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Simplexes on a Locally Compact Space

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§1. Introduction. The object composed of a compact Hausdorff space X and a convex cone C of continous functions on X has been studied in a large number of works, particularly in [2], [1], [8], [4], [6], [3]. In those papers, to the cone C is associated a preorder relation, denoted by $\ll$, on the set of non-negative measures on X, and a subset of X, called the Choquet boundary. Further, the existence and a characterization of maximal measures with respect to the preorder relation $\ll$ are discussed. Assuming C containes sufficiently many functions, (X, C) is called a simplex if for any $x \in X$ a maximal measure μ satisfying $\varepsilon_x \ll \mu$ is unique. The necessary or sufficient conditions for (X, C) to be a simplex are discussed.

In this paper we shall obtain similar results in the case of a locally compact, σ-compact Hausdorff space Ω applying the theory of what is so-called an adapted cone.

§2. Preliminaries. Throughout this paper, Ω will be a locally compact, σ-compact Hausdorff space. We denote by $C(\Omega)$ the set of all continous real-valued functions on Ω and by $C^+(\Omega)$ the set of all non-negative functions of $C(\Omega)$. Let P be an adapted convex cone of $C^+(\Omega)$. P is called adapted if P satisfies the following two conditions (i) and (ii);

(i) for any $x \in \Omega$ there exists $u \in P$ such that $u(x) > 0$;

(ii) for any $u \in P$, there exists $v \in P$ such that for any $\varepsilon > 0$ the set $\{x \in \Omega\,;\, u(x) \geqq \varepsilon v(x)\}$ is compact.

Let us put for $u \in P$,

$$H_u = \{f \in C(\Omega)\,;\, \exists \lambda > 0,\ |f| \leqq \lambda u\}.$$

Then H_g is a Banach space with norm

$$||f||_u = \{\inf \lambda\,;\ |f| \leqq \lambda u\}.$$

We shall assign to the vector space $H_p = \bigcup_{g \in p} H_g$ the topology of inductive limits of Banach spaces $\{H_g\}_{g \in p}$.

Let μ be a Radon measure on Ω. We call μ P-integrable if $|\mu|(f)<+\infty$ for any $f\in P$. We denote by $\mathfrak{M}_p$ the space of all P-integrable Radon measures on Ω. Any positive linear form on H_p is represented by a measure of $\mathfrak{M}_p^+$. $\mathfrak{M}_p$ is dual of H_p and $\mathfrak{M}_p=\mathfrak{M}_p^+-\mathfrak{M}_p^+$. [7], [9]

§ 3. **Extremal measures.** Let C be a cone with $P\subset C\subset H_p$. For any two measures μ, $\nu\in\mathfrak{M}_p^+$ we denote by

$$\mu \underset{(C)}{\ll} \nu \text{ or simply } \mu\ll\nu$$

if $\nu(s)\leqq\mu(s)$ for any $s\in C$. A measure μ on Ω is called C-extremal (or simply extremal) if for any measure $\nu\in\mathfrak{M}_p^+$ with $\mu\ll\nu$ we have

$$\nu(s)=\mu(s)$$

for any $s\in C$. Using Zorn's lemma, for any $\mu\in\mathfrak{M}_p^+$, we may find an extremal measure $\nu\in\mathfrak{M}_p^+$ such that $\mu\ll\nu$.

We shall say that an extended real-valued function f is upper (resp. lower) P-bounded if there exists $u\in P$ satisfying $f\leqq u$ (resp. $-u\leqq f$).

A function f on Ω is called C-concave or simply concave if for any $x\in\Omega$ and any measure $\mu\in\mathfrak{M}_p^+$ with $\varepsilon_x\ll\mu$, we have

$$\mu(f)\leqq f(x).$$

We denote by $\hat{C}$ the set of all lower P-bounded lower semicontinous concave functions on Ω.

A set $\mathfrak{F}$ of extended real-valued functions on Ω is called min-stable if for any functions f_1, f_2 from $\mathfrak{F}$ the function $\min(f_1, f_2)$ belongs also to $\mathfrak{F}$.

Let μ be a measure of $\mathfrak{M}_p^+$ and S be a closed subset of Ω. For any function f defined on a set containing S we denote by

$$Q_\mu^{S,C}(f)=Q_\mu^C(f)=Q_\mu^S(f)=Q_\mu(f)$$

the extended real number

$$\inf\{\mu(s)\,;\ s\in C,\ s\geqq f \text{ on } S\}.$$

For any $x\in\Omega$, we set $Q_x(f)$ instead of $Q_{\varepsilon_x}(f)$. We denote also by Qf the function $x\to Q_x(f)$. Obviously Qf is a concave function. If C is min-stable, then Qf is an upper semicontinous function on Ω, and we have

$$Q_\mu(f)=\mu(Qf),$$

since P is adapted and $P\subset C\subset H_p$.

LEMMA 1. *Let f be an upper P-bounded upper semicontinuous function on a closed set S. Then for any $\mu\in\mathfrak{M}_p^+$ there exists a measure*

ν such that $\mu \ll \nu$, $\nu(\Omega - S) = 0$ and

$$\nu(f) = Q_\mu^S(f) .$$

PROOF. Suppose first $f \in H_p$. Since the mapping $g \to Q_\mu^S(g)$ from $H_p(S)$ into R is sublinear, we may find, using Hahn-Banach's theorem, a linear functional ν_f on $H_p(s)$ such that

$$\nu_f \leqq Q_\mu^S$$

on $H_p(s)$ and $\nu_f(f) = Q_\mu(f)$. Obviously we see that

$$g \leqq 0 \Rightarrow Q_u^S(g) \leqq 0 .$$

Hence we may consider ν_f a non-negative measures on S. Particularly, for any $g \in C$, we have

$$\nu_f(g) \leqq Q_\mu^S(g) = \mu(g) ,$$

whence $\mu \ll \nu_f$.

For an upper P-bounded upper semicontinous function f, we can prove similarly in [3], by observing that the set $\{\lambda \in \mathfrak{M}_p^+ ; \mu \ll \lambda\}$ is compact under the topology $\sigma(\mathfrak{M}_p, H_p)$.

COROLLARY. *Under the same conditions, we have*

$$Q_\mu^S(f) = \sup\{\nu(f) ; \nu \in \mathfrak{M}_p^+, \nu(\Omega - S) = 0, \mu \ll \nu\} .$$

Applying lemma 1 we can prove easily the following proposition.

PROPOSITION 1. *A measure $\mu \in \mathfrak{M}_p^+$ is extremal if and only if for any $t \in -C$, we have*

$$Q_\mu^S(t) = \mu(t) .$$

§4. **Simplexes.** In this section P will be an adapted cone of $C^+(\Omega)$ and C a min-stable convex cone of $C(\Omega)$ such that $P \subset C \subset H_p$.

Let S be a closed subset of Ω. A function f on S is called C-affine or simply affine on S if for any $x \in S$ and any measure $\mu \in \mathfrak{M}_p^+$ on S satisfying $\varepsilon_x \ll \mu$, we have

$$\mu(f) = f(x) .$$

A closed subset S of Ω is called C-determining or simply determining if any element of C is non-negative, if it is non-negative on S.

PROPOSITION 2. *Let S be a determining set and g be an upper P-bounded upper semicontinous on S. Then for any concave function f on a closed set T containing S such that $f \geqq g$ on S, we have*

$$f \geqq Q^S g \text{ on } T.$$

PROOF. Let $x \in T$. By lemma 1 we may find a measure $\mu \in \mathfrak{M}_p^+$

such that $\varepsilon_x \ll \mu$, $\mu(\Omega - S) = 0$ and $\mu(g) = Q_x^S(g)$. Then for any concave function f on T such that $f \geqq g$ on S we get

$$Q_x^S(g) = \mu(g) \leqq \mu(f) \leqq f(x)\,.$$

COROLLARY 1. *Let S be a determining set and h be an upper P-bounded upper semicontinous concave function on S. Then we have*

$$h = Q^S h \ \text{ on } S.$$

COROLLARY 2. *Let S be a determining set and h be a P-bounded affine function on Ω. Then, h is continous on Ω if it's restriction on S is continous.*

PROOF. From proposition 2, we have $h \geqq Q^S h$ on Ω and $-h \geqq Q^S(-h)$ on Ω. Since $Q^S h \geqq -Q^S(-h)$, we get $h = Q^S(h) = -Q^S(-h)$. Hence h is continous on Ω.

PROPOSITION 3. *Let S be a determining set and h be an upper P-bounded upper semicontinous concave function on Ω. Then h is non-negative on Ω if it is non-negative on S.*

PROOF. Since h is non-negative on S, we have $Q^S h \geqq 0$ on Ω. By proposition 2, h is non-negative on Ω.

The pair (Ω, C) is called a simplex if for any $x \in \Omega$ and any two C-extremal measures $\nu, \nu' \in \mathfrak{M}_p^+$ such that $\varepsilon_x \ll \nu$ and $\varepsilon_x \ll \nu'$ we have

$$\nu(f) = \nu'(f)$$

for any $f \in C$.

Let us denote by $\mathfrak{A}$ the set of all upper P-bounded upper semicontinous affine functions on Ω. Then we have the following theorem which is an extention of theorem 3.1 in Boboc and Cornea [ ].

THEOREM 1. *Let S be a determining set, $\mathfrak{G}$ be a cone of functions on S such that*

$$-C_S{}^{*)} \subset \mathfrak{G} \subset -\widehat{C_S}$$

and $\mathfrak{F}$ (resp. $\mathfrak{H}$) be a set of concave (resp. affine) functions on S (resp. Ω) such that

$$C_S \subset \mathfrak{F} \ (\textit{resp. } \mathfrak{A} \subset \mathfrak{H}).$$

Then, the following assertion are equivalent.

a) (Ω, C) *is a simplex,*

b) $Q^S g \in \mathfrak{A}$ *for any* $g \in \mathfrak{G}$,

c) $Q^S(g + g') = Q^S(g) + Q^S(g')$ *for any* $g, g \in \mathfrak{G}$,

*) We denote by C_S the set of all restriction on S of elements of C.

d) for any $g \in \mathfrak{G}$ and any $f \in \mathfrak{F}$ such that $g \leqq f$ there exists $h \in \mathfrak{H}$ such that

$$g \leqq h \leqq s \text{ on } S.$$

PROOF. a)$\to$b) Let $x \in \Omega$ and $g \in \mathfrak{G}$. For any $\mu \in \mathfrak{M}_p^+$ with $\varepsilon_x \ll \mu$, we have $Q_\mu^S(g) \leqq Q_x^S(g)$. On the other hand we may find a measure $\nu \in \mathfrak{M}_p^+$ such that $\varepsilon_x \ll \nu$ and $Q_x^S(g) = \nu(g)$ by lemma 1. Let ν' be an extremal measure $\nu \ll \nu'$ (resp. $\mu \ll \mu'$). Then we get

$$\nu'(g) = \mu'(g),$$

since (Ω, C) is a simplex. Hence

$$Q_x^S(g) = \nu(g) \leqq \nu'(g) = \mu'(g) \leqq Q_\mu^S(g).$$

Therefore, we have

$$Q_x^S(g) = Q_\mu^S(g) = \mu(Q^S g).$$

This implies that the function $x \to Q_x^S(g)$ is affine and $Q^S g \in \mathfrak{A}$.

b)$\to$c) Let ν be an extremal measure such that $\varepsilon_x \ll \nu$. For any $g \in \mathfrak{G}$, we get

$$Q_x^S(g) = \nu(Q^S g) = Q_\nu^S(g) = \nu(g).$$

Therefore, we have

$$Q_x^S(g+g') = \nu(g+g') = \nu(g) + \nu(g') = Q_x^S(g) + Q_x^S(g').$$

c)$\to$a) Let $x \in \Omega$. For any $f \in H_p$ we define

$$p(f) = \sup\{Q_x^S(t)\,;\ t \in \mathfrak{G},\ t \leqq f \text{ on } S\}.$$

Then we get $-\infty < p(f) < +\infty$ and $p(f) \leqq Q_x^S(f)$.

Now we shall prove

$$-p(-f) \leqq Q_x^S(f). \quad \cdots\cdots\cdots\cdots (1)$$

From the definition of p it follows that

$$-p(-f) = \inf\{\nu(t)\,;\ \varepsilon_x \ll \nu,\ \nu(\Omega - S) = 0,\ t \in -\mathfrak{G},\ t \geqq f\},$$

by applying lemma 1.

For any measure $\nu \in \mathfrak{M}_p^+$ on S such that $\varepsilon_x \ll \nu$ and $f \in H_p$, we get

$$\inf\{\nu(t)\,;\ t \in -\mathfrak{G},\ t \geqq f \text{ on } S\} = \inf\{\nu(g)\,;\ g \in C,\ g \geqq f \text{ on } S\}$$
$$= Q_\nu^S(f) \leqq Q_x^S(f),$$

whence follows the relation (1).

Since the function $f \to -p(-f)$ is a sublinear function on H_p, we can find, by Hahn-Banach's extention theorem, a linear functional λ on H_p such that

$$\lambda(f) \leqq -p(-f)$$

for any $f \in H_p$. If $f \leqq 0$, we have

$$\lambda(f) \leqq -p(-f) \leqq Q_x^S(f) \leqq 0.$$

Therefore, λ is positive and we may suppose $\lambda \in \mathfrak{M}_p^+$.

Further we get

$$p(f) \leqq \lambda(f) \leqq -p(-f) \leqq Q_x(f)$$

for any $f \in H_p$.

Particularly for any $t \in -C$, we get

$$p(t) = Q_x^S(t),$$

whence $Q_x^S(t) = \lambda(t)$.

For any extremal measure $\nu \in \mathfrak{M}_p^+$ with $\varepsilon_x \ll \nu$, we have

$$\nu(t) \leqq Q_x^S(t) = \lambda(t)$$

for any $t \in -C$. Hence $\nu \ll \lambda$. Since ν is extremal, we have

$$\nu(g) = \lambda(g)$$

for any $g \in C$. Therefore (Ω, C) is a simplex.

b)$\to$d) For any $g \in \mathfrak{G}$, and $f \in \mathfrak{F}$ such that $g \leqq f$, using proposition 3, we get

$$g \leqq Q^s g \leqq f \text{ on } S.$$

From the assumption b), we have $Q^s g \in \mathfrak{A}$.

d)$\to$b) Let $g \in \mathfrak{G}$, $x \in \Omega$ and $\mu \in \mathfrak{M}_p^+$ with $\varepsilon_x \ll \mu$. We denote by ν a measure of $\mathfrak{M}_p^+$ on S such that $\mu \ll \nu$. From proposition 2, we have

$$Q_x^S(g) \geqq \mu(Q^s g) \geqq \nu(Q^s g) = \inf_{\substack{f \in C \\ f \geqq g \text{ on } S}} \nu(f) \geqq \inf_{\substack{h \in \mathfrak{H} \\ h \geqq g \text{ on } S}} (h) = \inf_{\substack{h \in \mathfrak{H} \\ h \geqq g \text{ on } S}} h(x) \geqq Q_x^S(g),$$

which shows that the function $x \to Q_x^S(g)$ is affine and upper P-bounded.

PROPOSITION 4. *Suppose that (Ω, C) is a simplex. Then any upper P-bounded upper semicontinous (resp. P-bounded continous) affine function on a determining set S may be uniquely extended to an element of $\mathfrak{A}$ (resp. $\mathfrak{A} \cap H_p$).*

PROOF. By corollary, in §3, and theorem 1, we get $Q^s h \in \mathfrak{A}$ and $Q^s h = h$ on S. Especially, if h is continous on S, $Q^s h$ is also continous on Ω. Applying proposition 3, it follows that such an extention is unique.

§5. Choquet boundary.

In this section P will be an adapted cone of $C^+(\Omega)$ and C be a min-stable cone of $C(\Omega)$ with $P \subset C \subset H_p$.

A closed subset $A \subset \Omega$ is called stable if for any $x \in A$ and any $\mu \in \mathfrak{M}_p^+$ satisfying $\varepsilon_x \ll \mu$, we have $\mu(\Omega - A) = 0$. We denote by $\Omega^-(C) = \Omega^-$ the open set $\bigcup_{v \in C} \{x \in \Omega\,;\ v(x) < 0\}$.

We call the Choquet boundary of C, denoted by $\partial(C)$, the set of all points x of Ω^- which is an element of a minimal compact stable set. By Mokobozki and Sibony [7], we know that if Ω^- is not empty, then the Choquet boundary is not empty and it's closure is a determining set.

We say that C is linearly separating if for any two different x, y of Ω and any $\lambda \geqq 0$ there exists a $v \in C$ such that $f(x) \neq \lambda f(y)$. By Pradelle [9], we have the following proposition;

PROPOSITION 5. *If P is an adapted cone of $C^{+}(\Omega)$ and C is a min-stable, linearly separating cone of H_p with $P \subset C$, then the vector space $C-C$ is dense in H_p.*

Using this proposition, we can prove easily the following proposition;

PROPOSITION 6. *If C is a linearly separating cone, the following assertions are equivalent;*

(a) (Ω, C) is a simplex,

(b) for any $x \in \Omega$ there exists uniquely an extremal measure $\mu \in \mathfrak{M}_p^+$ with $\varepsilon_x \ll \mu$.

THEOREM 2. *Suppose that C is a min-stable linearly separating cone of $C(\Omega)$ such that $P \subset C \subset H_p$ and $\Omega^- \neq 0$. Then the following two assertions are equivalent;*

(a) (Ω, C) is a simplex and $\delta(C)$ is closed,

(b) any P-bounded continous function on $\overline{\delta(C)}$ is uniquely extended to an element of $\mathfrak{A} \cap H_p$.

PROOF. (a)$\rightarrow$(b) Put $\delta(C)=S$. Then S is a determining set. For any $x \in S$, any measure $\mu \in \mathfrak{M}_p^+$ with $\varepsilon_x \ll \mu$ is equal to ε_x. Therefore, it follows that any continous P-bounded function h on S is affine. By proposition 3, h is uniquely extended to an element of $\mathfrak{A} \cap H_p$.

(b)$\rightarrow$(a) Put $\overline{\delta(C)}=S$. Then S is a determining set. Let $x \in S$ and $\mu \in \mathfrak{M}_p^+$ with $\varepsilon_x \ll \mu$. By lemma 1, we may find a measure $\nu \in \mathfrak{M}_p^+$ such that $\mu(\Omega-S)=0$ and $\mu \ll \nu$. Further, for any $f \in C$ there exists a $h \in H_p \cap \mathfrak{A}$ such that $h=f$ on S. By the assumption we have

$$f(x)=h(x)=\nu(h)=\nu(f) \leqq \mu(f) \leqq f(x).$$

Hence

$$f(x)=\mu(f)$$

for any $f \in C$ and any $\mu \in \mathfrak{M}_p^+$ with $\varepsilon_x \ll \mu$. This implies $x \in \delta(C)$, whence $\delta(C)$ is closed.

Suppose that μ, ν are extremal measures of $\mathfrak{M}_p^+$ satisfying $\varepsilon_x \ll \mu$ and $\varepsilon_x \ll \nu$. By lemma 1, we may find a measure μ_1 (resp. ν_1) of $\mathfrak{M}_p^+$ such that $\mu \ll \mu_1$ (resp. $\nu \ll \nu_1$) and $\mu_1(\Omega-S)=0$ (resp. $\nu_1(\Omega-S)=0$). Let $f \in C$ and h be an element of $\mathfrak{A} \cap H_p$ such that $f=h$ on S. Then we have

$$\mu(f)=\mu_1(f)=\mu_1(h)=h(x).$$

Similarly, we have

$$\nu(f)=h(x).$$

Hence $\mu(f)=\nu(f)$ for any $f\in C$. This implies that (Ω, C) is a simplex.

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