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On a Riemannian Manifold Admitting a Framed f-3-Structure

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Introduction

Recently, Y.Y. Kuo has introduced the notion of an almost contact 3-structure. An almost contact 3-structure in a differentiable manifold is a structure analogous to an almost quaternion structure, the former consists of three almost contact structures while the latter consists of three almost complex structures. On the other hand, a framed f-structure has been introduced as a generalization of an almost complex structure and an almost contact structure. In this paper we shall discuss a differentiable manifold with certain three framed f-structures and show that the theorems obtained by Kuo are generalized to our case.

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§1. **Framed f-structure.** Let M be an n dimensional differentiable manifold of class C^∞ satisfying second axiom of countability and let there be given a non-null tensor field f of type (1.1) and of class C^∞ satisfying

$$f^3+f=0 . \tag{1.1}$$

We call such a structure an f-structure of rank $2r$ when the rank of f is constant on M and is equal to $2r$.[1)]

If we put

$$s=-f^2, \qquad t=f^2+1$$

where 1 denotes the unit tensor, it is known that the tensors s, t acting in the tangent space at each point of M are complementary projection operators defining complementary distributions S and T. Then the distribution S is $2r$ dimensional and T is $n-2r$ dimensional.

If there are $n-2r$ contravariant vector fields F_a spanning the

Presented by S. Tachibana.

1) The rank of f is necessarily even, K. Yano [7].

distribution T at each point of M,[2] and if in addition, there are $n-2r$ covariant vector fields (1-form) ξ^a such that

$$\text{(1.2)} \qquad \begin{aligned} &\xi^a(F_b)=\delta_b^a\,, \\ &f^2+1=\xi^a\otimes F_a\,, \end{aligned}$$

where $\otimes$ denotes the tensor product, the set $(f, \{F_a\}, \{\xi^a\})$ is called a framed f-structure, and in this case, M is called a globally framed f-manifold or framed f-manifold.

From (1.1) and (1.2), we have easily

$$\text{(1.3)} \qquad fF_a=0\,, \qquad \xi^a\circ f=0\,.$$

Denoting by $f_j^{\ i}$, $F_a^{\ i}$, $\xi^a_{\ i}$ respectively the components of f, F_a, ξ^a with respect to local coordinates defined in an arbitrary coordinate neighborhood,[3] we find that

$$\xi^a_{\ k}F_b^{\ k}=\delta_b^a\,, \qquad f_i^{\ k}f_k^{\ j}=-\delta_i^j+\xi^a_{\ i}F_a^{\ j}\,,$$

$$f_k^{\ i}F_a^{\ k}=0\,, \qquad f_i^{\ k}\xi^a_{\ k}=0\,.$$

It is known [7] that if M is a framed f-manifold, there exists a positive definite Riemannian metric g with respect to which S and T are orthogonal and such that

$$g_{ij}F_a^{\ j}=\xi^a_{\ i}\,,$$

$$f_j^{\ k}f_i^{\ h}g_{kh}=g_{ji}-\xi^a_{\ j}\xi^a_{\ i}\,,$$

hold good. In this case, putting $f_{ij}=f_i^{\ r}g_{rj}$, we have $f_{ij}=-f_{ji}$. This metric is called an associated metric of $(f, \{F_a\}, \{\xi^a\})$.

EXAMPLE. Every n dimensional Lie group G admits a framed f-structure of rank $2r$, where r is any positive integer smaller than $(n+1)/2$.

We shall show this. Let $\mathfrak{g}$ be the Lie algebra of G and let X_1, X_2, $\cdots$, X_n be n left invariant vector fields obtained naturally from a base of $\mathfrak{g}$. Next, we take the base $\{\omega^1, \omega^2, \cdots, \omega^n\}$ which is dual to $\{X_1, X_2, \cdots, X_n\}$, then $\omega^1, \omega^2, \cdots, \omega^n$ are n left invariant 1-form of G. If we put

$$f=\omega^\lambda\otimes X_{\lambda+r}-\omega^{\lambda+r}\otimes X_\lambda$$

where the right side means the sum with respect to λ from 1 to r, and

$$F_a=X_{2r+a}\,, \qquad \xi^a=\omega^{2r+a}\,,$$

we know easily that $(f, \{F_a\}, \{\xi^a\})$ is a framed f-structure of rank $2r$.

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2) The indices $a, b, \ldots$ run over the rang $1, 2, \ldots, n-2r$. We shall adopt the summation convention for the index which appears twice times in a term.

3) The indices $i, j, k, \ldots$ run over the rang $1, 2, 3, \ldots, n$.

§ 2. **Framed f-3-structure.** Let M be a differentiable manifold with two framed f-structures $(p, \{P_a\}, \{\xi^a\})$, $(q, \{Q_a\}, \{\eta^a\})$ of same rank $2r$ which satisfy following equations.

$$\eta^a(P_b)=0\,, \qquad \xi^a(Q_b)=0\,,$$

$$p_k{}^i Q_a{}^k=-q_k{}^i P_a{}^k\,, \tag{2.1}$$

$$p_j{}^k \eta^a{}_k=-q_j{}^k \xi^a{}_k\,,$$

$$p_j{}^k q_k{}^i-\xi^a{}_j Q_a{}^i=-q_j{}^k p_k{}^i+\eta^a{}_j P_a{}^i\,.$$

Define tensor fields, l, L_a and ζ^a by

$$l_j{}^i=p_j{}^k q_k{}^i-\xi^a{}_j Q_a{}^i\,,$$

$$L_a{}^i=P_a{}^k q_k{}^i=-Q_a{}^k p_k{}^i\,, \tag{2.2}$$

$$\zeta^a{}_i=-q_i{}^k \xi^a{}_k=p_i{}^k \eta^a{}_k\,.$$

Then they satisfy the following equations:

$$\zeta^a(P_b)=\xi^a(L_b)=0\,, \qquad \zeta^a(Q_b)=\eta^a(L_b)=0\,,$$

$$P_a{}^i=Q_a{}^k l_k{}^i=-L_a{}^k q_k{}^i\,, \qquad \xi^a{}_i=q_i{}^k \zeta^a{}_k=-l_i{}^k \eta^a{}_k$$

$$Q_a{}^i=L_a{}^k p_k{}^i=-P_a{}^k l_k{}^i\,, \qquad \eta^a{}_i=l_i{}^k \xi^a{}_k=-p_i{}^k \zeta^a{}_k\,, \tag{2.3}$$

$$p_j{}^i=q_j{}^k l_k{}^i-\eta^a{}_j L_a{}^i=-l_j{}^k q_k{}^i+\zeta^a{}_j Q_a{}^i\,,$$

$$q_j{}^i=l_j{}^k p_k{}^i-\zeta^a{}_j P_a{}^i=-p_j{}^k l_k{}^i+\xi^a{}_j L_a{}^i\,,$$

and

$$l^3+l=0\,,$$

$$\zeta^a(L_b)=\delta^a_b\,, \qquad l^2+l=\zeta^a\otimes L_a\,. \tag{2.4}$$

L_a are linearly independent and satisfy $lL_a=0$. By (2.4), any vector X which satisfies $lX=0$ is a linear combination of L_a. Thus the rank of l is $2r$. Hence we have

THEOREM 1. If a space M admits two framed f-structures $(p, \{P_a\}, \{\xi^a\})$ and $(q, \{Q_a\}, \{\eta^a\})$ of same rank $2r$ satisfying (2.1), then M admits another framed f-structure of rank $2r$ defined by (2.2).

We call the collection of three framed f-structures satisfying these relations a framed f-3-structure.

§ 3. **Metric.** We obtain easily

LEMMA 2. In a differentiable manifold with framed f-3-structure, if a metric g associates to two of the framed f-structures, then it associates to the other.

In fact, if g associates to $(p, \{P_a\}, \{\xi^a\})$ and $(q, \{Q_a\}, \{\eta^a\})$, then we have

$$g_{ij}L_a{}^j = g_{ij}P_a{}^r q_r{}^j = -g_{rj}P_a{}^r q_i{}^j = -q_i{}^j \xi^a{}_j = \zeta^a{}_i$$

$$\begin{aligned} g_{ij}l_m{}^i l_n{}^j &= g_{ij}(p_m{}^r q_r{}^i - \xi^a{}_m Q_a{}^i)(p_n{}^t q_t{}^j - \xi^b{}_n Q_b{}^j) \\ &= p_m{}^r p_n{}^t (g_{rt} - \eta^a{}_r \eta^a{}_t) + \xi^a{}_m \xi^a{}_n \\ &= g_{mn} - \zeta^a{}_m \zeta^a{}_n . \end{aligned}$$

Now, we shall show the following

THEOREM 3. In a space with framed f-3-structure, there exists a Riemannian metric associated to each of the three structures at a time.

PROOF. Assume h to be an associated metric of $(p, \{P_a\}, \{\xi^a\})$ and define a metric m by

$$m_{ij} = \bar{h}_{rs}(\delta_i^r - L_a{}^r \zeta^a{}_i)(\delta_j^s - L_b{}^s \zeta^b{}_j) + \zeta^a{}_i \zeta^a{}_j ,$$

where we put

$$\bar{h}_{rs} = h_{ij}(\delta_r^i - Q_a{}^i \eta^a{}_r)(\delta_s^j - Q_b{}^j \eta^b{}_s) + \eta^a{}_r \eta^a{}_s .$$

Then m satisfies

$$m_{ij}P_a{}^j = \xi^a{}_i , \qquad m_{ij}Q_a{}^j = \eta^a{}_i , \qquad m_{ij}L_a{}^j = \zeta^a{}_i .$$

Next, we define another new metric g from m by

$$g_{ij} = 1/2(m_{ij} + m_{kh}p_i{}^k p_j{}^h + m_{kh}q_i{}^k q_j{}^h + m_{kh}l_i{}^k l_j{}^h - \xi^a{}_i \xi^a{}_j - \eta^a{}_i \eta^a{}_j - \zeta^a{}_i \zeta^a{}_j)$$

This metric g is associated to the three structures.

§4. **Normality.** Let $(f, \{F_a\}, \{\xi^a\})$ be a framed f-structure of rank $2r$ in M^n. It is known that the product manifold $M^{2(n-r)} = M^n \times R^{n-2r}$ admits an almost complex structure $\tilde{F}$ defined by

$$\tilde{F} = \begin{pmatrix} f & \xi^a \\ -F_a & 0 \end{pmatrix} .$$

If the induced almost complex structure $\tilde{F}$ is complex, then the framed f-structure is said to be normal.[4)]

Consider a framed f-3-structure $(p, \{P_a\}, \{\xi^a\})$, $(q, \{Q_a\}, \{\eta^a\})$, $(l, \{L_a\}, \{\zeta^a\})$ in M^n and

$$\tilde{P} = \begin{pmatrix} p & \xi^a \\ -P_a & 0 \end{pmatrix}, \quad \tilde{Q} = \begin{pmatrix} q & \eta^a \\ -Q_a & 0 \end{pmatrix}, \quad \tilde{L} = \begin{pmatrix} l & \zeta^a \\ -L_a & 0 \end{pmatrix}$$

4) S. Ishihara [2], H. Nakagawa [5].

in $M^{2(n-r)}$. We have that

$$\tilde{P}^2=\tilde{Q}^2=\tilde{L}^2=-1\,, \qquad \tilde{P}\tilde{Q}=-\tilde{Q}\tilde{P}=\tilde{L}\,,$$

i.e. they define an almost quaternion structure in $M^{2(n-r)}$. Hence, by virtue of the well known theorem for almost quaternion structure,[5] the following is shown.

THEOREM 4. If two structures of a framed f-3-structure are normal, then so is the third structure.

Next, let $(n, \{N_a\}, \{\omega^a\})$ be another framed f-structure of rank $2r$ satisfying

$$\omega^a(P_b)=0\,, \qquad \xi^a(N_b)=0\,,$$

$$(4.1)\qquad p_k{}^i N_a{}^k=-n_k{}^i P_a{}^k\,,$$

$$p_j{}^k \omega^a{}_k=-n_j{}^k \xi^a{}_k\,,$$

$$p_j{}^k n_k{}^i-\xi^a{}_j N_a{}^i=-n_j{}^k p_k{}^i+\omega^a{}_j P_a{}^i\,,$$

and (4.1)′, (4.1)″ which are obtained by replacing $(p, \{P_a\}, \{\xi^a\})$ in (4.1) with $(q, \{Q_a\}, \{\eta^a\})$, $(l, \{L_a\}, \{\zeta^a\})$ respectively. As $(p, \{P_a\}, \{\xi^a\})$ and $(n, \{N_a\}, \{\eta^a\})$ produce another framed f-3-structure, we have $\tilde{P}\tilde{N}=-\tilde{N}\tilde{P}$ and similarly $\tilde{Q}\tilde{N}=-\tilde{N}\tilde{Q}$, $\tilde{L}\tilde{N}=-\tilde{N}\tilde{L}$ hold good.

On the other hand the following is known.

LEMMA.[6] Let M be a differentiable manifold with almost quaternion structures $\Phi_{(\lambda)}$, $(\lambda=1, 2, 3)$. Then there does not exist an almost complex structure $\Phi_{(4)}$ such that

$$\Phi_{(\lambda)}\Phi_{(4)}=-\Phi_{(4)}\Phi_{(\lambda)}\,, \qquad \lambda=1, 2, 3.$$

Thus we have

THEOREM 5. There does not exist a framed f-structure of rank $2r$ $(n, \{N_a\}, \{\omega^a\})$ which satisfies the relations (4.1), (4.1)′, (4.1)″ with a framed f-3-structure $(p, \{P_a\}, \{\xi^a\})$, $(q, \{Q_a\}, \{\eta^a\})$, $(l, \{L_a\}, \{\zeta^a\})$ of rank $2r$.

§5. Structure group of tangent bundle. In a space with framed f-3-structure M^n, let g be an associated metric of the framed f-3-structure and let $\{U_\lambda\}$ be an open covering of M^n by coordinate neighborhoods. Let X be a unit vector field over U_λ, orthogonal to P_a, Q_a, L_a with respect to g. Then P_a, Q_a, L_a, X, pX, qX and lX are orthonormal. Next, let Y be a unit vector field over U_λ, orthogonal to P_a, Q_a, L_a, X, pX, qX, and lX. Then these vector fields and Y, pY, qY, lY

5) Y.Y. Kuo and S. Tachibana [4].
6) S. Tachibana and W.N. Yu [6].

constitute an orthonormal field. Consequently, in every U_λ, we can choose n mutually orthogonal unit vector fields

$$X_\alpha,\ pX_\alpha,\ qX_\alpha,\ -lX_\alpha,\ P_a,\ Q_a,\ -L_a,\quad (\alpha=1, 2, \cdots, (3r-n)/2(=k)).$$

We call this the adapted frame and denote it (u).

REMARK. Since $(3r-n)/2$ must be integer, we know that r is even or odd according as n is even or odd.

With respect to the adapted frame, the tensor g, p, q, l have components:

$$g=\begin{pmatrix} I_k & & & & \\ & I_k & & 0 & \\ & & I_k & & \\ & 0 & & I_k & \\ & & & & I_{3(n-2r)} \end{pmatrix}$$

(5.1)
$$p=\left(\begin{array}{cccc|ccc} 0 & I_k & 0 & 0 & & & \\ -I_k & 0 & 0 & 0 & & 0 & \\ 0 & 0 & 0 & I_k & & & \\ 0 & 0 & -I_k & 0 & & & \\ \hline & & & & 0 & 0 & 0 \\ & 0 & & & 0 & 0 & I_{n-2r} \\ & & & & 0 & -I_{n-2r} & 0 \end{array}\right)$$

$$q=\left(\begin{array}{cccc|ccc} 0 & 0 & I_k & 0 & & & \\ 0 & 0 & 0 & -I_k & & 0 & \\ -I_k & 0 & 0 & 0 & & & \\ 0 & I_k & 0 & 0 & & & \\ \hline & & & & 0 & 0 & -I_{n-2r} \\ & 0 & & & 0 & 0 & 0 \\ & & & & I_{n-2r} & 0 & 0 \end{array}\right)$$

$$l=\left(\begin{array}{cccc|ccc} 0 & 0 & 0 & -I_k & & & \\ 0 & 0 & -I_k & 0 & & 0 & \\ 0 & I_k & 0 & 0 & & & \\ I_k & 0 & 0 & 0 & & & \\ \hline & & & & 0 & -I_{n-2r} & 0 \\ & & 0 & & I_{n-2r} & 0 & 0 \\ & & & & 0 & 0 & 0 \end{array}\right) \tag{5.1}$$

where I_k denotes the $k\times k$ unit matrix.

Now take another adapted frame $(\bar{u})$, then we have

$$\bar{u}=\gamma u$$

where γ is an orthogonal matrix such that

$$\gamma=\begin{pmatrix} A_{4k} & 0 \\ 0 & I_{3(n-2r)} \end{pmatrix}.$$

As the tensor g, p, q, l have the same components as (5.1) with respect to (u), we can easily see that A must have the form

$$A_{4k}=\begin{pmatrix} a_k & b_k & c_k & d_k \\ -b_k & a_k & -d_k & c_k \\ -c_k & d_k & a_k & -b_k \\ -d_k & -c_k & b_k & a_k \end{pmatrix}$$

Thus the group of tangent bundle of M^n can be reduced to $Sp(k)\times I_{3(n-2r)}$.

Conversely, suppose M^n be a differentiable manifold such that the group of its tangent bundle reduces to $Sp(k)\times I_{3(n-2r)}$, $(k=(3r-n)/2)$. By assumption, we can take an open covering $\{U_\lambda\}$ of M^n by coordinate neighborhoods and a frame field over each U_λ so that, if $U_\lambda\cap U_\nu$ is not empty, the transformation of components of vector with respect to frames of U_λ and U_ν is given by a matrix of $Sp(k)\times I_{3(n-2r)}$. In each U_λ, take a tensor field g of type (0.2), tensor fields p and q of type (1.1) having (5.1) as components. We define contravariant and covariant vector fields P_a, Q_a, ξ^a, η^a by

$$P_a{}^i=\delta^i_{a+4k}, \qquad Q_a{}^i=\delta^i_{a+4k+(n-2r)},$$

$$\xi^a{}_i=\delta_i^{a+4k}, \qquad \eta^a{}_i=\delta_i^{a+4k+(n-2r)},$$

with respect to the frame. As the components of g satisfy

$$g=\gamma g^t\gamma \qquad \gamma\in Sp(k)\times I_{3(n-2r)}\ ,$$

all such tensor fields g over U_λ's constitute a single positive definite tensor field over M^n. The same is true for p, q, P_a, Q_a. It is easily shown that equations analogous to (1.1), (1.2) for $(p, \{P_a\}, \{\xi^a\})$, $(q, \{Q_a\}, \{\eta^a\})$, and (2.1) hold good with respect to the frames. Since these equations are all tensor equations, they hold good for every natural frames too. Thus we obtain

THEOREM 6. A necessary and sufficient condition for an n dimensional manifold to admit a framed f-3-structure of rank $2r$ is that the group of tangent bundle of the manifold be reduced be the group $Sp((2r-n)/2)\times I_{3(n-2r)}$, where r is even if n is even and r is odd if n is odd.

EXAMPLE. Every n dimensional Lie group G^n admits a framed f-3-structure of rank $2r$, where r is even or odd according as n is even or odd.

In fact, we shall constitute a framed f-3-structure of rank $2r$ in G^n. Let $\mathfrak{g}$, X_i, ω^i $(i=1, \cdots, n)$ be the same notations in Example of framed f-structure of § 1. Putting $k=(3r-n)/2$, if we define

$$\begin{aligned} p=&\ \omega^\alpha\otimes X_{\alpha+k}-\omega^{\alpha+k}\otimes X_\alpha+\omega^{\alpha+2k}\otimes X_{\alpha+3k}-\omega^{\alpha+3k}\otimes X_{\alpha+2k}\\ &+\omega^{4k+(n-2r)+a}\otimes X_{4k+2(n-2r)+a}-\omega^{4k+2(n-2r)+a}\otimes X_{4k+(n-2r)+a}\ ,\\ q=&\ \omega^\alpha\otimes X_{\alpha+2k}-\omega^{\alpha+k}\otimes X_{\alpha+3k}-\omega^{\alpha+2k}\otimes X_\alpha+\omega^{\alpha+3k}\otimes X_{\alpha+k}\\ &-\omega^{4k+a}\otimes X_{4k+2(n-2r)+a}+\omega^{4k+2(n-2r)+a}\otimes X_{4k+a}\ ,\\ l=&-\omega^\alpha\otimes X_{\alpha+3k}-\omega^{\alpha+k}\otimes X_{\alpha+2k}+\omega^{\alpha+2k}\otimes X_{\alpha+k}+\omega^{\alpha+3k}\otimes X_\alpha\\ &-\omega^{4k+a}\otimes X_{4k+(n-2r)+a}+\omega^{4k+(n-2r)+a}\otimes X_{4k+a}\end{aligned}$$

where the right sides mean the sum with respect to α from 1 to k and with respect to a from 1 to $n-2r$, and

$$P_a=X_{4k+a}\ ,\qquad Q_a=X_{4k+(n-2r)+a}\ ,\qquad L_a=-X_{4k+2(n-2r)+a}\ ,$$
$$\xi^a=\omega^{4k+a}\ ,\qquad \eta^a=\omega^{4k+(n-2r)+a}\ ,\qquad \zeta^a=-\omega^{4k+2(n-2r)+a}\ ,$$

then we know easily that $(p, \{P_a\}, \{\xi^a\})$, $(q, \{Q_a\}, \{\eta^a\})$, $(l, \{L_a\}, \{\zeta^a\})$ is a framed f-3-structure of rank $2r$.

Bibliography

[1] S.I. Goldberg and K. Yano: On normal globally framed f-manifolds, Tôhoku Math. Journ., 22 (1970), 362–370.

[2] S. Ishihara: Normal structure satisfying $f^3+f=0$, Kōdai Math. Sem. Rep., 18 (1966), 36–47.

[3] Y.Y. Kuo: On almost contact 3-structure, Tôhoku Math. Journ., 22 (1970), 325–332.

[4] Y.Y. Kuo and S. Tachibana: On the distribution appeared in contact 3-structure, to appear.

[5] H. Nakagawa: On framed f-manifolds, Kōdai Math. Sem. Rep., 18 (1966), 293–306.

]6] S. Tachibana and W.N. Yu: On a Riemannian space admitting more than one Sasakian structure. Tôhoku Math. Journ., 22 (1970), 536–540.

[7] K. Yano: On a structure defined by a tensor field f of type (1,1) satisfying $f^3+f=0$, Tensor, 14 (1963), 99–109.