Liftings and A Generalized Strassen's Theorem

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(Received Sep. 10, 1971)
(Revised Nov. 1, 1971)

§1. Preliminaries.

Let $(\Omega, \mathfrak{F}, \lambda)$ be a measure space with positive, complete and σ-finite measure λ. Let $M^\infty = M^\infty(\Omega, \mathfrak{F}, \lambda)$ be the set of all bounded $\mathfrak{F}$-measurable functions. The existence of lifting of M^∞, proved by D. Maharam [4] and subsequently by A.I. Tulcea and C.I. Tulcea [3], has important applications.

Let $\{K_i\}$ be an increasing sequence of $\mathfrak{F}$ such that $\lambda(K_i) < +\infty$ and $\Omega = \bigcup_{i=1}^\infty K_i$. We denote by $\mathfrak{K}^\infty(\Omega, \mathfrak{F}, \lambda)$ the set of all λ-measurable functions f such that $f \mid K_i$[1] are almost everywhere bounded for each $i \in \mathbf{N}$.[2]

We shall prove the existence of a lifting of $\mathfrak{K}^\infty(\Omega, \mathfrak{F}, \lambda)$ by which we shall further extend a theorem of Strassen on an integral representation of a continuous linear functional dominated by a support function in integral form.

§2. Liftings.

Let $(\Omega, \mathfrak{F})$ be a measure space and λ be a positive, σ-finite and complete measure on $(\Omega, \mathfrak{F})$. We shall fix an increasing sequence $\{K_i\}$ of elements of $\mathfrak{F}$ such that $\bigcup_{i=1}^\infty K_i = \Omega$, $\lambda(K_i) < +\infty$ and $\lambda(K_i) < \lambda(K_{i+1})$. If, for two $\mathfrak{F}$-measurable functions f and g, holds $f = g$ a.e., then we shall write $f \sim g$. We shall also denote by $\mathfrak{K}^\infty(\Omega, \mathfrak{F}, \lambda)$ or simply $\mathfrak{K}^\infty(\mathfrak{F})$ the set of all $\mathfrak{F}$-measurable functions f such that $f \mid K_i$ are a.e. bounded for each $i \in \mathbf{N}$.

We say that $\mathfrak{K}^\infty(\mathfrak{F})$ has a lifting ρ if there exists a mapping ρ from the space $\mathfrak{K}^\infty(\mathfrak{F})$ into $\mathfrak{K}^\infty(\mathfrak{F})$ having the following properties;

    (a) ρ is linear,

    (b) $f \geq 0$ a.e., $f \in \mathfrak{K}^\infty(\mathfrak{F}) \Rightarrow \rho(f)(\omega) \geq 0$ for each $\omega \in \Omega$,

    (c) $\rho(a) = a$ for every constant a,

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  1) For a subset $A \subset \Omega$ and a function f on Ω, we denote by $f \mid A$ the restriction of f on A.

  2) We denote by $\mathbf{N}$ the set of all natural numbers.

(d) $f \in \mathfrak{R}^\infty(\mathfrak{F})$, $i \in \mathbf{N} \Rightarrow \quad \rho(f)(\omega) \leq \|\rho(f) \mid K_i\|_\infty$ for every $\omega \in K_i$,

(e) $f \in \mathfrak{R}^\infty(\mathfrak{F}) \Rightarrow \quad \rho(f) \sim f$,

(f) $f, g \in \mathfrak{R}^\infty(\mathfrak{F}) \Rightarrow \quad \rho(fg) = \rho(f)\rho(g)$.

THEOREM 1. $\mathfrak{R}^\infty(\mathfrak{F})$ *has a lifting* ρ.

PROOF. $H_1 = K_1$ and $H_i = K_i - K_{i-1}$ $(i \geq 2)$. Then we have $\Omega = \bigcup_{i=1}^\infty H_i$ with $H_i \in \mathfrak{F}$, $\lambda(H_i) < +\infty$ and $(H_i \cap H_j) = \phi$ if $i \neq j$. $M^\infty(H_i, \mathfrak{F}, \lambda)$ has a lifting ρ_i by Tulcea [2]. We define for $f \in \mathfrak{R}^\infty(\mathfrak{F})$ as follows;

$$\rho(f)(\omega) = \rho_i(g)(\omega) \quad \text{for} \quad \omega \in H_i .$$

Here g is a function of $M^\infty(H_i, \mathfrak{F}, \lambda)$ such that $g \sim f \mid H_i$. Then ρ is well-defined and ρ has properties (a)~(f). Therefore ρ is a lifting of $\mathfrak{R}^\infty(\mathfrak{F})$.

§ 3. Integral representation.

Let E be a vector space. Suppose that $\{E_\alpha\}_{\alpha \in \mathfrak{A}}$ is a family of subspace of E with $E = \bigcup_{\alpha \in \mathfrak{A}} E_\alpha$, where $\mathfrak{A}$ is directed set with order $\prec$. Further, assume that for each $\alpha \in \mathfrak{A}$, E_α is a Banach space with norm $\| \ \|_\alpha$ and $E_\alpha \subset E_\beta$ if $\alpha \prec \beta$. Let us assign to E the topology of inductive limit of $\{E_\alpha\}_{\alpha \in \mathfrak{A}}$.

Let $(\Omega, \mathfrak{F})$ be a measure space with a positive, complete and σ-finite measure λ. We shall fix an increasing $\{K_i\}$ such that $K_i \in \mathfrak{F}$, $\Omega = \bigcup_{i=1}^\infty K_i$ and $\lambda(K_i) < +\infty$. Assume that there exists a mapping from $\mathfrak{A}$ into $\mathfrak{R}^\infty(\Omega, \mathfrak{F}, \lambda)$ such that for each $\alpha \in \mathfrak{A}$, the corresponding g_α is non-negative and λ-integrable.

Let D be the set of all functions φ such that $\varphi \in L^1(\Omega, \mathfrak{F}, \lambda)$ of each of which support is contained in some K_i. We denote by B the vector space generated by D and all constant functions on Ω. Let us notice that for any $\varphi \in B$ we can write $\varphi = \varphi_1 + \varphi_2$ where $\varphi_1 \in D$ and φ_2 is constant. For each $\varphi \in B$ and each $x \in E$, we denote by φx the function ; $\omega \to \varphi(\omega)x$ with values in E. Put

$$Q_\alpha = \{\varphi_1 x_1 + \cdots + \varphi_n x_n ; \ \varphi_i \in B, \ x_i \in E_\alpha, \ n \in \mathbf{N}\}$$

and $Q = \bigcup_\alpha Q_\alpha$. If for each $f = \sum_{i=1}^n \varphi_i x_i \in Q_\alpha$, we put

$$\|f\|_\alpha = \int \|\varphi_1(\omega)x_1 + \cdots + \varphi_n(\omega)x_n\|_\alpha \, g_\alpha(\omega) \, d\lambda(\omega) ,$$

then we have

$$\|f\|_\alpha \leq \sum_{i=1}^n \int |\varphi_j(\omega)| \, \|x_i\|_\alpha \, g_\alpha(\omega) \, d\lambda(\omega)$$

$$= \sum_{i=1}^n \|x_i\|_\alpha \int |\varphi_i(\omega)| \, g_\alpha(\omega) \, d\lambda(\omega) < +\infty .$$

We shall denote by E' the set of all continuous linear forms on E.

THEOREM 2. *Assume that a linear form G on the vector space Q satisfies the following condition;*
for each $\alpha \in \mathfrak{A}$ there exists a constant $M(\alpha)$ such that

$$|G(\varphi x)| \leq M(\alpha) \|\varphi x\|_\alpha$$

for any $x \in E_\alpha$ and any $\varphi \in B$.
Then there exists a mapping ξ from Ω into E' such that

$$G(F) = \int \langle \xi(\omega), F(\omega) \rangle \, d\lambda(\omega)$$

for any $F \in Q$, the mapping $\omega \to \langle \xi(\omega), x \rangle$ is λ-measurable for any $x \in E$, for each $\alpha \in \mathfrak{A}$, for each $i \in N$ and each $\omega \in K$, we have

$$|\langle \xi(\omega), x \rangle| \leq M(\alpha) \|x\|_\alpha \|g_\alpha | K_i\|_\infty .$$

PROOF. Let D_i the set of all $\varphi \in D$ of which supports are contained in K_i. From the assumption we have

$$|G(\varphi x)| \leq M(\alpha) \|x\|_\alpha \int |\varphi(\omega)| \, g_\alpha(\omega) \, d\lambda(\omega)$$

$$\leq M(\alpha) \|x\|_\alpha \|g_\alpha | K_j\|_\infty \int |\varphi(\omega)| \, d\lambda(\omega)$$

for any $x \in E_\alpha$ and $\varphi \in D_j$. Since the linear form $\varphi \to G(\varphi x)$ on $L^1(K_j, \mathfrak{F}, \lambda)$ is bounded, there exists a $h_x(\alpha, j) \in L^\infty(K_j, \mathfrak{F}, \lambda)$ such that

$$G(\varphi x) = \int h_x(\alpha, j)(\omega) \, \varphi(\omega) \, d\lambda(\omega) ,$$

$$\|h_x(\alpha, j)\|_\infty \leq M(\alpha) \|x\|_\alpha \|g_\alpha | K_j\|_\infty$$

for any $\varphi \in D_j$ and any $j \in N$. If $K_i \subset K_j$, we have $h_x(\alpha, i) = h_x(\alpha, j)$ a.e. on K_i. Therefore, we can define $h_x^\alpha(\omega) = h_x(\alpha, j)(\omega)$ for $\omega \in K_j$. The function h_x^α defined a.e. on Ω is λ-measurable and $h_x^\alpha | K_j$ is a.e. bounded. Hence $h_x^\alpha \in \mathfrak{R}^\infty(\Omega, \mathfrak{F}, \lambda)$. Assume that $x \in E_\alpha$. Put $\varphi_n = \chi_{k_n}$. Then we have $\varphi_n \to 1$. From $\varphi_n - 1 \in B$, we have

$$|G(x) - G(\varphi_n x)| = |G((1 - \varphi_n) x)|$$

$$\leq M(\alpha) \int \|x\|_\alpha (1 - \varphi_n(\omega)) \, g_\alpha(\omega) \, d\lambda(\omega) \to 0 \quad (n \to \infty) .$$

Hence,

$$G(x) = \lim_{n \to \infty} G(\varphi_n x) . \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (1)$$

Since for any $f \in L^\infty(\Omega, \mathfrak{F}, \lambda)$, $f\varphi_n$ is contained in D, we have

$$|G(f\varphi_n x)| \leq M(\alpha) \|f\varphi_n x\|_\alpha$$

from the assumption. This implies

$$\int h_x^\alpha(\omega)\, f(\omega)\, \varphi_n(\omega)\, d\lambda(\omega) \leqq \|x\|_\alpha\, M(\alpha)\, \|f\|_\alpha \int g_\alpha(\omega)\, d\lambda(\omega) \ .$$

Therefore, we have

$$h_x^\alpha \varphi_n \leqq \|x\|_\alpha\, M(\alpha)\, g_\alpha\ , \quad \text{a.e.}$$

Since g_α is λ-integrable, $h_x^\alpha = \lim_{n\to\infty} h_x^\alpha \varphi_n$ is also λ-integrable, and holds

$$\lim_{n\to\infty} \int h_x^\alpha(\omega)\, \varphi_n(\omega)\, d\lambda(\omega) = \int h_x^\alpha(\omega)\, d\lambda(\omega) \ . \quad\cdots\cdots\cdots\cdots\cdots\cdots(2)$$

By (1) and (2), we have for any $x \in E$

$$G(x) = \int h_x^\alpha(\omega)\, d\lambda(\omega)\ ,$$

whence

$$G(\varphi x) = \int h_x^\alpha(\omega)\, \varphi(\omega)\, d\lambda(\omega)$$

for any $x \in E_\alpha$ and any $\varphi \in B$.

For any $x \in E_\alpha$, h_x^α has the following properties ;

(i) $\|h_x^\alpha | K_j\|_\infty \leqq M(\alpha)\, \|x\|_\alpha \|g_\alpha | K_j\|_\infty$ for any $x \in E_\alpha$,

(ii) $h_{x+y}^\alpha(\omega) = h_x^\alpha(\omega) + h_x^\alpha(\omega)$ a.e.,

(iii) $h_{tx}^\alpha(\omega) = t h_x^\alpha(\omega)$ for real t.

iv) $E_\alpha \cap E_\beta \neq \phi \Rightarrow$ for any $x \in E_\alpha \cap E_\beta$, $h_x^\alpha = h_x^\beta$ a.e.

In particular we have $h_x^\alpha \in \mathfrak{K}^\infty(\Omega, \mathfrak{F}, \lambda)$ for $x \in E_\alpha$. From theorem 1 in § 2, there exists a lifting ρ of $\mathfrak{K}^\infty(\Omega, \mathfrak{F}, \lambda)$. Put $\xi(\omega)(x) = \rho(h_x^\alpha)(\omega)$ for any $x \in E_\alpha$ and any $\omega \in \Omega$. Then $\xi(\omega)$ is well-defined and $\xi(\omega) \in E'$ for each $\omega \in \Omega$, and the function ; $\omega \to \xi(\omega)(x)$ is, for each $x \in E$, λ-measurable. Further we have

$$G(\varphi x) = \int \xi(\omega)(x)\, \varphi(\omega)\, d\lambda(\omega) = \int \langle \xi(\omega),\, \varphi(\omega)x \rangle\, d\lambda(\omega)$$

for any $\varphi \in B$ and $x \in E$.   Therefore we have the conclusion.

Samely, we can prove the following theorem.

THEOREM 3. *Let E be an ordered vector space. Assume that, under the same notations in theorem 2, a linear form G on the vector space Q satisfies the following conditions ;*

(i) *for each $\alpha \in \mathfrak{A}$ there exists a constant $M(\alpha)$ such that*

$$G(\varphi x) \leqq M(\alpha)\, \|\varphi x\|_\alpha$$

for any $x \in E_\alpha$ and any $\varphi \in B$,

(ii) $G(\varphi x) \geqq 0$ *for any $x \geqq 0$ and for any $\varphi \geqq 0$.*
Then there exists a mapping ξ from Ω into $(E')^+$ such that

$$G(F) = \int \langle \xi(\omega), \ F(\omega) \rangle \ d\lambda(\omega)$$

for any $F \in Q$.

In theorem 3 $(E')^+$ is the set of all positive continous linear forms on E.

§4. A generalized Strassen's theorem.

Let S_E be the set of all sublinear functions on E and p be a mapping ; $\omega \to p_\omega$ from Ω into S_E. We shall say that p is weakly measurable if the mapping ; $\omega \to p_\omega(x)$ is λ-measurable for each $x \in E$.

We shall also say that p satisfies the condition (c) when p satisfies the following conditions ;

For each $\alpha \in \mathfrak{A}$, there exist a non-negative and λ-integrable function $g_\alpha \in \mathfrak{R}^\infty(\Omega, \mathfrak{F}, \lambda)$, and a constant $M(\alpha) > 0$ such that

$$|p_\omega(x)| \leqq M(\alpha) \ ||x||_\alpha \ g_\alpha(\omega)$$

for any $x \in E_\alpha$ and any $\omega \in \Omega$.

THEOREM 4. *Let* $(\Omega, \mathfrak{F}, \lambda)$ *be a measure space with positive, complete and σ-finite measure. Assume that a vector space E is separable under the topology of inductive limit of Banach spaces $\{E_\alpha\}_{\alpha \in \mathfrak{A}}$. Let p be a weakly measurable mapping from Ω into S_E satisfying the condition* (c). *Put*

$$s(x) = \int p_\omega(x) \ d\lambda(\omega)$$

for each $x \in E$, *then s is also sublinear function on E.*

For $x' \in E'$ the following (a) *and* (b) *are equivalent ;*

(a) $\langle x', x \rangle \leqq s(x)$ *for any* $x \in E$,

(b) *there exists a weakly measurable mapping ; $\omega \to x_\omega' \in E'$ such that x_ω' is dominated by p_ω and*

$$\langle x', x \rangle = \int \langle x_\omega', \ x \rangle \ d\lambda(\omega)$$

for any $x \in E$.

PROOF. It is clear that s is well-defined and sublinear since p satisfies the condition (c). It is also clear that (b) implies (a). Conversely we shall prove that (a) implies (b). We consider the space Q of theorem 3. For any function $F = \sum_{i=1}^{n} \varphi_i x_i$ of Q where $x \in E_\alpha$, $0 \leqq \varphi_i \in B$, the function ; $\omega \to p_\omega(\sum_{i=1}^{n} \varphi_i(\omega)x_i)$ is λ-measurable and we have

$$p_\omega(\sum_{i=1}^{n} \varphi_i(\omega)x_i) \leqq \sum_{i=1}^{n} \varphi_i(\omega) \ p_\omega(x_i) \leqq \sum_{i=1}^{n} \varphi_i(\omega) \ ||x||_\alpha \ g_\alpha(\omega) \ M(\alpha) \ .$$

Hence $\left| \int p_\omega (\sum_{i=1}^{n} \varphi_i(\omega) \, x_i) \, d\lambda(\omega) \right| < + \infty$. Put $S(F) = \int p_\omega(F(\omega)) \, d\lambda(\omega)$ for any $F \in Q$. Then S is a sublinear function on Q. Since $B \ni 1$, we may identity E a subspace of Q. By the Hahn-Banach's extention theorem, $x' \in E'$ dominated by S on E can be extended to a linear form ξ' dominated by S on Q. For any $x \in E_\alpha$ and $\varphi \in B$, we have

$$\langle \xi', \varphi x \rangle \leqq S(\varphi x) = \int p_\omega(\varphi(\omega) \, x) \, d\lambda(\omega)$$

$$\leqq M(\alpha) \int |\varphi(\omega)| \, \|x\|_\alpha \, g_\alpha(\omega) \, d\lambda(\omega) = M(\alpha) \, \|\varphi x\|_\alpha \, .$$

Hence, applying theorem 2, there exists a weakly measurable mapping ; $\omega \to x_\omega'$ from Ω into E' such that

$$\langle \xi', F \rangle = \int \langle x_\omega', F(\omega) \rangle \, d\lambda(\omega)$$

and

$$\int \langle x_\omega', F(\omega) \rangle \, d\lambda(\omega) \leqq \int p_\omega(F(\omega)) \, d\lambda(\omega)$$

for any $F \in Q$. In particular for any $x \in E$, we have

$$\langle x', x \rangle = \langle \xi', x \rangle = \int \langle x_\omega', x \rangle \, d\lambda(\omega) \, .$$

Since for any $x \in E$ and any λ-measurable set A contained in K_i, $\chi_A \in B$, and $\chi_A x \in Q$, we have $\langle \xi', \chi_A x \rangle \leqq S(\chi_A x)$, whence

$$\int_A \langle x_\omega', x \rangle \, d\lambda(\omega) \leqq \int_A p_\omega(x) \, d\lambda(\omega) \, .$$

Consequently, we have $\langle x_\omega', x \rangle \leqq p_\omega(x)$ a.e. on K_j, whence $\langle x_\omega', x \rangle \leqq p_\omega(x)$ a.e. on Ω.

Since E is separable, there exists a sequence $\{x_n\}$ dense in E. For any $n \in N$, we have $\langle x_\omega', x_n \rangle \leqq p_\omega(x_n)$ a.e. on Ω. Since p_ω and x_ω' are bounded on E_α for each $\alpha \in \mathfrak{A}$, they are continuous in E. Therefore, there exists a negligible set A such that $\langle x_\omega', x \rangle \leqq p_\omega(x)$ for any $x \in E$ and any $\omega \in A^c$. If we define $y_\omega' = x_\omega'$ if $\omega \in A^c$ and $y_\omega' = z_\omega'$ if $\omega \in A$ where for $\omega \in A$, z_ω' is any element of E' dominated by p_ω, then $y_\omega' \in E'$.

Thus $\langle y_\omega', x \rangle \leqq p_\omega(x)$ for any $x \in E$ and the mapping $\omega \to y_\omega'$ is weakly measurable and holds

$$\langle x', x \rangle = \int \langle y_\omega', x \rangle \, d\lambda(\omega)$$

for any $x \in E$. Hence the theorem is proved.

Samely we can prove the following theorem by applying theorem 3.

THEOREM 5. *Let E be an ordered vector space. Assume that $(\Omega, \mathfrak{F}, \lambda)$, E and p satisfy the same conditions in theorem 4. Then, for $x' \in (E')^+$ the following* (a) *and* (b) *are equivalent;*

(a) $\langle x', x \rangle \leqq s(x)$ *for any* $x \in E$,

(b) *there exists a weakly measurable mapping;* $\omega \rightarrow x_\omega' \in (E')^+$ *such that* x_ω' *is dominated by* p_ω *and*

$$\langle x', x \rangle = \int \langle x_\omega', x \rangle \, d\lambda(\omega)$$

for any $x \in E$.

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