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On Examples of Riemannian Spaces Harmonic relative to Killing Vectors

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Introduction. Let M^n be a Riemannian space with positive definite metric. If M^n admits a unit Killing vector $X=(\eta^i)$ satisfying

$$\nabla_k \nabla_j \eta^i = \eta_j \delta_k^i - \eta^i g_{kj},$$

it is called a Sasakian space, and lots of works have been done about it.[1] On the other hand, a Riemannian space M^n is said to be harmonic at a point O if Δs is a function of s only, where Δ is the Laplace-Beltrami operator and s denotes the geodesic distance measured from O. When M^n is harmonic at any point, it is called harmonic. It is easily seen that if a Sasakian space is harmonic it is a space of constant curvature. This comes from the following fact. The "harmonic property" requires the homogeneity for the directions in a sense contrary to what a Sasakian space has a speciality with the direction X. The purpose of this paper is to generalize harmonic property so that it is special with given directions, and we shall show that the Euclidean n-space and the $(2m+1)$-sphere have such a property.

1. The definition of K_r-harmonic space.

Let M^n be an n dimensional Riemannian space of C^∞ with positive definite metric g_{ij}. A Killing vector $X=(\eta^i)$ in M^n is a vector field satisfying

$$\nabla_j \eta_i + \nabla_i \eta_j = 0,$$

where ∇_i denotes the operator of Riemannian derivation and $\eta_i = g_{ij}\eta^j$.[2]

If a geodesic c is orthogonal to a Killing vector X at a point, then it is orthogonal to X at any point of c.

Now we shall assume that M^n admits an orthonormal r-field $(0 \leqq r \leqq n)$ of Killing vectors $X_\alpha = (\eta_\alpha{}^i)$. They satisfy

1) For examples, Sasaki and Hatakeyama [4], Okumura [5].

2) The summation convention is assumed for Latin indices throughout the paper. All functions, vector and tensor fields will be supposed to be of C^∞.

$$\nabla_j \eta_{\alpha i}+\nabla_i \eta_{\alpha j}=0\,, \qquad (\eta_{\alpha i}=g_{ij}\eta_\alpha{}^j)\,,$$

$$\eta_\alpha{}^i \eta_{\beta i}=\delta_{\alpha\beta}\,.$$

Let O be any point of M^n and U a normal neighbourhood of origin O. We shall denote by U_X the set consisting of all points in U each point of which is on a geodesic through O and orthogonal to $X_1, \cdots, X_r$. If we choose U sufficiently small, U_X becomes a $n-r$ dimensional submanifold. In fact, let $Y_1, \cdots, Y_{n-r}$ be vectors at O which constitute an orthonormal base together with $X_1(O), \cdots, X_r(O)$. Then U_X is given by $\mathrm{Exp}(u_1Y_1+\cdots+u_{n-r}Y_{n-r})$ in a small U, where $u_1, \cdots, u_{n-r}$ are parameters and Exp denotes the exponential map from the tangent space at O into U.

The Laplace-Beltrami operator Δ is defined by

$$\Delta=g^{ij}\nabla_i\nabla_j\,.$$

Defining $\bar{\nabla}_i$ by

$$\bar{\nabla}_i=\nabla_i-\sum_{\alpha=1}^{r}\eta_{\alpha i}\eta_\alpha{}^j\nabla_j\,,$$

we shall introduce an operator $\bar{\Delta}$ by

$$\bar{\Delta}=g^{ij}\bar{\nabla}_i\bar{\nabla}_j\,.$$

It is easy to see the following equation to be valid.

$$\bar{\Delta}=\Delta-\sum_{\alpha=1}^{r}\eta_\alpha{}^i\eta_\alpha{}^j\nabla_i\nabla_j\,.$$

A Riemannian space M^n admitting an orthonormal r-field of Killing vectors X_α will be called K_r-harmonic at O, if $\bar{\Delta}s$ is a function of s only provided that it is evaluated on a U_X of origin O, where s denotes the geodesic distance measured from O. If M^n is K_r-harmonic at any point O, it will be called K_r-harmonic. When $r=0$ we have $\bar{\Delta}=\Delta$ and K_0-harmonic of C^ω is nothing but harmonic in the sense of [1].

2. The Euclidean space E^n. Let E^n be the Euclidean n-space with orthogonal coordinates $\{x^i\}$ of origin O. In this section we shall show that E^n is K_r-harmonic at O for any orthonormal r-field of Killing vectors X_α.

Any geodesic c through O is of the form

(2.1) $$x^i=\xi_i s\,, \qquad \xi_i\xi_i=1\,,$$ [3]

where ξ_i are constant for c, and s the arc length. Thus the coordinate x^i of point in E^n are functions of ξ_i and s, and ξ_i and s are functions of x^i.

3) By our convention, $\xi_i\xi_i$ means $\sum \xi_i\xi_i$.

From $x^i x^i = s^2$, it follows that

$$x^i = ss_i \qquad (s_i = \partial_i s)$$

and hence

$$\xi_i = s_i \tag{2.2}$$

are valid. On the other hand, we have from (2.1)

$$\delta_j^i = s\,\partial_j \xi_i + s_j \xi_i .$$

Hence, if we put

$$\Delta_{ij} = \Delta_{ji} = \delta_{ij} - \xi_i \xi_j ,$$

then it holds that

$$\partial_j \xi_i = \frac{1}{s} \Delta_{ij} . \tag{2.3}$$

Δ_{ij} satisfy the following identities:

$$\Delta_{ij}\Delta_{jk} = \Delta_{ik} , \qquad \Delta_{ij}\xi_j = 0 ,$$
$$\Delta_{ii} = n-1 .$$

From (2.2) and (2.3) we have

$$\nabla_j \nabla_i s = \partial_j \partial_i s = \partial_j \xi_i = \frac{1}{s} \Delta_{ij} . \tag{2.4}$$

Thus we get

$$\Delta s = \nabla_i \nabla_i s = \frac{n-1}{s} \tag{2.5}$$

which shows E^n to be harmonic at O.

Now, let $X_\alpha = (\eta_\alpha{}^i)$ be an orthonormal r-field of Killing vectors. Then we have from (2.4)

$$\eta_\alpha{}^i \eta_\alpha{}^j \nabla_j \nabla_i s = \frac{1}{s} \{1 - (\eta_\alpha{}^i \xi_i)^2\} ,$$

from which and (2.5) it follows that

$$\bar{\Delta} s = \frac{1}{s} \{n - 1 - r + \sum_{\alpha=1}^{r} (\eta_\alpha{}^i \xi_i)^2\} .$$

Consider a point $P(\xi_i s_0)$ in U_X. As P is on a geodesic c: $x^i = \xi_i s$ orthogonal to $X_1, \cdots, X_r$, $\eta_\alpha{}^i \xi_i = 0$ hold good. Hence

$$\bar{\Delta} s = \frac{n-1-r}{s} \quad \text{on } U_X$$

is valid. Thus we know that E^n is K_r-harmonic at O.

3. Killing vectors in spaces of constant curvature. The next purpose of this paper is to show that the unit sphere of any odd dimension is K_1-harmonic for a Killing vector. As a preparation we shall determine in this section the form of Killing vector in spaces of constant curvature in terms of normal coordinates.

Let N^n be a non-flat space of constant curvature. The curvature tensor $R_{ijl}{}^h$ of N^n satisfies

$$R_{ijl}{}^h = k(g_{jl}\delta_i^h - g_{il}\delta_j^h),$$

where $k = R/n(n-1)$ is constant, R being the scalar curvature.

Let O be any point of N^n and consider a normal neighbourhood U of origin O. We suppose that the normal coordinate $\{x^i\}$ in U has been taken as $g_{ij}(0) = \delta_{ij}$, where g_{ij} denotes the metric tensor. In U any geodesic c through O is of the form

$$x^i = \xi_i s, \qquad \xi_i \xi_i = 1,$$

where ξ_i are constant for c, and s means the geodesic distance measured from O. In a similar way as in § 2, we know that $\xi_i = s_i$ and

$$\Delta_{ij} = \Delta_{ji} = \delta_{ij} - \xi_i \xi_j$$

satisfy the following identities:

$$\Delta_{ij}\Delta_{jk} = \Delta_{ik}, \qquad \Delta_{ij}\xi_j = 0,$$

$$\Delta_{ii} = n-1,$$

$$\partial_j s_i = \partial_j \xi_i = \frac{1}{s}\Delta_{ij}. \tag{3.1}$$

As is well known [2], the metric tensor are given by

$$g_{ij}(x) = \xi_i \xi_j + \gamma(s)\Delta_{ij},$$

$$g^{ij}(x) = \xi_i \xi_j + \gamma(s)^{-1}\Delta_{ij},$$

where $\gamma(s)$ is the function defined by

$$\gamma(s) = \begin{cases} \left(\dfrac{\sin as}{as}\right)^2, & \text{if } k = a^2, \\ \left(\dfrac{\sinh as}{as}\right)^2, & \text{if } k = -a^2, \end{cases}$$

a being positive.

The Christoffel's symbols are

$$\left\{ {h \atop ij} \right\} = \left(\frac{1-\gamma}{s} - \frac{\gamma'}{2}\right)\Delta_{ij}\xi_h + \frac{\gamma'}{2\gamma}(\Delta_{ih}\xi_j + \Delta_{jh}\xi_i). \tag{3.2}$$

LEMMA 3.1. *For any constants* $a_{ij} = -a_{ji}$,

$$u_i = \gamma(s) a_{ij} x^j$$

are covariant components of a Killing vector and satisfy

$$u_i(O) = 0, \qquad (\nabla_j u_i)_O = a_{ij}.$$

In fact, we have

$$\nabla_j u_i = \gamma a_{ij} + \frac{\gamma'}{2\gamma}(u_i \xi_j - u_j \xi_i).$$

The initial value $(\nabla_j u_i)_O$ follows from taking account of

$$\frac{\gamma'}{\gamma} = \begin{cases} 2\left(a \cot as - \dfrac{1}{s}\right) & \text{if } k = a^2, \\ 2\left(a \coth as - \dfrac{1}{s}\right), & \text{if } k = -a^2. \end{cases}$$

Next, if we put

$$\lambda(s) = \begin{cases} \dfrac{1}{as} \sin as \cos as, & \text{if } k = a^2, \\ \dfrac{1}{as} \sinh as \cosh as, & \text{if } k = -a^2, \end{cases}$$

the following lemma holds good.

LEMMA 3.2. *For any constants* b_i,

$$v_i = \lambda b_i + (1-\lambda)\beta(\xi)\xi_i$$
$$= b_i + (\lambda - 1)\Delta_{ij} b_j$$

are covariant components of a Killing vector and satisfy

$$v_i(O) = b_i, \qquad (\nabla_j v_i)_O = 0,$$

where

$$\beta(\xi) = b_i \xi_i.$$

In fact, we have

$$\nabla_j v_i = \mu(s)(\Delta_{ih}\xi_j - \Delta_{jh}\xi_i)v_h,$$

where

$$\mu(s) = \begin{cases} -a \tan as, & \text{if } k = a^2, \\ a \tanh as, & \text{if } k = -a^2. \end{cases}$$

Now, consider a Killing vector w^i in U satisfying

$$w_i(O) = b_i, \qquad (\nabla_j w_i)_O = a_{ij}.$$

Then, it is shown that w_i is written as

$$w_i = u_i + v_i$$

with u_i in Lemma 3.1 and v_i in Lemma 3.2. Because, a Killing vector w^i satisfies

$$\nabla_h \nabla_j w_i = k(w_j g_{hi} - w_i g_{hj})$$

and is determined by its values w_i and $\nabla_j w_i$ at O.

4. S^{2m+1} as a K_1-harmonic space. Consider the sphere S^{2m+1} of radius 1 in the Euclidean E^{2m+2}. It is a space of constant curvature

($k=1$) and admits a unit Killing vector $X=(\eta^i)$ globally. η_i satisfy

$$\nabla_h \nabla_j \eta_i = \eta_j g_{hi} - \eta_i g_{hj} .$$

S^{2m+1} is a typical example of Sasakian space and X is called a Sasakian structure of S^{2m+1}, [5].

If we put

$$\varphi_{ji} = \nabla_j \eta_i ,$$

then the following equations hold good:

$$\varphi_{ji}\eta^i = 0 , \tag{4.1}$$

$$\varphi_{ji}\varphi_h{}^i = g_{jh} - \eta_j \eta_h . \tag{4.2}$$

We shall show that S^{2m+1} is K_1-harmonic for the Sasakian structure $X=(\eta^i)$.

Let O be a point of S^{2m+1} and U a normal neighbourhood of origin O. η^i being Killing, it is written as

$$\eta_i = u_i + v_i$$

with u_i in Lemma 3.1 and v_i in Lemma 3.2. As η^i is unit, we have

$$b_i b_i = 1 \tag{4.3}$$

by taking account of $u_i(O)=0$ and $v_i(O)=b_i$. If we consider

$$\varphi_{ji} = \nabla_j \eta_i = \nabla_j u_i + \nabla_j v_i$$

at O, then it follows that

$$\varphi_{ji}(O) = a_{ij}$$

and b_i satisfy

$$a_{ij} b_j = 0 , \tag{4.4}$$

$$a_{ij} a_{ih} = \delta_{jh} - b_j b_h \tag{4.5}$$

by virtue of (4.1) and (4.2).

Thus, η_i is given by

$$\eta_i = \gamma a_{ih} x^h + \lambda b_i + (1-\lambda)\beta(\xi)\xi_i ,$$

where $a_{ih} = -a_{hi}$ and b_i satisfy (4.3), (4.4) and (4.5) and

$$\gamma(s) = \left(\frac{\sin s}{s}\right)^2, \qquad \lambda(s) = \frac{1}{s}\sin s \cos s , \qquad \beta(\xi) = b_h \xi_h .$$

The contravariant components of X are

$$\eta^i = a_{ih} x^h + \gamma^{-1}\lambda b_i + (1-\gamma^{-1}\lambda)\beta\xi_i . \tag{4.6}$$

Now we shall calculate $\bar{\Delta} s$ in U for $X=(\eta^i)$. From (3.1) and (3.2) it follows that

$$\nabla_j s_i = \partial_j s_i - \left\{ {h \atop ji} \right\} s_h = \left(\frac{\gamma}{s} + \frac{\gamma'}{2}\right)\Delta_{ij} ,$$

$$\Delta s = g^{ji}\nabla_j s_i = (n-1)\left(\frac{1}{s}+\frac{\gamma'}{2\gamma}\right) = (n-1)\cot s\,.$$

Taking account of (4.6) we can get

$$\eta^i\eta^j\nabla_j s_i = (1-\beta^2)\left(\frac{1}{s}+\frac{\gamma'}{2\gamma}\right) = (1-\beta^2)\cot s\,,$$

and hence

$$\bar{\Delta} s = \Delta s - \eta^i\eta^j\nabla_j s_i = (n-2+\beta^2)\cot s$$

follows. As $\beta(\xi)=b_i\xi_i=0$ is valid on U_X,

$$\bar{\Delta} s = (n-2)\cot s \qquad \text{on } U_X$$

holds good, which shows S^{2m+1} to be K_1-harmonic.

5. Remarks. (i) Let S^n be the n dimensional sphere of constant curvature. If $n=1$, 3 or 7, S^n admits an orthonormal n-field of Killing vectors, [3]. It is also known that S^{4m+3} admits an orthonormal 3-field of Killing vectors, [6], [7].

(ii) E^n and S^n are harmonic Riemannian spaces in the sense of H. S. Ruse. It is an open problem to find a K_1-harmonic Riemannian space which is not harmonic.

(iii) A harmonic Riemannian space (of C^ω) is an Einstein space, i.e., $R_{ij}=(n-1)kg_{ij}$ holds good. Does a K_1-harmonic Riemannian space of C^ω satisfy $R_{ij}=ag_{ij}+b\eta_i\eta_j$ for some constants a and b?

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