A Symmetry of Order 4 on Sasakian Spaces

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Introduction. In a Riemannian space, the covariant derivative of the Riemannian curvature tensor vanishes if and only if the geodesic symmetry is an isometry of a neighbourhood at each point. This local isometry is involutive and has an isolated fixed point. A Riemannian space is called an s-manifold if it admits a symmetry s_x at any point x of M, that is an isometry s_x of M with an isolated fixed point x, and the symmetry s_x is called a symmetry of order k if there exists a positive integer k such that $s_x{}^k=$ identity (A.J. Ledger and M. Obata [2]). Clearly a locally symmetric space is an s-manifold of order 2. On the other hand, E. Cartan proved the following result: Let ϕ be a tensor of type (1,1) satisfying $g_{ab}\phi_i{}^a\phi_j{}^b=g_{ij}$ and $R_{abcd}\phi_h{}^a\phi_i{}^b\phi_j{}^c\phi_k{}^d=R_{hijk}$ where g_{ij} and R_{hijk} denote the metric and curvature tensors. If the space M is locally symmetric, then ϕ induces a local isometry $\bar{\phi}$ at any point x of M. Taking a sufficiently small normal neighbourhood U of x, the isometry $\bar{\phi}$ is given by $(t\alpha^i)\to(t\phi_j{}^i\alpha^j)$.

We consider a Kählerian space with complex structure tensor $\varphi_i{}^j$. If $\varphi_i{}^j$ induces a local isometry $\bar{\varphi}$ of a normal neighbourhood at any point of M as above, then $\bar{\varphi}^2$ is a geodesic symmetry, and hence the space is locally symmetric. Conversely, if a Kählerian space is locally symmetric, then E. Cartan's theorem says that $\varphi_i{}^j$ induces an local isometry $\bar{\varphi}$ which is a symmetry of order 4. We show in this note that an analogous fact holds good in a Sasakian space.

1. Preliminaries. A $(2n+1)$-dimensional Riemannian space M^{2n+1} is called a Sasakian space if it admits a unit Killing vector field η^i satisfying

$$\nabla_i\nabla_j\eta_k=\eta_j g_{ik}-\eta_k g_{ij}.$$

We define a 2-form by $\varphi_{ij}=\nabla_i\eta_j$. Then the following relations are well known.

$$\eta^h R_{hijk}=\eta_k g_{ij}-\eta_j g_{ik}, \tag{1}$$

$$\varphi_i{}^a R_{ajkh}-\varphi_j{}^a R_{aikh}=\varphi_{ih}g_{jk}-\varphi_{ik}g_{jh}+\varphi_{jk}g_{ih}-\varphi_{jh}g_{ik}. \tag{2}$$

We can obtain easily that

(3) $$R_{ijab}\,\varphi_k{}^a\,\varphi_h{}^b = R_{ijkh} + \varphi_{ih}\,\varphi_{jk} - \varphi_{ik}\,\varphi_{jh} - g_{ih}\,g_{jk} + g_{ik}\,g_{jh}\,,$$

(4) $$R_{iabc}\,\varphi_j{}^a\,\varphi_k{}^b\,\varphi_h{}^c = \varphi_i{}^a\,R_{ajkh} + \varphi_{ih}\,\eta_j\,\eta_k - \varphi_{ik}\,\eta_j\,\eta_h\,,$$

(5) $$R_{abcd}\,\varphi_i{}^a\,\varphi_j{}^b\,\varphi_k{}^c\,\varphi_h{}^d = R_{ijkh} + g_{ik}\,\eta_j\,\eta_h + g_{ih}\,\eta_j\,\eta_h + g_{jh}\,\eta_i\,\eta_k - g_{jk}\,\eta_i\,\eta_h\,,$$

(6) $$R_{iabj}\,\eta^a\,\eta^b = g_{ij} - \eta_i\,\eta_j\,.$$

We define a tensor ϕ of type (1,1) by

$$\phi_i{}^j = \varphi_i{}^j - \eta_i\,\eta^j\,.$$

Then the following equations

$$g_{ab}\,\phi_i{}^a\,\phi_j{}^b = g_{ij}\,,$$

$$\phi_a{}^j(\varphi_i{}^a + \eta_i\,\eta^a) = -\delta_i^j$$

hold good.

We take a point x_0 of M^{2n+1} and a sufficiently small normal neighbourhood U of x_0. Since the exponential mapping $\mathrm{Exp}_{x_0} : T_{x_0}(M) \to U$ is a diffeomorphism, we can define a diffeomorphism $\bar{\phi} : U \to U$ by

$$\bar{\phi}(x) = \mathrm{Exp}_{x_0} \circ \phi_{x_0} \circ \mathrm{Exp}_{x_0}^{-1}(x)$$

where ϕ_{x_0} denotes the linear transformation on T_{x_0} induced by the tensor ϕ. If a point x of U has the normal coordinates $x^i = t\alpha^i$, where (α^i) is a unit vector of T_{x_0} and t denotes the distance between x and x_0, then it follows that the point $\bar{\phi}(x)$ in U has the coordinates $y^i = t(\phi_a{}^i)_{x_0}\,\alpha^a$.

2. Theorem.

First we prepare some lemmas for later use.

LEMMA 1. *In a Sasakian space, we have*

(7) $$\phi_h{}^a\,\phi_i{}^b\,\phi_j{}^c\,\phi_k{}^d\,R_{abcd} = R_{hijk}\,.$$

PROOF. We see that the terms of the left hand side of (7) vanish except those of the types of $\varphi_h{}^a\,\varphi_j{}^b\,\varphi_i{}^c\,\varphi_k{}^d\,R_{abcd}$, $\eta_j\,\eta_h\,\varphi_i{}^a\,\varphi_k{}^b\,\eta^c\,\eta^d\,R_{abcd}$. Making use of the equations (5) and (6), we have easily (7).

LEMMA 2. *In a Sasakian space, we have*

(8) $$\eta^a\,\nabla_b\,R_{acde} = -\varphi_b{}^a\,R_{acde} + \varphi_{be}\,g_{cd} - \varphi_{bd}\,g_{ce}\,,$$

(9) $$\eta^a\,\nabla_a\,R_{bcde} = 0\,.$$

PROOF. By virtue of (1), we get

$$\eta^a\,\nabla_b\,R_{acde} = \nabla_b(\eta_e\,g_{cd} - \eta_d\,g_{ce}) - \varphi_b{}^a\,R_{acde}$$
$$= -\varphi_b{}^a\,R_{acde} + \varphi_{be}\,g_{cd} - \varphi_{bd}\,g_{ce}\,.$$

Taking account of Bianchi's identity and (2), we have

$$\begin{aligned}\eta^a \nabla_a R_{bcde} &= \eta^a \nabla_b R_{acde} - \eta^a \nabla_c R_{abde} \\ &= -(\varphi_b{}^a R_{acde} - \varphi_c{}^a R_{abde}) + \varphi_{be} g_{cd} - \varphi_{bd} g_{ce} \\ &\quad - \varphi_{ce} g_{bd} + \varphi_{cd} g_{be} \\ &= 0 .\end{aligned}$$

The next lemma can be proved similarly.

LEMMA 3. *In a Sasakian space, we have*

$$\eta^b \eta^d \nabla_a R_{bcde} = \eta_e \varphi_{ac} - \eta_c \varphi_{ae} . \tag{10}$$

Let the Sasakian space be locally symmetric. Then it is well known that the space is of constant curvature. In such a space, the mapping $\bar{\phi}: U \to U$ is a local isometry which is a symmetry of order 4 by the preceding E. Cartan's theorem and Lemma 1.

Now we consider the converse problem, and prove the following

THEOREM. *In a Sasakian space M, we assume that the mapping $\bar{\phi}$: $U \to U$ induced from the tensor $\phi_{x_0} = \varphi_i{}^j - \eta_i \eta^j$ is an isometry at each point x_0 of M. Then the space is locally symmetric, and consequently M is a space of constant curvature.*

PROOF. Putting $y = \bar{\phi}(x)$ for x of U, we see that the induced mapping ϕ^* on the tangent space has the component

$$\frac{\partial y^a}{\partial x^i} = \varphi_i{}^a - \eta_i \eta^a \quad (= \phi_i{}^a)$$

where $\varphi_i{}^a$ and η_i are the values at x_0. As $\bar{\phi}$ is an isometry, it follows that ϕ^* preserves the curvature tensors and its covariant derivatives. Therefore we have

$$\frac{\partial y^a}{\partial x^i} \frac{\partial y^b}{\partial x^j} \frac{\partial y^c}{\partial x^k} \frac{\partial y^d}{\partial x^h} \frac{\partial y^e}{\partial x^m} \nabla_a R_{bcde}(y) = \nabla_i R_{jkhm}(x)$$

for any x of U. Considering this equation at $x = x_0$, we have

$$\phi_i{}^a \phi_j{}^b \phi_k{}^c \phi_h{}^d \phi_m{}^e \nabla_a R_{bcde}(x_0) = \nabla_i R_{jkhm}(x_0)$$

because of $\bar{\phi}(x_0) = x_0$. Since we take the normal coordinate, the covariant derivative coincides with the partial derivative at the origin x_0, and hence we obtain

$$\phi_i{}^a \phi_j{}^b \phi_k{}^c \phi_h{}^d \phi_m{}^e \partial_a R_{bcde}(x_0) = \partial_i R_{jkhm}(x_0) .$$

It follows that

$$\begin{aligned}-\partial_a R_{bcde} = (\varphi_a{}^p + \eta_a \eta^p)(\varphi_b{}^q + \eta_b \eta^q)(\varphi_c{}^r + \eta_c \eta^r)(\varphi_d{}^s + \eta_d \eta^s) \cdot \\ (\varphi_e{}^t + \eta_e \eta^t) \partial_p R_{qrst}\end{aligned}$$

(in the following, all components are considered at the origin x_0). Then it becomes

$$-\partial_a R_{bcde}=\varphi_a^{\ p}\varphi_b^{\ q}\varphi_c^{\ r}\varphi_d^{\ s}\varphi_e^{\ t}\partial_p R_{qrst}$$
$$+\sum{}^{*)}\eta_b\eta^q(\varphi_a^{\ p}\varphi_c^{\ r}\varphi_d^{\ s}\varphi_e^{\ t})\partial_p R_{qrst}$$
$$+\sum\eta_b\eta_d\eta^q\eta^s(\varphi_a^{\ p}\varphi_c^{\ r}\varphi_e^{\ t})\partial_p R_{qrst}.$$

Making use of Lemmas 2 and 3 at the origin x_0, we have

$$\eta^q(\varphi_a^{\ p}\varphi_c^{\ r}\varphi_d^{\ s}\varphi_e^{\ t})\partial_p R_{qrst}=-\varphi_a^{\ p}R_{pcde}+\varphi_{ae}g_{cd}-\varphi_{ad}g_{ce},$$
$$\eta^q\eta^r(\varphi_a^{\ p}\varphi_c^{\ r}\varphi_e^{\ t})\partial_p R_{qrst}=0.$$

Thus it holds that

$$-\partial_a R_{bcde}=\varphi_a^{\ p}\varphi_b^{\ q}\varphi_c^{\ r}\varphi_d^{\ s}\varphi_e^{\ t}\partial_p R_{qrst}+A_{abcde} \tag{11}$$

where

$$A_{abcde}=\sum\eta_b(-\varphi_a^{\ p}R_{pcde}+\varphi_{ae}g_{cd}-\varphi_{ad}g_{ce}).$$

Now transvecting (11) by $\varphi_i^{\ a}\varphi_j^{\ b}\varphi_k^{\ c}\varphi_h^{\ d}\varphi_m^{\ e}$, we obtain

$$-\varphi_i^{\ a}\varphi_j^{\ b}\varphi_k^{\ c}\varphi_h^{\ d}\varphi_m^{\ e}\partial_a R_{bcde}=(-\delta_i^{\ p}+\eta_i\eta^p)(-\delta_m^{\ t}+\eta_m\eta^t)$$
$$(-\delta_j^{\ b}+\eta_j\eta^b)(-\delta_k^{\ c}+\eta_k\eta^c)(-\delta_h^{\ d}+\eta_h\eta^d)\partial_p R_{qrst}.$$

Substituting (11) into the above equation, it follows that

$$\partial_i R_{jkhm}+A_{ijkhm}=-\partial_i R_{jkhm}+\sum\eta_j(\eta^b\partial_i R_{bkhm})$$
$$-\sum\eta_j\eta_h\eta^b\eta^d\partial_i R_{bkdm}.$$

Taking account of Lemmas 2 and 3 again, we see that

$$\sum\eta_j\eta^b\partial_i R_{bkhm}=A_{ijkhm},$$
$$\sum\eta_j\eta_h\eta^b\eta^d\partial_i R_{bkdm}=0.$$

Hence it follows that

$$\partial_i R_{jkhm}=0$$

at the origin x_0 of the normal neighbourhood. Since x_0 is an arbitrary point, we can conclude that the space is locally symmetric, and therefore is of constant curvature.

Bibliography

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[3] M. Okumura, Some remarks on space with a certain contact structure, Tôhoku Math. J., 14 (1962), 135-145.

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*) The notation $\sum$ denotes the sum of the terms of the same type with suitable signs.