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Concentration of Suspended Dusts and Amount of Sedimented Dusts emitted from a Stack in the Atmosphere

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Introduction. Dusts suspended in the atmosphere and those sedimented in leeward areas from industrial stacks are the most important factors for public nuisances. So we made formulae for the concentration of suspended dusts and the amount of sedimented ones.

2-dimensional case. We assume that dusts have uniformly the same radius r and the same density ρ, so they have the same falling velocity v_g. The vertical (z-direction) distribution of concentration of suspended dusts can be expressed by the next equation:

$$\frac{\partial C}{\partial t}=v_g\frac{\partial C}{\partial z}+\frac{\partial}{\partial z}\left(kz\frac{\partial C}{\partial z}\right) \qquad 1),$$

where C is the concentration, t is time, and kz is the vertical diffusion coefficient, so k is a proportional constant, and the ground surface is $z=0$. [1]

When we put $C\propto\exp(-\lambda^2 t)\varphi(z)$, we get

$$\frac{d^2\varphi}{dz^2}+\frac{(1+v_g/k)}{z}\frac{d\varphi}{dz}+\frac{\lambda^2}{kz}\varphi=0 \qquad 2).$$

Putting

$$1+\frac{v_g}{k}=p\,,\quad q=\frac{\lambda^2}{k} \qquad 3),$$

then we get

$$\frac{d^2\varphi}{dz^2}+\frac{1+p}{z}\frac{d\varphi}{dz}+q\frac{\varphi}{z}=0 \qquad 4).$$

The particular solution of this equation is

$$\varphi=z^{-\frac{p}{2}}Z_{\pm p}\left(\frac{2\lambda}{\sqrt{k}}\sqrt{z}\right) \qquad 5),$$

where $Z_{\pm p}$ is the Bessel function of the order $\pm p$. As $0\leqq z<\infty$, we can adopt only J_p. So we get

$$C=\int_0^\infty e^{-\lambda^2 t} z^{-\frac{p}{2}} J_p\left(2\sqrt{\frac{z}{k}}\lambda\right) f(\lambda) d\lambda \tag{6}$$

We assume that, when $t=0$, an instantaneous point source is at the height h, so

$$C=\delta(z-h)\,, \quad t=0 \tag{7}$$

and

$$\delta(z-h)=\int_0^\infty z^{-\frac{p}{2}} J_0\left(\frac{2\lambda}{\sqrt{k}}\sqrt{z}\right) f(\lambda) d\lambda \tag{8}$$

Multiplying $z^{\frac{p}{2}} J_p(a\sqrt{z})$ on both sides, and considering that

$$\sqrt{\alpha\beta}\int_0^\infty J_p(\alpha\zeta) J_p(\beta\zeta)\zeta d\zeta=\delta(\alpha-\beta) \tag{9}$$

namely,

$$\int_0^\infty J_p(\alpha\zeta) J_p(\beta\zeta)\zeta d\zeta=\frac{\delta(\alpha-\beta)}{\sqrt{\alpha\beta}} \tag{10}$$

we get after integration

$$h^{\frac{p}{2}} J_p(a\sqrt{h})=\int_0^\infty f(\lambda) d\lambda \int_0^\infty J_p(a\sqrt{z})\, J_p\left(\frac{2\lambda}{\sqrt{k}}\sqrt{z}\right) dz$$

$$=\int_0^\infty f(\lambda) 2\delta\left(a-\frac{2\lambda}{\sqrt{k}}\right)\frac{1}{a}\, d\lambda=\int_0^\infty f(\lambda)\frac{\sqrt{k}}{a}\,\delta\left(\lambda-\frac{a\sqrt{k}}{2}\right)$$

$$=\frac{\sqrt{k}}{a} f\left(\frac{a\sqrt{k}}{2}\right) \tag{11}$$

So we get

$$f\left(\frac{a\sqrt{k}}{2}\right)=\frac{a}{\sqrt{k}} h^{\frac{p}{2}} J_p(a\sqrt{h}) \tag{12}$$

and putting $\frac{a\sqrt{k}}{2}=\lambda$, we get

$$f(\lambda)=\frac{a}{\sqrt{k}} h^{\frac{p}{2}} J_p\left(\frac{2\lambda}{\sqrt{k}} h^{\frac{1}{2}}\right)=\frac{2\lambda}{k} h^{\frac{p}{2}} J_p\left(\frac{2\lambda}{\sqrt{k}} h^{\frac{1}{2}}\right) \tag{13}$$

Then we obtain

$$C=\int_0^\infty e^{-\lambda^2 t} z^{-\frac{p}{2}} J_p\left(2\lambda\sqrt{\frac{z}{k}}\right)\frac{2\lambda}{k} h^{\frac{p}{2}} J_p\left(\frac{2\lambda}{\sqrt{k}}\sqrt{h}\right) d\lambda$$

$$=\frac{1}{kt}\left(\frac{h}{z}\right)^{\frac{p}{2}} e^{-\frac{z+h}{kt}} I_p\left(\frac{2\sqrt{hz}}{kt}\right) \tag{14}$$

where I_p is the modified 1st kind Bessel function of the order p.

3-dimensional case. For 3-dimensional diffusion from a continuous point source, in the case when the diffusion coefficients depend on travelling time, we obtain the next formula for concentration. Detailed explanations can be found in the former paper. [2]

$$\left.\begin{aligned} C&=\frac{q}{u}\frac{e^{-\frac{y^2}{A}}}{\sqrt{A\pi}}\frac{e^{-\frac{h+z}{B}}}{B}\left(\frac{h}{z}\right)^{\frac{p}{2}}I_p\left(\frac{2\sqrt{h}}{B}\right)\\ k&=q_B\varphi_B u(1-e^{-\varphi_B x})\\ p&=\frac{v_g}{k}=\frac{v_g}{q_B\varphi_B u(1-e^{-\varphi_B x})}\\ A&=q_A(\varphi_A x+e^{-\varphi_A x}-1)\\ B&=q_B(\varphi_B x+e^{-\varphi_B x}-1) \end{aligned}\right\} \qquad 15)$$

where q is the source intensity, namely the emitting amount of dusts per unit time, u is the wind speed at the height of the source, and q_A, q_B, φ_A and φ_B are diffusion parameters, the values of which are shown in Table 1. [3]

As when z is small,

$$I_p(\xi)\sim\frac{1}{\Gamma(p+1)}\left(\frac{\xi}{2}\right)^p \qquad 16)$$

so we get

$$\left(\frac{h}{z}\right)^{\frac{p}{2}}I_p\left(\frac{2\sqrt{hz}}{B}\right)=\left(\frac{h}{z}\right)^{\frac{p}{2}}\frac{1}{\Gamma(p+1)}\left(\frac{\sqrt{hz}}{B}\right)^p=\frac{h^p}{B^p}\frac{1}{\Gamma(p+1)} \qquad 17)$$

then the ground level concentration is expressed by

$$C=\frac{q}{u}\frac{e^{-\frac{y^2}{A}}}{\sqrt{A\pi}}\frac{e^{-\frac{h}{B}}}{B}\left(\frac{h}{B}\right)^p\frac{1}{\Gamma(p+1)} \qquad 18)$$

Amount of sedimented dusts. The amount of sedimented dusts M_{y_1} which pass through the plane $y_1-\frac{\Delta y}{2}\sim y_1+\frac{\Delta y}{2}$, $0\leqq z<\infty$ at $x=x_1$, per unit time is expressed by

$$\begin{aligned} M_{y_1}&=\int_{y_1-\frac{\Delta y}{2}}^{y_1+\frac{\Delta y}{2}}dy\int_0^\infty Cudz\\ &=q\frac{e^{-\frac{y_1^2}{A_1}}}{\sqrt{A_1\pi}}\Delta y\, e^{-\frac{h}{B_1}}h^{\frac{p}{2}}\int_0^\infty\frac{e^{-\frac{z}{B_1}}}{B_1}z^{-\frac{p}{2}}I_p\left(\frac{2\sqrt{hz}}{B_1}\right) \end{aligned} \qquad 19)$$

where A_1 and B_1 are the values of A and B at $x=x_1$.

If we put $\sqrt{z}=\xi$, we get

$$M_{y_1}=q\frac{e^{-\frac{y_1^2}{A_1}}}{\sqrt{A_1\pi}}\Delta y\frac{e^{-\frac{h}{B_1}}h^{\frac{p}{2}}}{B_1}\int_0^\infty e^{-\frac{\xi^2}{B_1}}\xi^{-p}I_p\left(\frac{2\sqrt{h}}{B_1}\xi\right)2\xi d\xi \qquad 20)$$

Table 1. Values of q_A, q_B, φ_A and φ_B for several values of ζ and h. In this table, 4.78 (−2), for example, means 4.78×10^{-2}.

ζ	h	φ_A	$\sqrt{q_A}$	φ_B	q_B
0.4	0.5	4.78 (−2)	1.29 (1)	4.20 (−2)	3.50 (−1)
	10	4.78 (−2)	1.29 (1)	4.60 (−2)	2.93 (−1)
	20	4.78 (−2)	1.29 (1)	4.71 (−2)	2.86 (−1)
	30	4.78 (−2)	1.29 (1)	4.77 (−2)	2.83 (−1)
	50	4.78 (−2)	1.29 (1)	4.80 (−2)	2.78 (−1)
	70	4.78 (−2)	1.29 (1)	4.81 (−2)	2.75 (−1)
	100	4.78 (−2)	1.29 (1)	4.82 (−2)	2.70 (−1)
	150	4.78 (−2)	1.29 (1)	4.83 (−2)	2.69 (−1)
	200	4.78 (−2)	1.29 (1)	4.84 (−2)	2.67 (−1)
	300	4.78 (−2)	1.29 (1)	4.84 (−2)	2.64 (−1)
0	0.5	1.48 (−2)	4.72 (1)	1.10 (−2)	5.30
	10	1.09 (−2)	6.60 (1)	2.46 (−2)	1.02
	20	1.01 (−2)	7.18 (1)	3.00 (−2)	7.00 (−1)
	30	9.7 (−3)	7.50 (1)	3.29 (−2)	5.65 (−1)
	50	9.2 (−3)	7.95 (1)	3.79 (−2)	4.41 (−1)
	70	8.9 (−3)	8.20 (1)	4.02 (−2)	3.80 (−1)
	100	8.6 (−3)	8.60 (1)	4.27 (−2)	3.39 (−1)
	150	8.3 (−3)	8.91 (1)	4.40 (−2)	3.08 (−1)
	200	8.0 (−3)	9.21 (1)	4.63 (−2)	2.93 (−1)
	300	7.7 (−3)	8.80 (1)	4.78 (−2)	2.78 (−1)
−0.1	0.5	4.50 (−3)	2.30 (2)	4.25 (−3)	3.48 (1)
	10	2.12 (−3)	4.82 (2)	1.48 (−2)	2.87
	20	1.80 (−3)	5.70 (2)	1.98 (−2)	1.61
	30	1.61 (−3)	6.33 (2)	2.34 (−2)	1.14
	50	1.40 (−3)	7.20 (2)	2.87 (−2)	7.55 (−1)
	70	1.29 (−3)	7.80 (2)	3.30 (−2)	5.78 (−1)
	100	1.17 (−3)	8.65 (2)	3.70 (−2)	4.59 (−1)
	150	1.06 (−3)	9.30 (2)	4.20 (−2)	3.57 (−1)
	200	9.8 (−4)	1.03 (3)	4.44 (−2)	3.18 (−1)
	300	8.8 (−4)	1.11 (3)	4.78 (−2)	2.79 (−1)
−0.2	0.5	1.12 (−3)	8.40 (2)	1.30 (−3)	3.73 (2)
	10	2.52 (−4)	3.75 (3)	7.20 (−3)	1.18 (1)
	20	1.78 (−4)	5.25 (3)	1.10 (−2)	5.19
	30	1.44 (−4)	6.48 (3)	1.40 (−2)	3.21
	50	1.11 (−4)	8.40 (3)	1.93 (−2)	1.69
	70	9.50 (−5)	1.00 (4)	2.38 (−2)	1.11
	100	7.90 (−5)	1.19 (4)	2.95 (−2)	7.22 (−1)
	150	6.50 (−5)	1.48 (4)	3.74 (−2)	4.50 (−1)
	200	5.60 (−5)	1.68 (4)	4.28 (−2)	3.41 (−1)
	300	4.54 (−5)	2.07 (4)	4.78 (−2)	2.94 (−1)

As

$$J=\int_0^\infty e^{-\frac{\xi^2}{B_1}}\xi^{-p+1}I_p\left(\frac{2\sqrt{h}}{z}\xi\right)d\xi=\frac{\Gamma(1)\left(\frac{2\sqrt{h}}{B_1}\right)^p}{2^{1+p}\Gamma(1+p)}\frac{\Gamma(1+p)}{\Gamma(1)}\sum_{n=0}^{\infty}\frac{\Gamma(1+n)}{\Gamma(1+p+n)}\frac{\left(\frac{h}{B_1}\right)^n}{n!}$$

$$=\frac{h^{\frac{p}{2}}}{2B_1^{p-1}}\sum_{n=0}^{\infty}\frac{1}{\Gamma(1+p+n)}\left(\frac{h}{B_1}\right)^n \qquad 21),$$

we get

$$M_{y_1}=2q\frac{e^{-\frac{y_1^2}{A_1}}}{\sqrt{A_1\pi}}\Delta y\frac{e^{-\frac{h}{B_1}}h^{\frac{p}{2}}}{B_1}\frac{h^{\frac{p}{2}}}{2B_1^{p-1}}\sum_{n=0}^{\infty}\frac{1}{\Gamma(1+p+n)}\left(\frac{h}{B_1}\right)^n$$

$$=q\frac{e^{-\frac{y_1^2}{A_1}}}{\sqrt{A_1\pi}}\Delta y\, e^{-\frac{h}{B_1}}\sum_{n=0}^{\infty}\frac{1}{\Gamma(1+p+n)}\left(\frac{h}{B_1}\right)^{n+p} \qquad 22).$$

The amount of deposited dusts ΔM_y between 2 planes: $x=x_1$, $y_1-\frac{\Delta y}{2}\sim y_1+\frac{\Delta y}{2}$, $0\leqq z<\infty$ and $x=x_1+\Delta x$, $y_1-\frac{\Delta y}{2}\sim y_1+\frac{\Delta y}{2}$, $0\leqq z<\infty$, is given by

$$\Delta M_{y_1}=\frac{\partial M_{y_1}}{\partial x}\Delta x=\left(\frac{\partial M_{y_1}}{\partial A_1}\frac{\partial A_1}{\partial x}+\frac{\partial M_{y_1}}{\partial B_1}\frac{\partial B_1}{\partial x}\right)\Delta x \qquad 23).$$

In the present case considered, the change of ΔM_y along x caused by the lateral diffusion is smaller than that caused by sedimentation, so we can approximately assume the next equation

$$\Delta M_{y_1}=\frac{\partial M_{y_1}}{\partial B_1}\frac{\partial B_1}{\partial x}\Delta x \qquad 24),$$

as

$$\frac{\partial M_{y_1}}{\partial B_1}=q\frac{e^{-\frac{y_1^2}{A_1}}}{\sqrt{A_1\pi}}\Delta y\sum_{n=0}^{\infty}\frac{e^{-\frac{h}{B_1}}}{\Gamma(1+p+n)}\frac{1}{h}\left(\frac{h}{B_1}\right)^{p+1+n}\left[\frac{h}{B_1}-(n+p)\right] \qquad 25),$$

$$\frac{\partial B_1}{\partial x}=q_B\varphi_B(1-e^{-\varphi_B x})=\frac{k}{u} \qquad 26),$$

we get

$$\Delta M_{y_1}=q\frac{e^{-\frac{y_1^2}{A_1}}}{\sqrt{A_1\pi}}\frac{e^{-\frac{h}{B_1}}}{h}\frac{k}{u}\sum_{n=0}^{\infty}\left(\frac{h}{B_1}\right)^{p+1+n}\frac{\left[\frac{h}{B_1}-(n+p)\right]}{\Gamma(1+p+n)}\Delta x\Delta y \qquad 27).$$

Putting $h/B_1=\mu$, we get

$$\Delta M_{y_1}=\frac{q}{u}\frac{e^{-\frac{y_1^2}{A_1}}}{\sqrt{A_1\pi}}\frac{v_g e^{-\frac{h}{B_1}}}{h}\sum_{n=0}^{\infty}\frac{\mu^{p+1+n}[\mu-(n+p)]}{p\Gamma(p+1+n)}\Delta x\Delta y \qquad 28).$$

Hence

$$\sum_{n=0}^{\infty}\frac{\mu^{p+1+n}[\mu-(n+p)]}{p\Gamma(p+1+n)}=\frac{\mu^{p+1}}{p}\sum_{n=0}^{\infty}\frac{\mu^n[\mu-(n+p)]}{\Gamma(p+1+n)}$$

$$=\frac{\mu^{p+1}}{p}\left[\frac{\mu}{\Gamma(p+1)}-\frac{p}{\Gamma(p+1)}+\frac{\mu^2}{\Gamma(p+2)}-\frac{(p+1)\mu}{\Gamma(p+2)}+\cdots\right]$$

$$=\frac{\mu^{p+1}}{p}\left[\frac{\mu}{\Gamma(p+1)}-\frac{p}{\Gamma(p+1)}+\frac{\mu^2}{\Gamma(p+2)}-\frac{\mu}{\Gamma(p+1)}+\cdots\right]$$

$$=\frac{-\mu^{p+1}}{\Gamma(p+1)} \qquad 29),$$

so we get

$$|\Delta M_{y_1}|=\frac{q}{u}\frac{v_g}{h}\frac{e^{-\frac{y_1^2}{A_1}}}{\sqrt{A_1\pi}}\frac{e^{-\mu}\mu^{p+1}}{\Gamma(p+1)}\Delta x\Delta y$$

$$=\frac{qv_g}{u}\frac{e^{-\frac{y_1^2}{A_1}}}{\sqrt{A_1\pi}}\frac{e^{-\frac{h}{B_1}}}{B_1}\frac{\left(\frac{h}{B_1}\right)^p}{\Gamma(p+1)}\Delta x\Delta y=v_g C_{z=0}\Delta x\Delta y \qquad 30).$$

Therefore, the deposited amount, M, per unit area centered at (x_1, y_1) per unit time is given by

$$M=\frac{|\Delta M_{y_1}|}{\Delta x\Delta y}=\frac{qv_g}{u}\frac{e^{-\frac{y_1^2}{A_1}}}{\sqrt{A_1\pi}}\frac{e^{-\frac{h}{B_1}}}{B_1}\frac{\left(\frac{h}{B_1}\right)^p}{\Gamma(p+1)}=v_g C_{z=0} \qquad 31).$$

So the distribution of M along x is similar to that of the ground level concentration. It should be noticed, however, that C contains p, and p contains v_g and u.

Numerical calculation. The dust particle, assumed as spherical form, has falling velocity v_g.

$$v_g=\frac{4}{3}\frac{\pi r^3\rho g}{6\pi\eta r}=\frac{2}{9}\frac{r^2\rho}{\eta}g \qquad 32),$$

and if it is put in an air stream whose velocity U, it gains velocity $u(t)$ by air force, and $u(t)$ is expressed by the next equation:

$$u(t)=U(1-e^{-\frac{t}{T}}) \qquad 33),$$

$$T=\frac{4\pi r^3\rho}{3\times 6\pi\eta r}=\frac{2}{9}\frac{r^2\rho}{\eta}=\frac{v_g}{g} \qquad 34),$$

where ρ is the particle density, r is the radius and η is the viscosity coefficient of air. The relations between ρ, r, v_g and T are shown in Table 2. From this table, we can notice that, even when the particle radius is 30μ and the density is considerably large as 5, the value of T is smaller than 0.1 sec, so the dust particles $30\sim50\mu$ in radius can be deemed as they generally follow the turbulent motion of air fairly well, though they have falling velocities.

Table 2. Values of v_g and T for several values of ρ and r.
In this table, 1.20 (−2), for example, means 1.20×10^{-2}.

ρ	r (μ)	1	2	5	10	20	50	100
1	v_g (cm/s)	1. 20 (−2)	4. 80 (−2)	3. 00 (−4)	1. 20	4. 80	3. 00 (1)	1. 20 (2)
	T (s)	1. 23 (−5)	4. 90 (−5)	3. 07 (−4)	1. 23 (−3)	4. 90 (−3)	3. 07 (−2)	1. 23 (−1)
2	v_g (cm/s)	2. 40 (−2)	9. 60 (−2)	6. 00 (−1)	2. 40	9. 60	6. 00 (1)	2. 40 (2)
	T (s)	2. 46 (−5)	9. 80 (−5)	6. 14 (−4)	2. 46 (−3)	9. 80 (−3)	6. 14 (−2)	2. 46 (−1)
3	v_g (cm/s)	3. 60 (−2)	1. 44 (· ·1)	9. 00 (−1)	3. 60	1. 44 (1)	9. 00 (1)	3. 60 (2)
	T (s)	3. 68 (· ·5)	1. 56 (−4)	9. 21 (−4)	3. 68 (−3)	1. 56 (−2)	9. 21 (-2)	3. 68 (−1)
5	v_g (cm/s)	6. 00 (−2)	2. 40 (−1)	1. 50	6. 00	2. 40 (1)	1. 50 (2)	6. 00 (2)
	T (s)	6. 15 (−5)	2. 49 (−4)	1. 54 (−3)	6. 15 (−2)	2. 49 (−2)	1. 54 (- 1)	6. 15 (−1)

We calculated the amounts of deposited dusts along the line leeward from the stack, $M/\left(\frac{qv_g}{u}\right)$, in three stability conditions, namely stable ($\zeta=0.4$), neutral ($\zeta=0$) and unstable ($\zeta=-0.2$), and for several values of p. The stack height is assumed as 100 m. The value of k are generally deemed as constant when $h \doteqdot 100$ m and $x \geqq 200$ m, so the values of p only depend on v_g and u, but they are independent of x.

The results are shown in Fig. 1∼Fig. 3. The distributions of ground level concentration are similar to these figures, but the ordinates show the values of $C/(q/u)$. The calculations are carried out by using OKI-MINITAC 7000, Ochanomizu University.

Literature

[1] Sakagami, J.: On the Turbulent Diffusion in the Atmosphere near the Ground, 1954, Natural Science Report, Ochanomizu Univ., 5 (1), pp. 79–91.

[2] Sakagami, J.: On the Concentrations of Matter Emitted from a Source in the Atmosphere when a "Flux-Zero" Level Exists above the Source, 1971, ditto, 22 (2), pp. 153–176.

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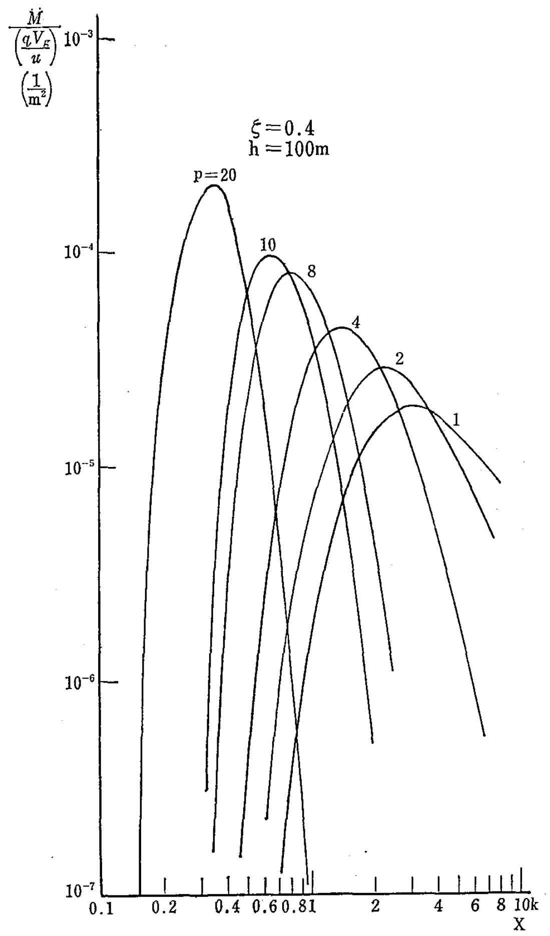

Fig. 1. Distributions of M along x for several values of p in stable condition ($\zeta=0.4$).

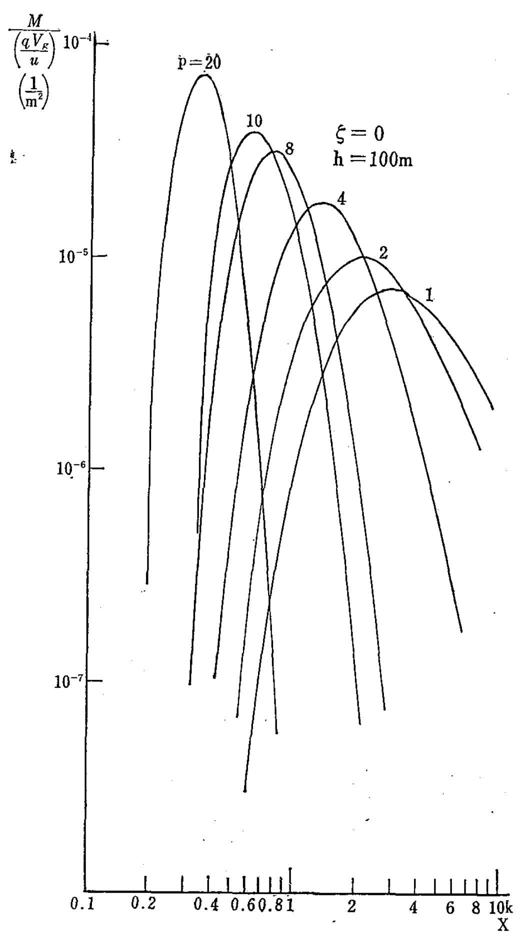

Fig. 2. Distributions of M along x for several values of p in stable condition ($\zeta=0$).

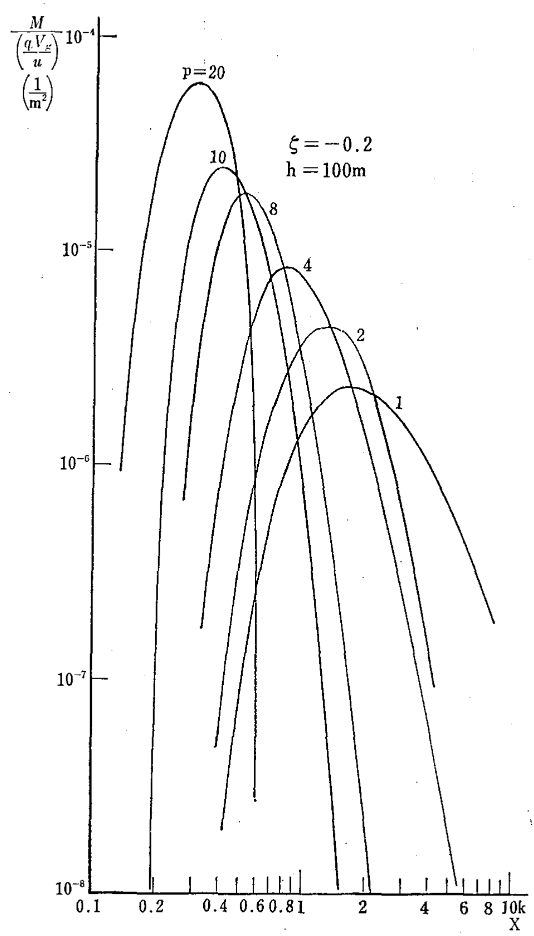

Fig. 3. Distributions of M along x for several values of p in unstable condition ($\zeta=-0.2$).