if it is carried by the set $\delta(C)$.

We shall call (Ω, C) a C-simplex (or simply simplex), if for any $x \in \Omega$, an extremal measure $\mu \in \mathfrak{M}_p^+$ with $\varepsilon_x \ll \mu$ exists and is unique. Let us denote the extremal measure with $\varepsilon_x \ll \mu$ by μ_x and write $Q_x(f) = \inf_{\substack{g \geqq f \\ h \in C}} g(x)$ provided that (Ω, C) be a simplex. Then we have $\mu_x(f) = Q_x(f)$ for any $f \in -C$. [7]

PROPOSITION 3. *If (Ω, C) is a simplex, then a function $x \to \mu_x(f)$ is Borel measurable for any $f \in H_p$.*

PROOF. Since $\mu_x(f) = Q_x(f)$ for any $f \in -C$ and $x \to Q_x(f)$ is upper semicontinuous, the function $x \to \mu_x(f)$ is Borel measurable, whence for any $f \in C - C$, $x \to \mu_x(f)$ is Borel measurable. Further for any $f \in H_p$, let us choose v_0 with $f \in H_{v_0}$. Then we can find $f_n \in (C-C) \cap H_{v_0}$ with $\|f - f\|_{v_0} < 1/n$, equivalently $|f(x) - f_n(x)| < (1/n) v_0(x)$ for any $x \in \Omega$, since $C - C$ is dense in H_p. Hence

$$|\mu_x(f) - \mu_x(f_n)| \leqq (1/n) \mu_x(v_0).$$

Since the function $x \to \mu_x(f_n)$ is Borel measurable, $x \to \mu_x(f)$ is also Borel measurable.

§4. **Dilations.** Throughout this section, Ω will be a locally compact Hausdorff space with a countable base. Let P be an adapted cone of $C^+(\Omega)$ and C be a min-stable, linearly separating cone with $P \subset C \subset H_p$. We shall say that an extended real-valued function f is upper P-bounded if there exists $v \in P$ satisfying $f \leqq v$. A function f on Ω is called C-concave (or simply concave) if for any $x \in \Omega$ and any measure $\mu \in \mathfrak{M}_p^+$ with $\varepsilon_x \ll \mu$, we have $\mu(f) \leqq f(x)$. If f and $-f$ concave, f is called affine. We denote by $\mathfrak{A}$ the set of all upper P-bounded upper semicontinuous affine functions on Ω.

A mapping D from Ω into $\mathfrak{M}_p^+$ is called a dilation if for any $x \in \Omega$, $\varepsilon_x \ll D(x)$ and for any $f \in H_p$, the function $x \to (Df)(x) = D(x)(f)$ is Borel measurable. We say that $x \in \Omega$ is D-regular if $D(x) = \varepsilon_x$, the set of which will be denoted by $\partial_r^D(\Omega)$. Further, a dilation D is called affine if for any $v \in -C$, Dv is the limit of a decreasing net of functions in $\mathfrak{A}$. A dilation D is called weakly affine if there exists a min-stable cone $C_1 \subset C$ with the following properties ;

(i) C_1 separates linearly points in Ω,
(ii) for any $v \in -C_1$, Dv is the limit of a decreasing net of functions in $\mathfrak{A}$.

THEOREM 1. *If D is a weakly affine dilation for C and $\Omega = \Omega^-(C)$, then (Ω, C) is a simplex and for any $x \in \Omega$ Dx is the extremal measure μ_x with $\varepsilon_x \ll \mu$. In particular, $\delta(C) = \partial_?^P(C)$.*

PROOF. Let μ, ν be extremal measures of $\mathfrak{M}_p^+$ with $\varepsilon_x \ll \mu$, $\varepsilon_x \ll \nu$. Since D is a weakly affine dilation, there exists a min-stable linearly separating cone $C_1 \subset C$ such that for any $v \in -C_1$, $D(v)$ is the limit of a decreasing net of $\mathfrak{A}$. Therefore, by proposition 1 we get $\mu(D(v)) = \nu(D(v))$. Since μ, ν are carried by $\delta(C)$ and $D(v) = v$ on $\delta(C)$, we have $\mu(v) = \nu(v)$, whence $\mu = \nu$. Thus, (Ω, C) is a simplex. To prove $\mu_x = D(x)$, let $v \in -C_1$. Then we have

$$\mu_x(v) = Q_x(v) = \sup\{\mu(v)\ ;\ \mu \in \mathfrak{M}_p^+,\ \varepsilon_x \ll \mu\} \geqq (D(v))(x). \quad \text{.........(1)}$$

Further, suppose that $(D(v))(x) \leqq a_1(x)$ for any $x \in \Omega$. Then we get $v(x) \leqq a_1(x)$ since $v(x) \leqq (D(x))(v)$. Hence $Q_x(v) \leqq Q_x(a_1) = a_1(x)$. It follows

$$\mu_x(v) \leqq (D(v))(x). \quad \text{................................(2)}$$

By (1) and (2) we have

$$\mu_x(v) = (D(v))(x) = D(x)(v)$$

for any $v \in -C_1$, whence $\mu_x = D(x)$.

THEOREM 2. *If (Ω, C) is a simplex, there exists an affine dilation for C.*

PROOF. For any $x \in \Omega$, we put μ_x the unique extremal measure with $\varepsilon_x \ll \mu_x$. When we define $D(x) = \mu_x$, D is a dilation, since, by proposition 3, for any $f \in H_p$ the mapping $x \to \mu_x(f)$ is Borel measurable.

Further, for any $v \in -C$, we get

$$\mu_x(v) = Q_x(v) = \inf\{h(x)\ ;\ h \in \mathfrak{A},\ h \geqq v\}. \quad \text{....................(1)}$$

Suppose that $h_1, h_2 \in \mathfrak{A}$ satisfy $h_1 \geqq v$ and $h_2 \geqq v$. Then the function $\min(h_1, h_2)$ is concave and satisfies $v \leqq \min(h_1, h_2)$. Therefore, there exists a $h \in \mathfrak{A}$ with $v \leqq h \leqq \min(h_1, h_2)$ by [7]. It follows that the family in the third equality of (1) is a decreasing net. Hence D is an affine dilation.

§5. **Balayaged measures.** Let Ω be a harmonic space which satisfies Bauer's axioms (I), (II), (III), (IV) in [1], and P be the set of all continuous potentials in Ω. According to Bauer, we call Ω a strong harmonic space, if Ω satisfies the following conditions;

(*) for any $x\in\Omega$, there exists a $f\in P$ with $f(x)>0$.

Hereafter we assume that Ω is a strong harmonic space.

THEOREM 3. *For any $\mu\in\mathfrak{M}_p^+$ and any $E\subset\Omega$, there exists uniquely a measure μ^E such that*

$$\int v d\mu^E=\int \hat{R}_v^E d\mu$$

for any $v\in P$.

PROOF. Since P is a min-stable, linearly separating adapted cone of $C^+(\Omega)$, $P-P$ is dense in H_p. [6] We put $N=P-P$. Assume that an element d of N has two representations; $d=u-v=u'-v'$ where u, v, u', $v'\in P$. Then we have $\hat{R}_u^E-\hat{R}_{u'}^E=\hat{R}_v^E-\hat{R}_{v'}^E$. Further $\hat{R}_u^E$ is P-integrable since it is dominated by u. Hence we get

$$\int(\hat{R}_u^E-\hat{R}_v^E)d\mu=\int(\hat{R}_{u'}^E-\hat{R}_{v'}^E)d\mu.$$

Thus, we can define for any $d=u-v$, $L(d)=\int(\hat{R}_u^E-\hat{R}_v^E)d\mu$. Since L, which is positive and linear, is well-defined on N. L is uniquely extended to a positive linear form on H_p. We know that P is adapted, and there exists a measure $\mu^E\in\mathfrak{M}_p^+$ such that $\mu^E(f)=L(f)$ for any $f\in H_p$. [5] Particularly, for any $d=u-v$ we have

$$\int(u-v)d\mu^E=\int(\hat{R}_u^E-\hat{R}_v^E)d\mu.$$

The measure μ^E is called the balayaged measure with respect to μ and E.

LEMMA 1. *Let U be an open set in Ω and $v\geqq 0$ be superharmonic in Ω. Then we have $\hat{R}_v^{U^c}(x)=\hat{R}_v^{\partial U}(x)$(*) for any $x\in\Omega$.*

PROOF. Let $w\geqq 0$ be hyperharmonic in Ω with $w\geqq v$ on ∂U. Then for any $z\in\partial U$ we have $\liminf_{x\to z, x\in U} w(x)\geqq w(z)\geqq v(z)$. Put

$$w_1(x)=\begin{cases}\inf(w(x), v(x)) & \text{for } x\in U\\ v(x) & \text{for } x\in U^c.\end{cases}$$

Then w_1 is non-negative and hyperharmonic in Ω, and $w_1=v$ in U^c. Hence we have $\hat{R}_v^{U^c}(x)\leqq w_1(x)\leqq w(x)$ for any $x\in U$. It follows $\hat{R}_v^{U^c}(x)\leqq\hat{R}_v^{\partial U}(x)$ in U. On the other hand, it is clear that $\hat{R}_v^{\partial U}(x)\leqq\hat{R}_v^{U^c}(x)$ in U.

Let U be an open set in Ω. For any $x\in U$ we shall call $(\varepsilon_x)^{U^c}$ the

(*) We denote by ∂U the topological boundary of U.

harmonic measure with respect to x and U, and denote by μ_x^U.

By lemma 1, we have the following proposition ;

PROPOSITION 4. *For any $x \in U$, μ_x^U is supported by ∂U.*

By Pradelle (6), we consider the Dirichle's problem for open sets U. If f is an extended real-valued function on ∂U, we put

$$\bar{H}_f^v = \inf\{v\ ;\ v \in \bar{\mathfrak{H}}_f^U\}$$

where $\bar{\mathfrak{H}}_f^U$ is a family of all hyperharmonic functions v in U satisfying the following conditions ;

i) $\liminf_{U \ni x \to z} v(x) \geqq f(z)$ and $> -\infty$ for any $z \in \partial U$,

ii) $v \geqq -p_v$ for an $p_v \in P$.

Similarly we define $\underline{H}_f^U = -H_{(f)}^U$. A function f is called resolutive if $\underline{H}_f^v = \bar{H}_f^v$.

By the similar method of Bauer [1], we have the following two propositions ;

PROPOSITION 5. *Any $f \in H_p(\partial U)$ is resolutive and for any $x \in U$ we have*

$$\bar{H}_f(x) = \int f d\mu^U.$$

PROPOSITION 6. *Suppose that for any $x \in U$, f is μ_x^U-integrable. Then a function $x \to \int f d\mu_x^U$ is harmonic in U.*

A point $x_0 \in \partial U$ is called regular if $\lim_{U \in x \to x_0} \bar{H}_\varphi(x) = \varphi(x_0)$ for any $\varphi \in H_p(\partial U)$, or

$$\lim_{x \to x_0} \mu_x^U = \varepsilon_{x_0} \text{ under the topology } \sigma(\mathfrak{M}_p(\partial U),\ H_p(\partial U)).$$

Then we have the following proposition ;

PROPOSITION 7. *For any $z \in \partial U$, the following assertions are equivalent ;*

(*a*) *z is regular,*

(*b*) *U^c is not thin at z,*

(*c*) $(\varepsilon_z)^{U^c} = \varepsilon_z$.

Using this proposition, we have the following lemma ;

LEMMA 2. *If $z \in \partial U$, there is a sequence (x_n) of U converging to z for which the measure $(\varepsilon_x)^{U^c}$ converges to $(\varepsilon_z)^{U^c}$ under the topology $\sigma(\mathfrak{M}_p(\bar{U}),\ H_p(\bar{U}))$.*

By this lemma we see that for any $z\in\bar{U}$, $(\varepsilon_z)^{U^c}$ is supported on ∂U.

Let U be an open set in a strict harmonic space Ω and C be the set of all P-bounded continuous functions on $\bar{U}$ which are superharmonic in U. We know that C is a min-stable, linearly separating cone and the set $P|\bar{U}$ of all restrictions on $\bar{U}$ of elements of P is an adapted cone. Hereafter we shall assume that for $x\in\bar{U}$, there exists $v\in C$ with $v(x)<0$. Then the Choquet boundary $\delta(C)$ of $\bar{U}$ with respect to C is not empty and by the maximum principle, the topological boundary ∂U is a determining set. Hence $\delta(C)\subset\partial U$.

We shall write $B(x)=(\varepsilon_x)^{U^c}$ for any $x\in\bar{U}$.

THEOREM 4. *A mapping $x\rightarrow B(x)$ from $\bar{U}$ into $\mathfrak{M}_p^+(\bar{U})$ is a dilation for C and the regular points of ∂U are just the B-regular points.*

PROOF. For any $p\in P$, the function $x\rightarrow B(x)(p)=\hat{R}_p^{U^c}(x)$ is lower semicontinuous and it is Borel measurable. Since $P|\bar{U}-P|\bar{U}$ is dense in $H_p(\bar{U})$, $x\rightarrow B(x)(f)$ is Borel measurable for any $f\in H_p(\bar{U})$. For any $g\in C$, $g|U$ is an upper function of $g|\partial U$. It follows for any $x\in U$

$$B(x)(g)=H_g(x)\leqq g(x)=\varepsilon_x(g).$$

Hence $\varepsilon_x\ll B(x)$.

If $z\in\partial U$, then by lemma 2, there exists a sequence (x_n) of U such that $B(x_n)$ converges $B(z)$ under $\sigma(\mathfrak{M}_p(\bar{U}), H_p(\bar{U}))$. Hence for any $g\in C$,

$$B(z)(g)=\lim B(x_n)(g)\leqq\varepsilon_x(g).$$

By proposition 7, $x\in\partial U$ is regular point if and only if $B(x)=\varepsilon_x$. It follows that $x\in\partial U$ is regular if and only if it is B-regular.

We say that Ω satisfies Axiom D if for all locally bounded superharmonic functions, the continuity of the restriction it's support implies the continuity of Ω everywhere. We have the following theorem by the same method in Effros and Kazdan [4].

THEOREM 5. *Suppose that Ω satisfies Axiom D, and U is an open set in Ω. Then balayage $x\rightarrow B(x)$ is an affine dilation for C.*

COROLLARY. *Suppose that Ω satisfies Axiom D and $U\subset\Omega$ is an open set such that for any $x\in\bar{U}$, there exists $v\in C$ with $v(x)<0$. Then the regular and Choquet boundary points coincide and $(\bar{U}, C)$ is a simplex. Further, for any $x\in\bar{U}$, the balayaged measure $(\varepsilon_x)^{U^c}$ is the extremal measure μ_x with $\varepsilon_x\ll\mu_x$.*

PROPOSITION 8. *Suppose that Ω is elliptic. If U is a connected* *open set in Ω, then for any $x, y \in U$, the harmonic measure μ_x^U and μ_y^U are mutually absolutely continuous. The support of each measure contains all regular points.*

PROOF. Let B is a Borel set contained in a compact set of ∂U. Put:

$$h_B(z)=\int \chi_B d\mu_z^U.$$

Then h_B is harmonic in U by proposition 6, since χ_B is μ_z^U-integrable. Since Ω is elliptic, we may apply Harnak's inequality in (1); there is a constant α such that for all B,

$$h_B(x) \leqq \alpha h_B(y) \quad \text{or}$$

$$\mu_x^U(B) \leqq \alpha \mu_y^U(B). \qquad \text{......................................(1)}$$

On the other hand, as Ω is σ-compact, there exists for any Borel set A an increasing sequence (B_n) of Borel sets where each B_n is contained in an compact set and $A=U_n B_n$. If $\mu_y^U(A)=0$, $\mu_y^U(B_n)=0$. Hence we have $\mu_x^U(B_n)=0$ by (1). Therefore $\mu_x^U(A)=0$ holds. By symmetry we have the first assertion.

Let z be any regular point and (x_n) be a sequence of converging to z. If z does not lie in the support of μ_x^U, we may find a positive continuous function f with compact support on ∂U which satisfies $f(z) \neq 0$ and vanishes on the support of μ_z^U. Then, by the absolute continuity, we get $\mu_{x_n}^U(f)=0$ for all n. Since z is regular, we have

$$f(z)=\lim_{n\to\infty} H_f(x)=\lim_{n\to\infty}\int f d\mu_{x_n}^U=0.$$

This is a contradiction.

REMARK. Axiom D implies that Ω is elliptic. Therefore, if Ω satisfies Axiom D, then by theorem 5 and proposition 8, the support of harmonic measure μ_x^U for any $x \in U$ coincides with $\overline{\delta(C)}$.

References

[1] H. Bauer, Harmonische Räume und ihre Potentieltheorie, Lecture Notes in Mathematics, 22. (Springer, 1966).

[2] M. Brelot, Lectures on potential theory, Tata Inst. of Fund. Research, Bombay (1960).

[3] G. Choquet, Lectures on analysis, 1, 2 (Benjamin, 1969).

[4] E.G. Effros & J.I. Kazdan, Applications of Choquet simplex to elliptic and parabolic boundary value problems, J. of Diffe. Equ. 8 (1970) 95–134.

[5] G. Mokobozki & D. Sibony, Cônes adaptés de fonctions continus et théorie du potentiel, Sémi. Choquet, 6 (1966/67).

[6] A. de La Pradelle, A propos du memoire de G.F. Vincent-Smith sur l'approximation des fonctions harmoniques, Ann. Fourier 19 (1969) 355–370.

[7] H. Watanabe, Simplexes on a locally compact space, Natu. Sci. Rep. Ochanomizu Univ. 23 (1972) 35–42.