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Extension of Maximal Ideals in Commutative Normed Algebras

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§1. Introduction

Let A be a commutative normed algebra and B an arbitrary commutative normed algebra containing A as a closed subalgebra. The problem of extension of a maximal ideal in A to a maximal ideal in B was studied by Šilov [3] and Rickart [7]. Šilov's extension theorem shows that every maximal ideal in the Šilov boundary for A can be extended to a maximal ideal in B. In this paper we study the extension problem of maximal ideals in commutative normed algebras in general situation. All algebras in the following will be commutative and complex, and have an identity.

Let A and B be commutative normed algebras. We shall say hereafter that B contains A algebraically if there exists an algebraic isomorphism φ of A into B which takes the identity of A onto the identity of B.

Let us denote by $\mathfrak{M}(A)$ and $\mathfrak{M}(B)$ the maximal ideal spaces of A and B respectively, and χ_φ the adjoint mapping of $\mathfrak{M}(B)$ into $\mathfrak{M}(A)$ induced by φ. Then $\chi_\varphi(\mathfrak{M}(B))$ is a closed subset of $\mathfrak{M}(A)$ in which every maximal ideal can be extended to a maximal ideal in B. Now, let us say that a closed subset F of $\mathfrak{M}(A)$ has the extension property for B if $\chi_\varphi(\mathfrak{M}(B))\supset F$. Thus, in this term, we can state Šilov's extension theorem as follows: the Šilov boundary for A has the extension property for B if B contains A as a closed subalgebra relative to the norm of B.

Our aim is to understand, under certain circumstances, to what part of $\mathfrak{M}(A)$ the extension property for normed algebras containing A should be admited.

In §3, we shall give a proof of Rickart's extension theorem, based on our characterization of the Šilov boundary in commutative normed algebras in §2, which is a generalization of Šilov's extension theorem.

If we consider arbitrary commutative normed algebras containing A algebraically with norm preserving isomorphisms, then, by definition, the cortex of A is the part of $\mathfrak{M}(A)$ which is contained in $\chi_\varphi(\mathfrak{M}(B))$

for every such B. Moreover, it is well-known that the cortex of A contains the Šilov boundary for A, and that it coincides with the Šilov boundary when A has the sup norm [1] [4].

On the other hand, as we shall show in Theorem 2 later, if there exists an algebraic isomorphism of A into B which preserves the spectrum of every element in A, at least one of the minimal unit-boundaries for A has the extension property for B, which, by definition [3], contains the Silov boundary. And we know that the Šilov boundary is the minimum unit-boundary if and only if every non-invertible element in A is a topological divisor of zero with respect to the spectral radius.

§ 2. A characterization of the Šilov boundary in commutative normed algebras

PROPOSITION. *Let A be a commutative normed algebra. Then a necessary and sufficient condition for a maximal ideal M in A to belong to the Šilov boundary for A is that, for every finite subset $\{m_1, \dots, m_k\}$ of M, there exists a sequence $\{a_n\}$ in A such that $\rho_A(a)=1$ and $\rho_A(a_n m_i) \to 0$ $(n \to \infty)$ for $i=1, \dots, k$, where $\rho_A(a)$ is the spectral radius of a.*

PROOF. Suppose that M_0 is in the Šilov boundary for A. Put $U_n(M_0)=\{M \in \mathfrak{M}(A)\,;\ |\hat{m}_i(M)^*| < \frac{1}{n}\ (i=1, \dots, k)\}$ for a given finite subset $\{m_1, \dots, m_k\}$ of M, then $U_n(M_0)$ is a neighborhood of M_0. Take $\mu = \max_{1 \leqq i \leqq k} \max_{M \in \mathfrak{M}(A)} |\hat{m}_i(M)| = \max_{1 \leqq i \leqq k} \rho_A(m_i)$. Since M_0 is in the Šilov boundary, there exists an element a_n in A such that $\rho_A(a_n) = \max_{M \in U_n} |\hat{a}_n(M)| = 1$ and $|\hat{a}_n(M)| < \frac{1}{\mu}$ for every M outside $U_n(M_0)$. Therefore, $\rho_A(a_n m_i) = \max_{M \in \mathfrak{M}(A)} |\hat{a}_n(M)||\hat{m}_i(M)| < \frac{1}{n}$. Thus, for every n, we can choose a_n in A, and $\{a_n\}$ will be the sequence which we require for the finite subset $\{m_1, \dots, m_k\}$.

Next, we prove the converse. Suppose $U(M_0)$ is an arbitrary neighborhood of M_0. Then it contains a neighborhood of M_0, $U_0(M_0)$, defined by k-inequalities $|\hat{m}_i(M)| < 1$ $(i=1, \dots, k)$, where $m_1, \dots, m_k$ are in M. Since, for the finite subset $\{m_1, \dots, m_k\}$, there exists a sequence $\{a_n\}$ in A and $\rho_A(a_n m_i) \to 0$ $(n \to \infty)$ $(i=1, \dots, k)$ as in Proposition, we have $\rho_A(a_{n_0} m_i) < 1$ $(i=1, \dots, k)$ for some n_0 which shows $\max_{M \notin U_0} |\hat{m}_i(M)||\hat{a}_{n_0}(M)| \leqq \rho_A(a_{n_0} m_i) = \max_{M \in \mathfrak{M}(A)} |\hat{m}_i(M)||\hat{a}_{n_0}(M)| < 1$ $(i=1, \dots, k)$. If M is outside

* We denote by $\hat{a}(M)$ the value of the Gelfand transform of a in A at a maximal ideal M.

$U_0(M_0)$, then there exists j; $1 \leqq j \leqq k$ such that $|\hat{m}_j(M)| \geqq 1$, whence $|\hat{a}_{n_0}(M)| < 1$. If M is in $U_0(M_0)$, then, since $\rho_A(a_{n_0}) = 1$, $|\hat{a}_{n_0}(M')| = 1$ for some M' in $U_0(M_0)$. Namely, M_0 is in the Šilov boundary which completes the proof.

REMARK. Żelazko has also obtained the same characterization of the Šilov boundary in function algebras [8] where he states that every maximal ideal in the Šilov boundary is non-removable for any normed algebra B, not necessarily being a function algebra, containing A algebraically. While, ours, as will be seen in the following section, is to find a condition for B that the Šilov boundary for A may have the extension property for B.

§ 3. Extension of maximal ideals

Here, we shall prove Rickart's extension theorem [7] applying Proposition in § 2.

THEOREM 1. *Let A be a commutative normed algebra and B an arbitrary commutative normed algebra containing A algebraically. Then the following two conditions are equivalent.*

1) *There exists an algebraic isomorphism φ satisfying that*

$$\rho_A(a) = \rho_B(\varphi(a)), \text{ for every } a \text{ in } A.$$

2) *Every maximal ideal in the Šilov boundary for A can be extended to a maximal ideal in B.*

PROOF. Suppose the condition (1) holds and that there exists a maximal ideal M in the Šilov boundary for A which cannot be extended to a maximal ideal in B. If we denote by J the set of all finite sums of the form $\sum \varphi(m_i)b_i$ ($m_i \in M$, $b_i \in B$), then, since J is a ideal in B containing $\varphi(M)$, J coincides with B. In particular, the identity e_B of B can expressed as $e_B = \sum_{i=1}^{k} \varphi(m_i)b_i$ by $m_1, \dots, m_k$ in M and $b_1, \dots, b_k$ in B. Since M is in the Šilov boundary for A, it follows from Proposition in § 2 that, for the finite subset $\{m_1, \dots, m_k\}$ of M, there exists a sequence $\{a_n\}$ in A such that $\rho_A(a_n) = 1$ for every n and $\rho_A(a_n m_i) \to 0$ $(n \to \infty)$ for $i = 1, \dots, k$. Multiply e_B by $\varphi(a_n)$ and consider the spectral radius. Put $C = \max_{1 \leqq i \leqq k} \rho_B(b_i)$. Then we have

$$1 = \rho_B(\varphi(a_n)e_B) = \rho_B\Big(\sum_{i=1}^{k} \varphi(a_n)\varphi(m_i)b_i\Big) \leqq \sum_{i=1}^{k} \rho_B(\varphi(a_n)\varphi(m_i)b_i)$$

$$\leqq C \sum_{i=1}^{k} \rho_A(a_n m_i) \to 0 \quad (n \to \infty).$$

This contradiction shows that the condition (2) holds.

Now, we shall prove the converse. Since $\mathfrak{M}(A)\supset\chi_{\varphi}(\mathfrak{M}(B))\supset\partial(A)$, by definitions, we have

$$\rho_B(\varphi(a))=\max_{M'\in\mathfrak{M}(B)}|\widehat{\varphi(a)}(M')|=\max_{M'\in\mathfrak{M}(B)}|\hat{a}(\chi_{\varphi}(M'))|$$

$$=\max_{M\in\chi_{\varphi}(\mathfrak{M}(B))}|\hat{a}(M)|=\max_{M\in\partial(A)}|\hat{a}(M)|, \text{ for every element a in } A,$$

where $\partial(A)$ is the Šilov boundary for A. Thus, we see that $\rho_A(a)=\rho_B(\varphi(a))$, for every a in A, which completes the proof.

If we consider, instead of general algebraic isomorphisms, those preserving the spectrum of every element in A, then the class of algebras containing A algebraically becomes smaller than the class of algebras considered in Theorem 1. But the consideration of the above smaller class, where the part of $\mathfrak{M}(A)$ with the extension property may become larger than the Šilov boundary for A, will lead to the next theorem.

THEOREM 2. *Let A and B be the same as in Theorem 1. Then the following two conditons are equivalent.*

1) *There exists an algebraic isomorphism φ of A into B satisfying that $\sigma_A(a)=\sigma_B(\varphi(a))$, for every a in A, where $\sigma_A(a)$ is the spectrum of a with respect to A.*

2) *There exists a minimal unit-boundary for A in which every maximal ideal can be extended to a maximal ideal in B.*

PROOF. Since the condition (1) is equivalent to the fact that the image of A, $\varphi(A)$, under the isomorphism φ is a " sous-algèbre pleine " of B [2] [6], we have to see that the condition (2) is equivalent to the fact $\varphi(A)$ that is a " sous-algèbre pleine " of B.

If $\varphi(A)$ is a " sous-algèbre pleine " of B, then, since $\mathfrak{M}(B)$ is a unit-boundary for B, $\chi_{\varphi}(\mathfrak{M}(B))$ is a unit-boundary for A and consequently contains a minimal one for A. Thus the condition (2) follows from the condition (1).

To see the converse, suppose that, for an element a in A, $\varphi(a)$ is invertible in B. Then the Gelfand transform $\widehat{\varphi(a)}$ of $\varphi(a)$ vanishes nowhere on $\mathfrak{M}(B)$. Therefore, by the condition (2), a vanishes nowhere on the unit-boundary with the extension property. Hence a is invertible in A. This shows that $\varphi(A)$ is a " sous-algèbre pleine " of B. This completes the proof.

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