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Relations of Ergodicities to Topological Amenabilities for a Semigroup of Operators with Some of Their Applications

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§1. Introduction

Ergodicities for a semigroup of operators on a Banach space have been investigated by many authors. (For example, see [1] and [5]) We shall mention some of them which seem important to us (Prop. 1, Def. 1–3). Amenability of an abstract discrete semigroup $\mathfrak{S}$, the condition that there exists a translation invariant mean on the space of all bounded functions on $\mathfrak{S}$, is known to be important in connection with the theory of ergodicity (for example, see [3] § 8). Since some topologies are naturally induced in a semigroup of operators, we want to consider topological amenabilities, the conditions that there exists a translation invariant mean on some subspaces of $C(\mathfrak{S})$, the space of all bounded continuous functions on a topological semigroup $\mathfrak{S}$ (Def. 4–5, Prop. 2–3). It seems to be interesting to make clear the equivalence relations of ergodicities to topological amenabilities, which will be first discussed in § 2 (Th. 1).

Further we shall give an example of a group of operators Γ which has an invariant mean on $C(\Gamma)$ but fails to be amenable (§ 3).

Let X be a compact Hausdorff space and $C(X)$ the Banach algebra of all complex (or real)-valued continuous functions on X with the supremum norm. We call a bounded linear operator T of $C(X)$ into itself a *Markov operator* on $C(X)$ if $Te=e$ and $Tf \geqq 0$ whenever $f \geqq 0$, where e is the constant function with the value 1 on X. Let Σ be a fixed semigroup of Markov operators on $C(X)$.

As the applications of what will be obtained in § 2, we can improve several theorems on the relations of maximal Σ-ideals, extreme points of Σ-invariant probability measures and functionals which are multiplicative "modulo an averaging process"[1] on $C(X)$. We obtain first under the condition of right quasi-ergodicity of Σ (see Def. 2), that

1) This wording appears in [2].

each maximal *Σ-ideal*, Σ-invariant closed proper ideal in $C(X)$, is of the form $J_\varphi=\{f\in C(X):\varphi(|f|)=0\}$ for a suitable $\varphi\in K[C(X)]_\Sigma=\{\varphi\in C(X)^*:\varphi\geqq 0,\ \varphi(e)=1$, and $\varphi(Tf)=\varphi(f)$ for all $T\in\Sigma$ and $f\in C(X)\}$ (Th. 2). The result has been obtained by Schaefer [7] for one positive operator and by Takahashi [8] for an amenable semigroup of Markov operators. It is clear from Th. 1 and § 3 that our condition is weaker than the condition by Takahashi.

Next, we can obtain under the condition of ergodicity of Σ (see Def. 1) that $\varphi\to J_\varphi$ is a bijection of the set of extreme points of $K[C(X)]_\Sigma$ onto the family of all maximal Σ-ideals (Th. 3). It was proved by Schaefer [7] for one ergodic Markov operator and by Takahashi [8] under the conditions of amenability and his ergodicity. It is clear from Th. 1 and § 3 that our condition is weaker than these. Moreover, we define a Σ-invariant subset of X, for which we shall treat analogously as above.

We can also obtain under the condition that Σ is restrictedly right quasi-ergodic (see Def. 3) or ergodic, that for real-valued $C(X)$ the extreme points of $K[C(X)]_\Sigma$ are the functionals on $C(X)$ which are multiplicative "modulo an averaging process" (Th. 4 and Cor. 3). It was proved by Converse, Namioka and Phelps [2] under the condition that Σ is right amenable, or amenable and weakly almost periodic. It is clear from Th. 1 and § 3 that our condition is weaker than these.

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§ 2. Relations of ergodicities to topological amenabilities for a semigroup of operators

Throughout this section, let E be a fixed Banach space, E^* the dual of E and E^{**} the dual of E^*. We denote by $\mathfrak{L}(E)$ and $\mathfrak{L}(E, E^{**})$ the sets of all bounded linear operators of E into E and E^{**}, respectively, and by I the identity operator on E. A *semigroup of operators* means a subsemigroup of $\mathfrak{L}(E)$ under the composition of operators. A semigroup Σ of operators is called *uniformly bounded* if there exists $M>0$ such that $\|T\|\leqq M$ for all $T\in\Sigma$. Throughout this section, let Σ be a fixed uniformly bounded semigroup of operators and M be $\sup\{\|T\|: T\in\Sigma\}$. We denote by Σ^* the semigroup of the adjoint operators T^* of $T\in\Sigma$. A semigroup is called a *topological semigroup* if it is a semigroup with Hausdorff topology in which the product is separately continuous. Now in $\mathfrak{L}(E)$ its strong operator topology and weak operator topology are induced and we denote them by (so) and (wo), respectively. Therefore Σ is automatically a topological semigroup under (so) or (wo) in $\mathfrak{L}(E)$, and moreover, since Σ is uniformly bounded, the product is jointly continuous under (so) in $\mathfrak{L}(E)$. We denote Σ supplied with (so) $[(wo)]$ by $(\Sigma, (so))$ $[(\Sigma, (wo))]$.

Many authors have obtained various conditions for the ergodicity of semigroups of operators (for example, see [1], [5] and [3] § 8), among which special mentions will be made of the following;

$(\boldsymbol{E}_1)$ For each $f \in E$ there exists uniquely a common fixed point g of Σ in $\overline{co}\Sigma f$, the norm-closed convex hull of the orbit $\{Tf: T \in \Sigma\}$ of f in E.

$(\boldsymbol{E}_2)$ There exists a bounded idempotent linear operator P of E onto $\mathfrak{F}(\Sigma)=\{f \in E: Tf=f \text{ for all } T \in \Sigma\}$ satisfying the following conditions;

(i) $PT=TP=P$ for all $T \in \Sigma$.

(ii) P belongs to the (so)-closure of the convex hull $co\Sigma$ of Σ.

$(\boldsymbol{E}_3)$ There exists a $P \in \mathfrak{L}(E)$ whose adjoint operator P^* is a bounded linear idempotent operator of E^* onto $\mathfrak{F}(\Sigma^*)=\{\varphi \in E^*: T^*\varphi=\varphi$ for all $T^* \in \Sigma^*\}$ satisfying the following conditions;

(i) $T^*P^*=P^*T^*=P^*$ for all $T^* \in \Sigma^*$.

(ii) P^* belongs to the closure of $co\Sigma^*$ in the weak* operator topology in $\mathfrak{L}(E^*)$, where $co\Sigma^*$ denotes the convex hull of Σ^*.

$(\boldsymbol{E}_4)$ $[(\boldsymbol{E}_4')]$ There exists a net $\{U_\alpha\}$ in $\mathfrak{L}(E)$ satisfying the following conditions;

(i) $(so)-[(wo)-]\lim_\alpha U_\alpha(T-I)=0$ for all $T \in \Sigma$.

(ii) $(so)-[(wo)-]\lim_\alpha(T-I)U_\alpha=0$ for all $T \in \Sigma$.

(iii) For each $f \in E$ and U_α, $U_\alpha f$ belongs to $\overline{co}\Sigma f$.

(iv) For each $f \in E$, the net $\{U_\alpha f\}$ has a weakly convergent subnet.

PROPOSITION 1. *These above four conditions* $(\boldsymbol{E}_1), \cdots, (\boldsymbol{E}_4)$ *are equivalent.*

PROOF. $(\boldsymbol{E}_1) \Rightarrow (\boldsymbol{E}_2)$ We define the mapping P on E by $Pf=g$, where g is a unique fixed point in $\overline{co}\Sigma f$ in $(\boldsymbol{E}_1)$. Since $Pf \in \mathfrak{F}(\Sigma)$, $TP=P$ for each $T \in \Sigma$, and hence $\overline{co}\Sigma Pf=\{Pf\}$ thus $P^2=P$, and P is clearly an operator of E onto $\mathfrak{F}(\Sigma)$. For each $T \in \Sigma$ and $f \in E$,

$$\{PTf\}=\overline{co}\Sigma Tf \cap \mathfrak{F}(\Sigma) \subset \overline{co}\Sigma f \cap \mathfrak{F}(\Sigma)=\{Pf\}$$

thus $PTf=Pf$. For $f \in E$ and complex c, clearly $P(cf)=cPf$.

On the other hand, it is known that $(\boldsymbol{E}_1)$ implies the following,

$(*)$ "For every $\varepsilon>0$ and $U \in co\Sigma$, there exists a $V \in co\Sigma$ such that $\|WVUf-g\|<\varepsilon$ for all $W \in co\Sigma$."

From the property (*), for $f_1, f_2 \in E$ and $\varepsilon > 0$, there exists a $U_1 \in co\Sigma$ such that

$$\| U_1' U_1 f_1 - Pf_1 \| < \varepsilon/2 \text{ for all } U_1' \in co\Sigma ,$$

and there exists a $U_2 \in co\Sigma$ such that

$$\| U_2' U_2 U_1 f_2 - Pf_2 \| < \varepsilon/2 \text{ for all } U_2' \in co\Sigma .$$

Hence for $U = U_2' U_2 U_1 \in co\Sigma$,

$$\| U(f_1 + f_2) - (Pf_1 + Pf_2) \| < \varepsilon ,$$

from which follows

$$Pf_1 + Pf_2 \in \overline{co}\Sigma(f_1 + f_2) \cap \mathfrak{F}(\Sigma) = \{P(f_1 + f_2)\}$$

that is $P(f_1+f_2) = Pf_1 + Pf_2$. Thus P is linear. Since Σ is uniformly bounded and $Pf \in \overline{co}\Sigma f$ for each $f \in E$, P is bounded.

We shall prove that, for each finite system of elements of E, $f_1, \cdots, f_n \in E$ and $\varepsilon > 0$, there exists a $U \in co\Sigma$ such that

$$\| Uf_i - Pf_i \| < \varepsilon \text{ for all } i = 1, 2, \cdots, n .$$

To prove this, we shall make use of the property (*). For $\varepsilon > 0$ there exists a $U_1 \in co\Sigma$ such that

$$\| U_1' U_1 f_1 - Pf_1 \| < \varepsilon \text{ for all } U_1' \in co\Sigma .$$

Next, there exists a $U_2 \in co\Sigma$ such that

$$\| U_2' U_2 U_1 f_2 - Pf_2 \| < \varepsilon \text{ for all } U_2' \in co\Sigma .$$

After all there exists a $U_n \in co\Sigma$ such that

$$\| U_n' U_n \cdots U_1 f_n - Pf_n \| < \varepsilon \text{ for all } U_n' \in co\Sigma .$$

Thus we have for $U = U_n' U_n \cdots U_1$,

$$\| Uf_i - Pf_i \| < \varepsilon \text{ for all } i = 1, 2, \cdots, n ,$$

whence we have $(\boldsymbol{E}_2)$.

$(\boldsymbol{E}_2) \Rightarrow (\boldsymbol{E}_1)$ Conversely, by $(\boldsymbol{E}_2)$, Pf is a common fixed point of Σ in $\overline{co}\Sigma f$ and if there exists a common fixed point g of Σ in $\overline{co}\Sigma f$, then $g = Pg \in \overline{co}\Sigma Pf = \{Pf\}$, which implies $g = Pf$. Thus $(\boldsymbol{E}_2) \Leftrightarrow (\boldsymbol{E}_1)$ is proved.

$(\boldsymbol{E}_2) \Leftrightarrow (\boldsymbol{E}_3)$ It is obvious from direct computation.

The equivalence between $(\boldsymbol{E}_4)$ and $(\boldsymbol{E}_4')$ depends on Lemma 5.2 in [3], which essentially depends on Mazur theorem that norm-closed convex sets are weakly closed.

$(\boldsymbol{E}_2) \Leftrightarrow (\boldsymbol{E}_4)$ If $(\boldsymbol{E}_2)$ holds, we have only to put $\{U_\alpha\} = \{P\}$ to prove $(\boldsymbol{E}_4)$. Conversely, if $(\boldsymbol{E}_4)$ holds, along the proof of the mean ergodic theorem we can prove that the net $\{U_\alpha f\}$ itself converges strongly to a fixed point. If we define Pf by the limit of $\{U_\alpha f\}$, then it has the property of P in $(\boldsymbol{E}_2)$.

DEFINITION 1. We call Σ *ergodic* if Σ satisfies any one of $(\boldsymbol{E}_i,\ i=1, 2, 3, 4)$.

DEFINITION 2. Σ is called *quasi-ergodic* {*right quasi-ergodic*} [*left quasi-ergodic*] if there exists a $\Pi\in\mathfrak{L}(E, E^{**})$ satisfying the following conditions;

(i) $\Pi T=T^{**}\Pi=\Pi\{\Pi T=\Pi\}\ [T^{**}\Pi=\Pi]$ for all $T\in\Sigma$ and

(ii) for each $f\in E$, Πf belongs to $w^*-\overline{co}\Sigma f$, the weak*-closed convex hull of the orbit $\{Tf\colon T\in\Sigma\}$ of f in E^{**}; which is equivalent to the condition that there exists a net $\{U_\alpha\}$ in $\mathfrak{L}(E)$ satisfying (i), (ii) and (iii) {(i) and (iii)} [(ii) and (iii)] in $(\boldsymbol{E}_4)$.

Quasi-ergodicity of Σ was examined in [3] §8 etc., where it was called ergodic.

DEFINITION 3. Σ is called *restrictedly quasi-ergodic* {*right restrictedly quasi-ergodic*} [*left restrictedly quasi-ergodic*] if there exists a $\Pi\in\mathfrak{L}(E, E^{**})$ satisfying the following conditions;

(i) $\Pi T=T^{**}\Pi=\Pi\ \{\Pi T=\Pi\}\ [T^{**}\Pi=\Pi]$ for all $T\in\Sigma$, and

(ii) Π is contained in $w_*-\overline{co}\Sigma$, the (w^*o)-closure of $co\Sigma$ in $\mathfrak{L}(E, E^{**})$; which is equivalent to the condition that there exists a net $\{U_\alpha\}$ in $co\Sigma$ satisfying (i) and (ii) {(i)} [(ii)] in $(\boldsymbol{E}_4)$.

There are following implications among the ergodicities mentioned above.

ERGODIC
↓
RESTRICTEDLY QUASI-ERGODIC

↙ ↓ ↘

LEFT RESTRICTEDLY QUASI-ERGODIC | QUASI-ERGODIC | RIGHT RESTRICTEDLY QUASI-ERGODIC

↓ ↙ ↘ ↓

LEFT QUASI-ERGODIC | RIGHT QUASI-ERGODIC

Let us now mention about a discrete semigroup $\mathfrak{S}$ that is not necessarily a semigroup of operators. Let $m(\mathfrak{S})$ be the Banach space of all bounded complex (or real)-valued functions on $\mathfrak{S}$ with the supremum norm. For $s\in\mathfrak{S}$ the right translation r_s {left translation l_s} of $m(\mathfrak{S})$ by s is given by $(r_sx)s'=x(s's)$ $\{(l_sx)s'=x(ss')\}$, where $x\in m(\mathfrak{S})$ and $s'\in\mathfrak{S}$. Let B be a subspace of $m(\mathfrak{S})$ closed under complex conjugation. Then B is *right* {*left*} *translation-invariant* if $r_sB\subset B\{l_sB\subset B\}$ for all $s\in\mathfrak{S}$. If B is both right and left translation-invariant, then B is called *translation-invariant*. Let B be a right {left} translation-invariant closed subspace of $m(\mathfrak{S})$ that contains $\mathbf{1}$, the constant function with the value 1 on $\mathfrak{S}$, closed under complex conjugation. An element $\mu\in B^*$ is a *mean* on B if $\mu(\mathbf{1})=1$ and $\mu\geqq 0$. A mean on B is *right* {*left*} *invariant* if $\mu(r_sx)=\mu(x)$

$\{\mu(l_s x)=\mu(x)\}$ for all $x\in B$ and $s\in\mathfrak{S}$. If μ is both right and left invariant, then μ is called *invariant*. Σ is called *amenable* {*right amenable*} [*left amenable*] if there exists an invariant {right invariant} [left invariant] mean on $m(\mathfrak{S})$ itself.

According to [3] § 8, $\mathfrak{S}$ is amenable if and only if any uniformly bounded representation or antirepresentation of $\mathfrak{S}$ over any Banach space is quasi-ergodic or restrictedly quasi-ergodic.

Here let E be an arbitrary fixed Banach space and Σ an arbitrary fixed uniformly bounded semigroup of operators, then the following representation functions on Σ for E are naturally introduced.

DEFINITION 4. We define a *representation function of the first type* or *r_1-function* $x_{f,\varphi}$ on Σ for each $f\in E$ and $\varphi\in E^*$ by $x_{f,\varphi}(T)=\varphi(Tf)$ for all $T\in\Sigma$, and denote by $R_1(\Sigma)$ the closed linear subspace of $m(\Sigma)$ generated by all r_1-functions and all the constant functions, closed under complex conjugation.

$C(\Sigma,(so))$ and $C(\Sigma,(wo))$ denote the Banach algebras of all bounded continuous functions on $(\Sigma,(so))$ and $(\Sigma,(wo))$, respectively.

We can easily show that $x_{f,\varphi}\in C(\Sigma,(wo))$ (hence $x_{f,\varphi}\in C(\Sigma,(so))$, too) for each $f\in E$ and $\varphi\in E^*$ and $R_1(\Sigma)$ is translation-invariant.

If Σ is a semigroup of Markov operators on $E=C(X)$, then $R_1(\Sigma)$ is the closed linear subspace of $C(\Sigma,(wo))$ (hence of $C(\Sigma,(so))$) closed under complex conjugation generated by r_1-functions. In fact, for $f=e$ and $\varphi_0(e)=1$, $x_{e,\varphi_0}(T)=\varphi_0(Te)=\varphi_0(e)=1$ for all $T\in\Sigma$, that is $x_{e,\varphi_0}=\mathbf{1}$.

When E is any Banach algebra, we want to consider another representation functions as follows.

DEFINITION 5. We define a *representation function of the second type* or *r_2-function* $x_{f,g,\varphi}$ on Σ for each f, $g\in E$ and $\varphi\in E^*$ by $x_{f,g,\varphi}(T)=\varphi(Tf\cdot g)$ for all $T\in\Sigma$, and denote by $R_2(\Sigma)$ the closed $*$-subalgebra of $m(\Sigma)$ generated by all r_2-functions and all the constant functions.

We can easily show that $x_{f,g,\varphi}\in C(\Sigma,(so))$ for each f, $g\in E$ and $\varphi\in E^*$, and $R_2(\Sigma)$ is right translation-invariant.

If Σ is a semigroup of Markov operators on $E=C(X)$, then $R_1(\Sigma)\subset R_2(\Sigma)$ and $R_2(\Sigma)$ is the closed $*$-subalgebra of $C(\Sigma,(so))$ generated by r_2-functions.

When amenability is topologized for an abstract semigroup, $RUC(\mathfrak{S})$ and $WRUC(\mathfrak{S})$ have been investigated by several authors (for example, see [6]), where $RUC(\mathfrak{S})$ the space of *right uniformly continuous* functions on $\mathfrak{S}$ is the set of all $x\in C(\mathfrak{S})$ satisfying the condition that for each $s\in\mathfrak{S}$, if $s_\alpha\to s$, then $r_{s_\alpha}x\to r_s x$ in the norm-topology of $C(\mathfrak{S})$, and $WRUC(\mathfrak{S})$ the space of *weakly right uniformly continuous* functions on $\mathfrak{S}$ is the set of all $x\in C(\mathfrak{S})$ satisfying the condition that for each $s\in\mathfrak{S}$, if $s_\alpha\to s$, then $r_{s_\alpha}x\to r_s x$ in the weak-topology of $C(\mathfrak{S})$.

PROPOSITION 2. *For each* $f\in E$ *and* $\varphi\in E^*$, $x_{f,\varphi}$ *is in* $RUC(\Sigma, (so))$. *Thus* $R_1(\Sigma)\subset RUC(\Sigma, (so))$. *If* E *is a Banach algebra, then* $x_{f,g,\varphi}$ *is in* $RUC(\Sigma, (so))$ *for each* $f, g\in E$ *and* $\varphi\in E^*$. *Thus* $R_2(\Sigma)\subset RUC(\Sigma, (so))$.

PROOF. For each $S\in\Sigma$, if $S_\alpha \underset{(so)}{\rightarrow} S$, then

$$\|r_{S_\alpha}x_{f,\varphi}-r_Sx_{f,\varphi}\|=\sup_{T\in\Sigma}|\varphi(TS_\alpha f)-\varphi(TSf)|$$

$$\leqq\sup_{T\in\Sigma}\|\varphi\|\,\|T\|\,\|S_\alpha f-Sf\|\leqq\|\varphi\|M\|S_\alpha f-Sf\|\to 0\,.$$

Therefore $x_{f,\varphi}\in RUC(\Sigma, (so))$ for $f\in E$ and $\varphi\in E^*$.

If E is a Banach algebra, then

$$\|r_{S_\alpha}x_{f,g,\varphi}-r_Sx_{f,g,\varphi}\|=\sup_{T\in\Sigma}|\varphi(TS_\alpha f\cdot g)-\varphi(TSf\cdot g)|$$

$$\leqq\|\varphi\|M\|S_\alpha f-Sf\|\,\|g\|\to 0\,.$$

Therefore $x_{f,g,\varphi}\in RUC(\Sigma, (so))$ for $f, g\in E$ and $\varphi\in E^*$.

Since $RUC(\Sigma, (so))$ is a closed $*$-subalgebra of $C(\Sigma, (so))$, containing **1**, which proves our proposition.

PROPOSITION 3. *For* $f\in E$ *and* $\varphi\in E^*$, $x_{f,\varphi}$ *is in* $WRUC(\Sigma, (wo))$. *Thus* $R_1(\Sigma)\subset WRUC(\Sigma, (wo))$.

PROOF. Let us define the mapping $\phi\in E^*$ by $\phi(f)=\nu(x_{f,\varphi})$ for each $f\in E$, depending on $\nu\in C(\Sigma, (wo))^*$ and $\varphi\in E^*$.

For each $S\in\Sigma$, if $S_\alpha \underset{(wo)}{\rightarrow} S$, then for each $\nu\in C(\Sigma, (wo))^*$,

$$|\nu(r_{S_\alpha}x_{f,\varphi})-\nu(r_Sx_{f,\varphi})|=|\nu(x_{S_\alpha f,\varphi})-\nu(x_{Sf,\varphi})|$$

$$=|\phi(S_\alpha f)-\phi(Sf)|\to 0\,.$$

Therefore $x_{f,\varphi}\in WRUC(\Sigma, (wo))$ for each $f\in E$ and $\varphi\in E^*$.

Since $WRUC(\Sigma, (wo))$ is a closed subspace of $C(\Sigma, (wo))$ containing **1**, closed under complex conjugation, follows this proposition.

The conditions with respect to left-translations will not work in the same way as right translations, because operator composition is written from the right.

Now, we shall state our main result about the relations of ergodicities to topological amenabilities.

THEOREM 1. *Let* E *be a Banach space and* Σ *an uniformly bounded semigroup of operators on* E. *Then* Σ *is (right) restrictedly quasi-ergodic if and only if* $R_1(\Sigma)$ *has a (right) invariant mean.*

When Σ *is a semigroup of Markov operators on* $E=C(X)$, Σ *is right restrictedly quasi-ergodic if and only if* $R_2(\Sigma)$ *has a right invariant mean.*

PROOF. The convex hull of the point evaluation functionals

$\{\delta_T : T\in\Sigma\}$[1] in $R_k(\Sigma)^*$ $(k=1, 2)$ is weak* dense in the set $\boldsymbol{M}_k$ of all means on $R_k(\Sigma)$, that is, $\boldsymbol{M}_k = w^* - \overline{co}\{\delta_T ; T\in\Sigma\}$.

Assume that Σ is restrictedly quasi-ergodic, that is, there exists a net $\{U_\alpha\}$ in $co\Sigma$ satisfying

(i) $(so)-\{(wo)-\}\lim_\alpha U_\alpha(T-I)=0$ for all $T\in\Sigma$,

(ii) $(so)-\{(wo)-\}\lim_\alpha(T-I)U_\alpha=0$ for all $T\in\Sigma$.

Let $U_\alpha=\sum_{i=1}^{n_\alpha}\lambda_{\alpha,i}T_{\alpha,i}$ where $T_{\alpha,i}\in\Sigma$, and $\sum_{i=1}^{n_\alpha}\lambda_{\alpha,i}=1$. Then we denote by δ_α the functional on $R_k(\Sigma)$ given by $\delta_\alpha(x)=\sum_{i=1}^{n_\alpha}\lambda_{\alpha,i}x(T_{\alpha,i})$ for each $x\in R_k(\Sigma)$. Clearly δ_α belongs to the convex hull of $\{\delta_T : T\in\Sigma\}$. Since $\{\delta_\alpha\}$ is in $\boldsymbol{M}_k$ and $\boldsymbol{M}_k$ is weak* compact, we can assume without loss of generality that the net $\{\delta_\alpha\}$ converges to some $\mu\in\boldsymbol{M}_k$ in the weak*-topology in $R_k(\Sigma)^*$.

From now on, we denote $\mu_T(x(T))=\mu(x)$ for $x\in R_k(\Sigma)$ and $\mu\in R_k(\Sigma)^*$.

We shall first consider $R_1(\Sigma)$. By (i) for each $f\in E$, $\varphi\in E^*$ and $S\in\Sigma$,

$$\begin{aligned}0&\leqq|\mu(r_S x_{f,\varphi})-\mu(x_{f,\varphi})|=|\mu_T(\varphi(T(S-I)f))|\\&=\lim_\alpha|(\delta_\alpha)_T(\varphi(T(S-I)f))|=\lim_\alpha|\varphi(U_\alpha(S-I)f)|=0\,.\end{aligned}$$

Hence $\mu(r_S x_{f,\varphi})=\mu(x_{f,\varphi})$. Similarly by (ii) $\mu(l_S x_{f,\varphi})=\mu(x_{f,\varphi})$ for each f, φ and S.

On the other hand it is trivial that $\mu(c\mathbf{1})=\mu(r_S c\mathbf{1})=\mu(l_S c\mathbf{1})$ for each $S\in\Sigma$ and complex c.

Let us consider the set $A_1=\{x\in R_1(\Sigma) : \mu(r_S x)=\mu(x)=\mu(l_S x)$ for all $S\in\Sigma\}$. Then A_1 contains all r_1-functions and constant functions.

Further A_1 is closed under complex conjugation. In fact,

$$\mu(r_S\bar{x})=\mu(\overline{r_S x})=\overline{\mu(r_S x)}=\overline{\mu(x)}=\mu(\bar{x})\,.$$

Similarly $\mu(l_S\bar{x})=\mu(\bar{x})$. Hence $\bar{x}\in A_1$.

Thus A_1 is a closed linear subspace of $R_1(\Sigma)$ closed under complex conjugation containing all $x_{f,\varphi}$ and $c\mathbf{1}$, which implies $A_1=R_1(\Sigma)$. Hence μ is an invariant mean on $R_1(\Sigma)$.

By examining the above proof carefully, we see that right restrictedly quasi-ergodicity of Σ implies that $R_1(\Sigma)$ has a right invariant mean.

When E is a Banach algebra, we have by (i) for each $f, g\in E$, $\varphi\in E^*$ and $S\in\Sigma$,

$$\begin{aligned}0&\leqq|\mu(r_S x_{f,g,\varphi})-\mu(x_{f,g,\varphi})|=|\mu_T(\varphi(TSf\cdot g)-\varphi(Tf\cdot g))|\\&\leqq\lim_\alpha|\varphi(U_\alpha Sf\cdot g)-\varphi(U_\alpha f\cdot g)|\\&\leqq\lim_\alpha\|\varphi\|\,\|U_\alpha(S-I)f\|\,\|g\|=0\,.\end{aligned}$$

1) We denote by δ_T the functional on $R_k(\Sigma)$ given by $\delta_T(x)=x(T)$ for all $x\in R_k(\Sigma)$.

Hence $\mu(r_S x_{f,g,\varphi})=\mu(x_{f,g,\varphi})$.

Let us next consider the set $A_2=\{x\in R_2(\Sigma): \mu(r_S x)=\mu(x)$ for all $S\in\Sigma\}$. Then A_2 contains all r_2-functions as proved above. Similarly as in the case of A_1, A_2 also contains all constant functions and is closed under complex conjugation. We shall prove that $x, y\in A_2$ imply $xy\in A_2$. It is sufficient to prove that $\mu(r_S z-z)=0$ implies $\mu(u(r_S z-z))=0$ for $u\in A_2$, since

$$\mu(r_S(xy))-\mu(xy)=\mu(r_S x(r_S y-y))+\mu((r_S x-x)y) .$$

Let $\mu(r_S z)-\mu(z)=0$. Then we have

$$0=\mu(r_S z-z)=\lim_\alpha \delta_\alpha(r_S z-z)$$

$$=\lim_\alpha \sum_{i=1}^{n_\alpha} \lambda_{\alpha,i}(r_S z(T_{\alpha,i})-z(T_{\alpha,i})) ,$$

whence

$$0\leqq|\mu(u(r_S z-z))|\leqq \lim_\alpha|\delta_\alpha(u(r_S z-z))|$$

$$=\lim_\alpha\Big|\sum_{i=1}^{n_\alpha} \lambda_{\alpha,i}u(T_{\alpha,i})(r_S z(T_{\alpha,i})-z(T_{\alpha,i}))\Big|$$

$$\leqq\|u\|\lim_\alpha\Big|\sum_{i=1}^{n_\alpha} \lambda_{\alpha,i}(r_S z(T_{\alpha,i})-z(T_{\alpha,i}))\Big|=0 .$$

Thus A_2 is a closed $*$-subalgebra of $R_2(\Sigma)$ containing all $x_{f,g,\varphi}$ and $c\mathbf{1}$, which implies $A_2=R_2(\Sigma)$. Hence μ is a right invariant mean on $R_2(\Sigma)$.

Conversely if $R_1(\Sigma)$ has a right invariant mean μ, then $\mu_T(\varphi(TSf))=\mu_T(\varphi(Tf))$ for each $f\in E$, $\varphi\in E^*$ and $S\in\Sigma$. Now we can choose a net $\{\delta_\alpha\}$ in the convex hull of $\{\delta_T: T\in\Sigma\}$ converging to μ in the weak* topology in $R_1(\Sigma)^*$. Thus for each $f\in E$, $\varphi\in E^*$ and $S\in\Sigma$,

$$\lim_\alpha(\varphi(U_\alpha(S-I)f))=\lim_\alpha(\delta_\alpha)_T(\varphi(T(S-I)f))$$

$$=\mu_T(\varphi(T(S-I)f))=\mu_T(\varphi(TSf))-\mu_T(\varphi(Tf))=0 .$$

Therefore the condition (i) is verified.

If the same μ is left invariant, then (ii) holds similarly.

For $R_2(\Sigma)$, let Σ be a semigroup of Markov operators on $E=C(X)$, then the same argument holds for $R_2(\Sigma)$ and right translations because $x_{f,e,\varphi}=x_{f,\varphi}$. Thus the theorem is proved.

The invariant mean μ on $R_1(\Sigma)$ constructed in the proof of Th. 1, really takes the following value at each r_1-function; for each $f\in E$ and $\varphi\in E^*$,

$$\mu(x_{f,\varphi})=\lim_\alpha\varphi(U_\alpha f)=(\Pi f)\varphi ,$$

where Π is the bounded linear operator of E into E^{**} and $\{U_\alpha\}$ is the net in $co\Sigma$ in Def. 3.

Even if we define $R_1(\Sigma)$ by the closed *$*$-subalgebra* of $m(\Sigma)$ generated by all r_1-functions and all constant functions, we shall see that this theorem is proved in the same way by examining the above proof carefully. Therefore Th. 1 is the generalization of a part of Th. 7.4, the equivalence between (i) of Th. 4.11 and III, in [4].

§ 3. An example of a group Γ of operators which has an invariant mean on $C(\Gamma)$ but fails to be amenable

A compact group $\mathfrak{G}$ has an invariant mean on $C(\mathfrak{G})$ given by normalized Haar measure over $\mathfrak{G}$ (see [3] § 10). On the other hand, no amenable group has a subgroup which is free on two generators (see [3] § 3 (3J')).

Let $\mathfrak{G}$ be a compact group having a subgroup which is free on two generators (for example, the (compact) real 3-dimensional orthogonal group $O(3)$ in its usual Lie group topology). We write $r(\mathfrak{G})$ for $\{r_g : g \in \mathfrak{G}\}$ the set of all right translations on $C(\mathfrak{G})$. This $r(\mathfrak{G})$ is a (wo)-(and (so)-) compact group of Markov operators on $C(\mathfrak{G})$ having a subgroup which is free on two generators, thus $r(\mathfrak{G})$ is a required group Γ.

Further $r(\mathfrak{G})$ is ergodic, for $r(\mathfrak{G})$ is restrictedly quasi-ergodic from Th. 1 and satisfies (iv) of $(\boldsymbol{E}_4)$ too.

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(i) In the above example, (so) and (wo) coincide on $r(\mathfrak{G})$. Therefore we may simply write $r(\mathfrak{G})$ for $(r(\mathfrak{G}), (so)) = (r(\mathfrak{G}), (wo))$.

(ii) If Σ is a (so) or (wo)-compact semigroup of operators, then $R_2(\Sigma) = C(\Sigma)$ by Stone-Weierstrass theorem. Hence in the above example, by (i) and Prop. 2,

$$R_2(r(\mathfrak{G})) = RUC(r(\mathfrak{G})) = WRUC(r(\mathfrak{G})) = C(r(\mathfrak{G}))\,.$$

§ 4. Σ-ideals and Σ-invariant probability measures

Throughout this section let $C(X)$ be the space of all complex (or real)-valued continuous functions on a compact Hausdorff space X and Σ a semigroup of Markov operators on $C(X)$.

For each closed ideal J in $C(X)$, there exists uniquely a compact subset S_J of X; $J = \{f \in C(X) : f(t) = 0 \text{ for all } t \in S_J\}$. This S_J is called the *support* of J.

The strong dual $C(X)^*$ of $C(X)$ is the space of complex (or real) Radon measures on X.

For $\varphi \in C(X)^*$ with $\varphi \geqq 0$,

$$J_\varphi = \{f \in C(X) : \varphi(|f|) = 0\}$$

is a closed ideal in $C(X)$ whose support is identical with the support S_φ of φ.

$\varphi \in C(X)^*$ is called a *probability measure* if $\varphi(e) = 1$ and $\varphi \geqq 0$, and the set of all probability measures is denoted by $K[C(X)]$.

DEFINITION 6. J is called a *Σ-ideal* in $C(X)$ if it is a closed proper ideal in $C(X)$ which is invariant under each $T\in\Sigma$, that is $TJ\subset J$ for each $T\in\Sigma$.

A Σ-ideal in $C(X)$ is *maximal* if it is not properly contained in any other Σ-ideal.

DEFINITION 7. A compact subset Y of X is called a *Σ-invariant set* if Tf is identically zero on Y for each $T\in\Sigma$ and $f\in C(X)$ vanishing on Y.

A Σ-invariant set Y is *minimal* if it does not properly contain any other Σ-invariant set.

Next, we shall mention without proof Proposition 4 which immediately follows from these definitions.

PROPOSITION 4. (i) *A Σ-invariant set is identical with the support of a Σ-deal in $C(X)$.*

(ii) *For two closed ideals J_1, J_2 and their supports S_{J_1}, S_{J_2}, $J_1\supset J_2$ if and only if $S_{J_1}\subset S_{J_2}$. Hence a Σ-ideal J is maximal if and only if its support S_J is a minimal Σ-invariant set.*

(iii) *If J_1, J_2 are distinct maximal Σ-ideals, then $S_{J_1}\cap S_{J_2}=\emptyset$.*

$K[C(X)]_\Sigma$ denotes the set of all $\varphi\in K[C(X)]$ satisfying $\varphi(Tf)=\varphi(f)$ for all $T\in\Sigma$ and $f\in C(X)$. In general, $K[C(X)]_\Sigma$ may be empty.

The following theorem is an extension of Th. 1 in [8].

THEOREM 2. *Let Σ be a right quasi-ergodic semigroup of Markov operators on $C(X)$.*

Then

(i) $K[C(X)]_\Sigma\neq\emptyset$,

and,

(ii) *every maximal Σ-ideal is of the form $J_\varphi=\{f\in C(X):\varphi(|f|)=0\}$ for a suitable $\varphi\in K[C(X)]_\Sigma$.*

PROOF. (i) Since Σ is right quasi-ergodic, there exists a bounded linear operator Π of E into E^{**} satisfying that $\Pi T=\Pi$ for all $T\in\Sigma$ and for each $f\in E$, Πf is contained in $w^*\text{-}\overline{co}\Sigma f$. (see Def. 2)

Let $\varphi\in K[C(X)]$ be fixed. Define $\varphi_0\in C(X)^*$ by $\varphi_0(f)=(\Pi f)\varphi$ for each $f\in C(X)$. This φ_0 is the required one.

(ii) Let $\{U_\alpha\}$ be a net satisfying the conditions (i) and (iii) in $(\boldsymbol{E}_4)$ by right quasi-ergodicity of Σ. Since $TJ\subset J$ for each $T\in\Sigma$ and so $U_\alpha J\subset J$, each $T\in\Sigma$ and U_α induce naturally Markov operators T_J and $U_{\alpha,J}$ on $C(X)/J$ which also satisfy the conditions (i) and (iii) in $(\boldsymbol{E}_4)$. Hence the induced semigroup $\Sigma_J=\{T_J:T\in\Sigma\}$ on $C(X)/J$ is right quasi-ergodic. Thus it follows from (i) and the fact that $C(X)/J$ may be identified with $C(S_J)$ as a Banach algebra, that there exists a Σ-invariant probability measure $\hat{\varphi}_0$ on S_J. The remaining part of the proof goes along the same way as that of Th. 1 in [8].

To get the same conclusion in the above theorem, Takahashi [8] assumes amenability of Σ, necessary to use Day's fixed point theorem. When I proved this conclusion at the first time, I also assumed that there existed a right invariant mean on $RUC(\Sigma, (so))$ or on $WRUC(\Sigma, (wo))$ necessary to use Mitchell's fixed point theorems, generalizations of Day's fixed point theorem. However Th. 2 states that the same conclusion follows from the apparently less stronger condition, right quasi-ergodicity of Σ.

Corresponding to the above theorem, we have the following corollary concerning the minimal Σ-invariant sets.

COROLLARY 1. *Let Σ be a right quasi-ergodic semigroup of Markov operators on $C(X)$. Then every minimal Σ-invariant set is the support of a suitable $\varphi \in K[C(X)]_\Sigma$.*

PROOF. This corollary follows from Th. 2 and Prop. 4.

The following theorem is an extension of Th. 4 in [8].

THEOREM 3. *Let Σ be an ergodic semigroup of Markov operators on $C(X)$, then $\varphi \rightarrow J_\varphi$ is a bijection of the set of extreme points of $K[C(X)]_\Sigma$ onto the family of all maximal Σ-ideals. Moreover, every Σ-ideal of the form J_φ ($\varphi \in K[C(X)]_\Sigma$) is the intersection of all maximal Σ-ideals containing it.*

PROOF. This theorem follows from $(\boldsymbol{E}_3)$, Th. 2 and the proofs of Th. 2 in [7] and Th. 4 in [8].

COROLLARY 2. *Let Σ be an ergodic semigroup of Markov operators on $C(X)$. Then for a compact subset Y of X, Y is a minimal Σ-invariant set if and only if it is the support of an extreme point of $K[C(X)]_\Sigma$. Moreover, for every $\varphi \in K[C(X)]_\Sigma$, its support S_φ is the closure of the union of all minimal Σ-invariant set contained in S_φ.*

PROOF. This corollary follows from Th. 3 and Prop. 4.

§ 5. Extreme points of real-valued Σ-invariant probability measures

Throughout this section, let $C(X)$ be the space of all real-valued continuous functions on X.

We have already investigated that $R_2(\Sigma)$ has a right invariant mean if and only if Σ is right restrictedly quasi-ergodic (see Th. 1). Now we shall use this fact.

THEOREM 4. *Let Σ be a right restrictedly quasi-ergodic semigroup of Markov operators on $C(X)$. Then for $\varphi \in K[C(X)]_\Sigma$ the following conditions are equivalent.*

(a) *φ is an extreme point of $K[C(X)]_\Sigma$,*

(b) *For each right invariant mean μ on $R_2(\Sigma)$,*

$$\mu_T(\varphi(Tf\cdot g)) = \varphi(f)\varphi(g) \text{ for all } f, g \in C(X).$$

(c) *For some mean μ on $R_2(\Sigma)$,*

$$\mu_T(\varphi(Tf\cdot g)) = \varphi(f)\varphi(g) \text{ for all } f, g \in C(X).$$

(d) *For each $f \in C(X)$ there exists a sequence $\{U_n\} \subset co\Sigma$ such that*

$$\lim_{n\to\infty} \varphi(U_n f\cdot U_n g) = \varphi(f)\varphi(g) \text{ for all } g \in C(X).$$

PROOF. Since there exists a right invariant mean on $R_2(\Sigma)$ from Th. 1, (a)⇒(b) can be proved in the same way as (a)⇒(b) of Th. 2.1 in [2].

(b)⇒(c). Trivial.

Since the convex hull of the point evaluation functionals in $R_2(\Sigma)^*$ is weak* dense in the set of all means on $R_2(\Sigma)$ (see the proof of Th. 1), (c)⇒(d) can be proved in the same way as (c)⇒(d) of Th. 2.1 in [2].

We can prove (d)⇒(a) in the same way as (d)⇒(a) of Th. 2.1 in [2].

COROLLARY 3. *Let Σ be an ergodic semigroup of Markov operators on $C(X)$, then for $\varphi \in K[C(X)]_\Sigma$ the following conditions are equivalent;*

(i) *φ is an extreme point of $K[C(X)]_\Sigma$.*

(ii) *For any $f, g \in C(X)$, $\varphi(f\cdot Pg) = \varphi(f)\varphi(g)$.*

(iii) *For any $f, g \in C(X)$, $\varphi(Pf\cdot Pg) = \varphi(f)\varphi(g)$.*

(iv) *For any $f, g \in \mathfrak{F}(\Sigma) = \{f \in C(X) : Tf = f \text{ for all } T \in \Sigma\}$,*

$$\varphi(f\cdot g) = \varphi(f)\varphi(g).$$

Here P is the projection of $C(X)$ onto $\mathfrak{F}(\Sigma)$ in ($\boldsymbol{E}_2$).

PROOF. If Σ is ergodic, then Σ is restrictedly quasi-ergodic, and so the assumption of Th. 4 is satisfied.

Because Σ is ergodic, $T^*\varphi = \varphi$ for all $T \in \Sigma$ if and only if $P^*\varphi = \varphi$ (see ($\boldsymbol{E}_3$)), and $\{P\}$ is a one-element semigroup, hence (b), (d) in Th. 4 applied to $\{P\}$ imply (ii), (iii). (iii)⇔(iv) is evident.

Since an (AM)-space whose cone has an interior point e such that the unit ball is the set $\{f: -e \leqq f \leqq e\}$ may be regarded as $C(X)$ by Kakutani representation theorem, Th. 4 and Cor. 3 become extensions of Th. 2.1 and Cor. 5.2 in [2].

If $\mathfrak{F}(\Sigma)$ is a subalgebra of $C(X)$, we obtain the following bijections.

THEOREM 5. *Let Σ be an ergodic semigroup of Markov operators on $C(X)$ and the closed subspace $\mathfrak{F}(\Sigma)$ of $C(X)$ be a subalgebra of $C(X)$, where $\mathfrak{F}(\Sigma)$ can be represented by $C(Z)$ where Z is a compact Hausdorff space. Then there exist the following bijections.*

(1) *There exists a bijection of the set of extreme points of $K[C(X)]_\Sigma$ onto the set of extreme points of $K[\mathfrak{F}(\Sigma)]=\{\phi\in\mathfrak{F}(\Sigma)^* : \phi(e)=1, \phi\geqq 0\}$ and this bijection is given by the restriction to $\mathfrak{F}(\Sigma)$.*

(2) *There exists a bijection of the family of all maximal Σ-ideals in $C(X)$ onto the family of all maximal ideals in $\mathfrak{F}(\Sigma)$.*

(3) *There exists a bijection of the family of all minimal Σ-invariant sets onto Z.*

PROOF. (1) is obtained by Cor. 3 (i)$\Leftrightarrow$(iv), and the fact that extreme points of $K[\mathfrak{F}(\Sigma)]$ are identical with multiplicative functionals on $\mathfrak{F}(\Sigma)$. We apply Th. 3 to $C(X)$ and Σ, and to $\mathfrak{F}(\Sigma)\cong C(Z)$ and the one point semigroup of an identity operator. Therefore, (2) is obtained from (1) and Th. 3. Similarly (3) is obtained from (1) and Cor. 2.

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