Natural Science Report, Ochanomizu University, Vol. 22, No. 2, 1971 111
お茶の水女子大学 自然科学報告 第22巻 第1号

Balayages of Measures and Dilations on Locally Compact Spaces

Hisako Watanabe

Department of Mathematics, Faculty of Science,
Ochanomizu University, Tokyo
(Received September 10, 1971)

§1. Preliminaries.

Let K be a compact, convex and metrizable set in a locally convex topological vector space. In [1] P. Cartier, J.M. Fell and P.A. Meyer proved that for two positive measures λ, μ on K, the following statements (a) and (b) are equivalent;

(a) μ is a balayage of λ,
(b) there exists a dilation T such that $\lambda T=\mu$.

Here we say that a Markov kernel T on the Borel σ-field of K is a dilation on K if $r(\varepsilon_\alpha T)=x$ for $x\in K$. P.A. Meyer extended this theorem by applying a theorem of Strassen [7];

Let K be a compact metrizable set and φ be a cone of continuous functions on K, containing positive constants and closed under the operation "inf". For two positive measures λ, μ on K, the following (a), (b) are equivalent;

(a) μ is a balayage of λ with respect to φ,
(b) there exists a φ-dilation on K such that $\lambda T=\mu$. [5]

In this paper we shall extend this theorem to the case of locally compact, σ-compact K. We shall use the theory of adapted cone introduced by G. Choquet [2] and developed by G. Mokobodzki and D. Sibony [6].

§2. The integrations of filtering families of continuous functions.

Let Ω be a locally compact, σ-compact space and u, v be two real-valued functions $\geqq 0$ on Ω. We shall write $u\uparrow < v$, if for any $\varepsilon>0$, there exists a compact $K\subset\Omega$ such that

$$x\in K^{c(*)}\Rightarrow v(x)\leqq\varepsilon u(x)\,.$$

PROPOSITION 1. *Let μ be a positive Radon measure on Ω, and f a lower semicontinuous function on Ω such that there exist a μ-integrable*

(*) For a subset $A\subset\Omega$, we denote by A^c the complement of A.

$g\geqq 0$ satisfying $|f|\uparrow<g$. Further, let $\{f_\alpha\}$ be an increasing filtering family of positive continuous functions on Ω such that $\sup_\alpha f_\alpha=f$. Then we have

$$\int f d\mu=\sup_\alpha \int f_\alpha d\mu .$$

PROOF. From $f\uparrow<g$, for any $\varepsilon>0$, there exists a compact K such that $f(x)\leqq\varepsilon g(x)$ on K^c. This implies

$$\int f d\mu=\int_k f d\mu+\int_{k^c} f d\mu\leqq\sup\int_k f_\alpha d\mu+\varepsilon\int g d\mu .$$

Hence

$$\int f d\mu-\sup\int f_\alpha d\mu<\varepsilon\int g d\mu .$$

from which follows

$$\int f d\mu=\sup\int f_\alpha d\mu .$$

Similarly we have the following proposition.

PROPOSITION 2. *Let μ be a positive measures on Ω, f an upper semicontinuous function on Ω and $\{\varphi_\alpha\}_{\alpha\in\mathfrak{A}}$ a decreasing filtering family of positive continuous functions on Ω such that $\inf\varphi_\alpha=f$. Assume that there exists a positive, μ-integrable function g such that $|f|\uparrow<g$. Then we have*

$$\inf\int \varphi_\alpha d\mu=\int f d\mu .$$

§ 3. Adapted cones and balayages of measures.

Let Ω be a locally compact, σ-compact space. Let P be a convex cone of positive continuous function on Ω. We call P adapted if P satisfies the following condition;

(i) for any $x\in\Omega$ there exists $g\in P$ such that $g(x)>0$,

(ii) for any $g\in P$ there exists $h\in P$ such that $g\uparrow<h$.

Let V be a vector space of continuous functions on Ω. We call V adapted if we can write $V=V^+-V^+$ where $V^+=C^+(\Omega)$ and V^+ is an adapted cone. Let us put for $g\in C^+(\Omega)$,

$$H_g=\{f\in C(\Omega)\,;\ \exists\lambda>0,\ |f|\leqq\lambda g\} .$$

Then H_g is a vector space and a Banach space with norm

$$||f||_g=\{\inf\lambda\,;\ \exists\lambda>0,\ |f|\leqq\lambda g\} .$$

Let $P\subset C^+(\Omega)$ be a cone. Then the space $H_p=\bigcup_{g\in p} H_g$ is a vector space. We shall assign to H_p the topology of inductive limit of Banach spaces $\{H_v\}_{g\in p}$.

Let μ be a positive Radon measure. We call μ P-integrable if $\mu(|f|)<+\infty$ for any $f\in P$. We denote by $\mathfrak{M}_p^+$ the space of all P-integrable positive measures. If P is adapted, any positive linear form on H_p is represented by a measure of $\mathfrak{M}_p^+$. [6]

Let C be a cone of $C(\Omega)$. For positive measures λ, μ on Ω, we shall write $\lambda\ll\mu$ if $\mu(f)\leqq\lambda(f)$ for any $f\in C$, and we shall say that μ is a balayage of λ with respect to C.

Assume that $P\subset C\subset H_p$. If for each $g\in H_p$, we define

$$\hat{g}(x)=\inf\{f(x)\ ;\ f\geqq g,\ f\in C\}\ ,$$

then we have $|\hat{g}(x)|<+\infty$. In particular if C is an min-stable cone, which means that C is closed under the operation "min", g is upper semicontinuous.

PROPOSITION 3. *Let $C\subset C(\Omega)$ be a* min-*stable cone and $P\subset C^+(\Omega)$ be an adapted cone such that $P\subset C\subset H_p$. Let be C-integrable positive measures. Then the following two conditions are equivalent ;*

(i) $\lambda\ll\mu$,

(ii) $\mu(\hat{g})\leqq\int g(x)\,d\lambda(x)$ *for any* $g\in H_p$.

PROOF. From proposition 2, we have

$$\int \hat{g}(x)\,d\lambda(x)=\inf\left\{\int f(x)\,d\lambda(x)\ ;\ f\geqq g,\ f\in C\right\}\ . \quad\cdots\cdots\cdots\cdots(*)$$

Applying $(*)$, it is clear that (i) and (ii) are equivalent.

PROPOSITION 4. *Let $C\subset C(\Omega)$ be a cone and $P\subset C^+(\Omega)$ be an adapted cone such that $P\subset C\subset H_p$. Then for each $x\in\Omega$ we have*

$$\hat{g}(x)=\sup_{\varepsilon_x\ll\mu}\int gd\mu$$

for any $g\in H_p$.

PROOF. If $\varepsilon_x\ll\mu$, for any $f\in C$ with $f\geqq g$, then we have $\mu(g)\leqq\mu(f)\leqq f(x)$. This implies $\sup_{\varepsilon_x\ll\mu}\mu(g)\leqq g(x)$.

Conversely, the mapping ; $\varphi\to\hat{\varphi}(x)$ from H_p into $\boldsymbol{R}$ is a sublinear function. Therefore, by the Hahn-Banach's extention theorem, there exists, for any $g\in H_p$, a linear functional L on H_p such that $L(g)=\hat{g}(x)$ and $L(\varphi)\leqq\hat{\varphi}(x)$ for $\varphi\in H_p$. It is clear that L is positive. Hence there exists a positive P-integrable measure μ such that $\mu(\varphi)=L(\varphi)$ for any $\varphi_p\in H$.

For any $f\in C$, we have

$$\int fd\mu=L(f)\leqq\hat{f}(x)=f(x)\ .$$

Hence $\varepsilon_x \ll \mu$. By $\mu(g)=\hat{g}(x)$, we have $\hat{g}(x) \leqq \sup_{\varepsilon_x \ll \mu} \mu(g)$.

§ 4. Separability of H_p.

We shall call strictly positive a real valued function f such that $f(x)>0$ for any $x \in \Omega$. We denote by $C_k(\Omega)$ the set of all continous functions with compact supports.

PROPOSITION 5. *Let Ω be a locally compact, σ-compact set and $P \subset C^+(\Omega)$ be an adapted cone containing a strictly positive function f. Then $C_k(\Omega)$ is dense in H_p.*

PROOF. First we remark $C_k(\Omega) \subset H_p$. Let U be a neighborhood of a $\varphi \in H_p$. For any $u \in P$ with $\varphi \in H_u$, $U \cap H_u$ is open in H_u. Hence there exists $\varepsilon>0$ such that $\{h\,;\, \|\varphi-h\|_u<\varepsilon\} \subset U \cap H_u$. In order to prove that $C_k(\Omega)$ is dense in H_p, we have only to find a $u \in P$ such that $H_u \ni \varphi$ and for any $\varepsilon>0$ there exists a $h \in C_k(\Omega)$ such that $\|f-h\|_u<\varepsilon$.

There exists $v \in P$ such that $H_v \ni \varphi$, since $H_p \ni \varphi$. This implies $|\varphi| \leqq \|\varphi\|_v v$. Since P is adapted, we can find $u \in P$ such that for any $\varepsilon>0$ that exists a compact K satisfying

$$v \leqq \varepsilon u \quad \text{on } K^c.$$

We may assume $u>0$. Thus there exists a $\lambda>0$ such that $|\varphi|<\lambda u$ on K. Hence we have

$$|\varphi| \leqq (\lambda+\varepsilon\|\varphi\|_v)u \quad \text{on } \Omega.$$

This implies $\varphi \in H_u$. Further, there exists a $g \in C_k^+(\Omega)$ such that $|\varphi| \leqq g$ on K. Put $h=\sup(-g, \inf(\varphi, g))$, then we have $h \in C_k^+(\Omega)$, $|\varphi-h|=0$ on K, and $|\varphi-h|<\varepsilon\|\varphi\|_v u$ on K^c, from which we have $\|\varphi-h\|_u<\varepsilon\|\varphi\|_v$ and the proof is completed.

PROPOSITION 6. *Let Ω be locally compact, σ-compact and metrizable. Let $P \subset C^+(\Omega)$ be a convex cone such that for any $x \in \Omega$ there exists a $f \in P$ such that $f(x)>0$. Then there exists an enumerable set $D \subset C_k(\Omega)$ such that D is dense in $C_k(\Omega)$ under the topology of H_p.*

PROOF. Let $\{K_n\}$ be a sequence of compact subsets of Ω satisfying $\bigcup_n K_n=\Omega$ and $K_n \subset K_{n+1}^i$(*) for any $n \in \boldsymbol{N}$. Since the space $C(K_n)$ is separable under the uniform norm, there exists an enumerable set $D_n \subset C_k(\Omega)$ such that for any $h \in D_n$ the support of h is contained in K_{n+1} and the set $\{h|K_n\,;\, h \in D_n\}$(**) is dense in $C(K_n)$ with the uniform norm. Put

(*) For a subset $A \subset \Omega$, we denote by A^i the interior of A.

(**) For a subset $A \subset \Omega$ and a function h on Ω, we denote by $h \mid A$ the restriction of h on A.

$$D'=\bigcup_{n=1}^{\infty}[\{-h\,;\,h\in D_n\}\cup D_n]$$

and $D=\{\sup(\inf(h_1, h_2), \inf(h_3, h_4))\,;\, h_i\in D'$ for $i=1, 2, 3, 4\}$. Then D is an enumerable set.

Given any $\in C_k(\Omega)$. Assume that the support of φ is contained in K_n. Then, for each $\varepsilon>0$, there exists a $h_1\in D_{n+1}$ such that $|\varphi-h|<\varepsilon$ on K_{n+1}. We can find a function h_2 such that $h_2\geqq|\varphi|$ on K_n and the support of $h_2\in D'$ is contained in K_{n+1}. Put $f=\sup(\inf((-h_1, -h_2), \inf(h_1, h_2))$, then we have $f\in D$ and $|\varphi-f|<\varepsilon$ on K_{n+1} and $|\varphi-f|=0$ on K_{n+1}^C. From the assumption made in the present proposition, there exists a $v\in P$ such that $v>1$ on $\dot{K}_{n+1}$. Hence we have $|\varphi-f|<\varepsilon v$ on Ω. This implies $||\varphi-f||_v<\varepsilon$.

From proposition 5 and 6, we have the following corollary.

COROLLARY. *Let Ω be locally compact, σ-compact and metrizable and $P\subset C^+(\Omega)$ an adapted cone containing a strictly positive function. Then H_p is separable.*

§ 5. The main Theorem. Let Ω be locally compact, σ-compact and $C\subset C(\Omega)$ a cone. We denote by $\mathfrak{M}_c^+$ the set of all C-integrable positive Radon measures and by $\ll$ the balayage with respect to C. Let T be a kernel with the Borel σ-field on Ω. We shall call T C-dilation if $\varepsilon_\omega<\varepsilon_\omega T$ for any $\omega\in\Omega$, where ε_ω is the point measure of ω.

In order to prove theorem 2, we need the following theorem by the author. [8]

THEOREM 1. *Let $(\Omega, \mathfrak{F}, \lambda)$ be a measure space with positive, comlete and σ-finite measure. Assume that an ordered vector space E is separable under the topology of inductive limit of Banach spaces $\{E_\alpha\}_{\alpha\in\mathfrak{A}}$, where E_α is a subspace of E such that $E=\bigcup_\alpha E_\alpha$. Let p be a weakly measurable mapping from Ω into S_E satisfying the condition* (c). *Put $s(x)=\int p_\omega(x)\,d\lambda(\omega)$ for each $x\in E$, then S is also sublinear function on E.*

For $x'\in L^+(E)$, the following condition (a) *and* (b) *are equivalent;*

(a) *$\langle x'\ x\rangle\leqq S(x)$ for any $x\in E$,*

(b) *there exists a weakly measurable mapping; $\omega\to x_\omega'\in E$ such that x_ω' is dominated by p_ω and*

$$\langle x', x\rangle=\int\langle x_\omega', x\rangle\,d\lambda(\omega)$$

for any $x\in E$.

Here, we denote by S_E the set of all sublinear functions on E and we shall say that p is weakly measurable if the mapping; $\omega\to p_\omega(x)$ is λ-measurable for each $x\in E$.

Further we say that p satisfies the condition (c) when p satisfies the following condition ;

for each $\alpha\in\mathfrak{A}$, there exist a non-negative, λ-integrable function $g_\alpha\in\mathfrak{R}^\infty(\Omega, \mathfrak{F}, \lambda)^{(*)}$ and a constant $M(\alpha)$ such that

$$|p_\omega(x)|\leqq M(\alpha)\|x\|_\alpha\, g_\alpha(\omega)$$

for any $x\in E$ and any $\omega\in\Omega$.

THEOREM 2. *Let Ω be locally compact, σ-compact and metrizable and $C\subset C(\Omega)$ be a* min-*stable cone. Assume that an adapted cone $P\subset C^+(\Omega)$ satisfies $H_p\supset C\supset P$ and contains a strictly positive function. Then for two C-integrable positive Radon measures λ, μ on Ω, the following two conditions are equivalent;*

(a) *μ is a balayage of λ with respect to C,*
(b) *there exists a C-dilation T on Ω such that $\lambda T=\mu$.*

PROOF. If, for each $f\in H_p$, we put $\hat{f}(\omega)=\inf_{\substack{g\geqq f\\ g\in C}} g(\omega)$, then $\hat{f}(\omega)=\sup\{\langle\theta, f\rangle, \varepsilon_\omega\ll\theta, \theta\in\mathfrak{M}_C^+\}$. Since $\hat{f}$ is upper semicontinuous and λ-integrable, we can define $p_\lambda(f)=\langle\lambda, \hat{f}\rangle$ for any $f\in H_p$. Thus, from proposition 3, (a) is equivalent to the following condition (a′) ;

(a′) $\mu(f)\leqq p_\lambda(f)$ for any $f\in H_p$.
Now we have only to prove that (a′) and (b) are equivalent.

(b) $\Rightarrow$(a′)

From $\mu=\lambda T$, we have $\varepsilon_\omega\ll\varepsilon_\omega T$ for any $\omega\in\Omega$ and

$$\langle\mu, f\rangle=\int\langle\varepsilon_\omega T, f\rangle d\lambda(\omega)$$

for any $f\in H_p$. Hence

$$\langle\mu, f\rangle=\int\hat{f}(\omega)\, d\lambda(\omega)=p_\lambda(f)\,.$$

(a′) $\Rightarrow$(b)

We apply theorem 1. Take for $(\Omega, \mathfrak{F})$ the space Ω with the σ-field of λ-measurable sets, for E the space H_p and set $p_\omega(f)=\hat{f}(\omega)$ for each $f\in H_p$ and each $\omega\in\Omega$. Let $\{K_n\}$ be an increasing family of compact sets such that $\Omega=\bigcup_n K_n$ and $K_n\subset K_n^i$. It is clear that for each $v\in P$, $v\in\mathfrak{R}(\Omega, \mathfrak{F}, \lambda)$ and v is λ-integrable. Since $|p_\omega(f)|\leqq\|f\|_v\, v(\omega)$ for any $f\in H_v$ and each $\omega\in\Omega$, all the assumptions of theorem 1 are completely satisfied. Therefore, there exists a weakly λ-measurable mapping t ;

(*) Let $\{K_i\}$ be an increasing sequence of $\mathfrak{F}$ such that $\lambda(K_i)<+\infty$ and $\Omega=\bigcup_{i=1}^{\infty}K_i$. We denote by $\mathfrak{R}^\infty(\Omega, \mathfrak{F}, \lambda)$ the set of all λ-measurable functions f such that $f\mid K_i$ are almost everywhere bounded for each $i\in\boldsymbol{N}$.

$\omega \to t_\omega$ from Ω into $L^+(H_p)$, where $L^+(H_p)$ is the set of all positive continous linear forms on H_p, such that

$$\langle \mu, f \rangle = \int \langle t_\omega, f \rangle \, d\lambda(\omega)$$

for any $f \in H_p$ and

$$\langle t_\omega, f \rangle \leqq p_\omega(f)$$

for any $\omega \in \Omega$ and any $f \in H_p$.

Since any positive linear form on H_p for an adapted cone is represented by a positive P-integrable Radon measure, we may regard t_ω as a positive P-integrable measure on Ω.

Let $\{f_n\}$ be a dense sequence in H_p. Then there exists a Borel set A such that $\lambda(A)=0$ and for any $n \in \boldsymbol{N}$ the λ-measurable function; $\omega \to \langle t_\omega, f_n \rangle$ is equal to a Borel function on A^c. Let

$$T_\omega = \begin{cases} t_\omega & \text{for} \quad \omega \in A^c \\ \varepsilon_\omega & \text{for} \quad \omega \in A \,, \end{cases}$$

then for any $n \in \boldsymbol{N}$ the function; $\omega \to \langle T_\omega, f_n \rangle$ is a Borel function and for any $f \in H_p$

$$\langle T_\omega, f \rangle \leqq p_\omega(f) \,.$$

Consequently,

$$\langle \mu, f_n \rangle = \int \langle T_\omega, f_n \rangle \, d\lambda(w)$$

for any $n \in \boldsymbol{N}$ and we have

$$\langle \mu, f \rangle = \int \langle T_\omega, f \rangle \, d\lambda(\omega) \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (1)$$

for any $f \in H_p$, since $\{f_n\}$ is dense in H_p.

Let B be an open set on Ω such that $\bar{B}$ is compact. Then there exists a sequence $\{\varphi_n\} \subset C_k(\Omega) \subset H_p$ such that $0 \leqq \varphi_n \leqq 1$ and $\varphi_n \uparrow \chi_B$ for any $n \in \boldsymbol{N}$, since Ω is metrizable. Therefore, for any Borel set B, the function; $\omega \to \langle T_\omega, \chi_B \rangle$ is a Borel function. Thus the mapping T; $\omega \to T_\omega$ is a kernel on the Borel σ-field on Ω and a C-dilation from $\varepsilon_\omega \ll T_\omega$ for any $\omega \in \Omega$. From (1) we have $\lambda T = \mu$.

References

[1] P. Cartier, J.M.G. Fell and P.A. Meyer: Comparaison des mesures portées par un ensemble convex compact. Bull. Soc. M. France, 92 (1964), 435–445.

[2] G. Choquet: Le problème des moments. Sémi. Choquet, 1 (1962).

[3] G. Choquet: Lectures on Analysis, Vol. 2. (Benjamin, 1969).

[4] A.I. Tulcea & C.I. Tulcea: Topics in the theory of liftings. (Springer, 1969).

[5] P.A. Meyer: Probability and potentials. (Blaisdell, 1966).

[6] G. Mokobodzki & D. Sibony: Cônes adaptés de fonctions continues et théorie du potentiel. Sémi. Choquet, **6** (1966/67).

[7] V. Strassen: The existence of probability measures with given marginals. Am. M. Stat, **36** (1965), 423-439.

[8] H. Watanabe: Liftings and a generalized Strassen's theorem. Natu. Sci. Rep. Ochanomizu Univ. To appear.