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On a Separation Property of a Function Algebra

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§1. Introduction

In this paper we shall consider a separation property of a function algebra which is obtained in studying unit-boundaries of a function algebra. [2]*, [3]

Let X be a compact Hausdorff space and $C(X)$ the Banach algebra of all complex valued continuous functions on X with the sup-norm. We consider a function algebra A on X, that is, A is a closed subalgebra of $C(X)$ which separates the points of X and contains the constant functions.

Throughout this paper, let $M(A)$ be the maximal ideal space, $\Gamma(A)$ the Silov boundary and $Cho(A)$ the Choquet boundary, of A, respectively. Further, let, for any set S, $f(S)$ be the set $\{f(x)\,;\,x\in S\}$.

DEFINITION. *A closed subset F of $M(A)$ is called a unit-boundary of A iff F satisfies the following condition; for any function f in A which does not vanish of F, there is a function g in A with $f\cdot g=1$.*

The above definition is written equivalently as follows;

A unit-boundary F is a closed subset of $M(A)$ with $f(F)=f(M(A))$.

In the paper [3], we have obtained the necessary & sufficient condition for F to be identical with the Silov boundary.

DEFINITION. *A function algebra A satisfied the condition (*) on a closed subset S in $M(A)$ iff for any closed proper subset K in S there is a function f in A and a point x in $S-K$ such that $f(x)=0$ and $(Re\ f)(y)>0$ (or <0) for all y in K.*

A function algebra A satisfies the condition (**) on a closed subset S in $M(A)$ iff for any closed proper subset K in S and any positive number ε, there is a function h in A and a point x in $S-K$ such that $h(x)=1$ and $|h(y)|<\varepsilon$ on K.

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* The number in bracket refers to the paper in the reference.

By the lemma in [3], we know that for a closed subset S in $M(A)$ the condition (*) is equivalent to the condition (**).

§ 2. On a separation property of A

The following definition we owe to D.R. Wilken [5].

DEFINITION. *A function algebra A is said to be "approximately regular on X" iff, for each point p in X and each closed set K in X not containing p and for any positive number ε, there is a function f in A such that $f(p)=1$ and $|f(y)|<\varepsilon$ on K. A is said to be "approximately normal on X" iff, for any two disjoint closed subset K_1, K_2 in X and any positive number ε, there is a function f in A such that $|f(x)-1|<\varepsilon$ on K_1 and $|f(y)|<\varepsilon$ on K_2.*

It is evident that if A is approximately normal, then A is approximately regular, and if A is approximately regular, then A satisfies the condition (**).

In general the condition (**) is weaker than "approximately regular".

In the following we shall construct an example for this.

EXAMPLE. Let $X=\{z:|z|\leqq 1\}$, $T=\{z:|z|=1\}$,
$\tilde{A}=\{f\in C(X)$: for the restriction of f to T, there is a founction $\tilde{f}$ which is analytic in X' (the interior of X) and $f(0)=\tilde{f}(0)\}$.

Then $\tilde{A}$ is a function algebra on X and its Silov boundary is X. By the proposition in [3], we know that $\tilde{A}$ has the condition (*) on X.

On the other hand $\tilde{A}$ in not approximately regular. In fact for the point 0, closed set T and any positive number $\varepsilon(<1)$, there is no function f in $\tilde{A}$ such that $f(0)=1$ and $|f(y)|<\varepsilon$ on T by the maximum modulus principle.

It is interesting that the condition (*) and (**) do not possess the property that if A satisfies the condition (*) (or **)) on a closed subset of S again. On the other hand the concept of "approximately regular" and "approximately normal" has this property.

To show this we shall make use of the above example.

We know already that $\tilde{A}$ satisfies the condition (*) on X. But on the closed subset $T\cup\{0\}$, $\tilde{A}$ does not satisfies the condition (*).

In fact if $\tilde{A}$ satisfied the condition (*) on $T\cup\{0\}$, then for the closed subset T of $T\cup\{0\}$ and any positive number ε, there would be a function f in $\tilde{A}$ such that $f(0)=1$ and $|f(y)|<\varepsilon$ on T. This contradicts the maximum modulus principle.

Now we try to strengthen the condition (**).

At first we exchange the condition (**) for the following condition (**)′ ; if A satisfies the condition (**) on S, then on any closed subset K in S A satisfies again the condition (**).

We can prove easily that the condition (**)′ is equivalent to " approximately regular ".

Therefore the stronger concept " (**)′ " is not interesting.

Now we define the condition (***) as follows, which is stronger than (**) and weaker than " approximately regular " and is equivalent on $M(A)$ to " approximately regular in a weak sense* " and " approximately normal in a weak sense* ".

DEFINITION. *A function algebra A satisfies the condition* (***) *on a closed subset S of $M(A)$ iff A satisfies the condition* (**) *on S and further on any closed subset K of S having no isolated points with respect to the relative topology of K, A satisfies the condition* (**) *on K again i.e. for any proper closed subset J of K and positive number ε, there is a function f in A and a point x in $K-J$ such that $f(x)=1$ and $|f(x)|<\varepsilon$ on J.*

To show that in general the condition (***) is weaker than approximately regular, we shall make use of the above example.

Already we know that the function algebra $\tilde{A}$ in the example is not approximately regular and satisfies the condition (**) on X. Then we shall show that $\tilde{A}$ satisfies the condition (***) on X.

Let K be any closed subset which has no isolated points with relative topology of K. Then for any x in K and an open nghd U of x, the set $U\cap K-\{x, 0\}$ is not empty. By the construction of $\tilde{A}$, any point except 0 in X is a peak point of $\tilde{A}$.

Let x' be a point of $U\cap K-\{x, 0\}$. Then the point x' is a peak point of $\tilde{A}$ i.e. there is a function f in $\tilde{A}$ such that $f(x')=1$ and $|f(y)|<1$ for any $y(\neq x)$.

Hence A satisfies the condition (***).

In spite of the weakness of the condition (***), on the maximal ideal sace this concept is equivalent to approximate normality and approximate regularity.

THEOREM. *If a function algebra A satisfies the condition* (***) *on $M(A)$, then A is approximately normal.*

In otherwords, on $M(A)$ the following three concepts are equivalent ;

(a) *A satisfies the condition* (***),

(b) *A is approximately regular in a weak sense,*

(c) *A is approximately normal in a weak sense.*

* The definition of " approximate regularity in a weak sense " and " approximate normality in a weak sense " is written after the references of this paper.

PROOF. It is sufficient to show that (a)$\Rightarrow$(c).

Let K_1, K_2 be disjoint closed subsets in $M(A)$ having at most finite isolated points, and A_0 the uniform closure of the restriction of A to $K_1 \cup K_2$ in $C(K_1 \cup K_2)$.

Then A_0 is a function algebra on $K_1 \cup K_2$.

It is well known that the maximal ideal space of A_0 is the set $\{x \in M(A) : |f(x)| \leqq \|f\|_{K_1 \cup K_2}$ for any f in $A\}$.

Hence the set $K_1 \cup K_2$ is contained in $M(A_0)$.

Now if $M(A_0)$ has isolated points $\{x_\alpha : \alpha \in \mathfrak{A}\}$, then by the Silov's theorem [4], there is a function f_α in A_0 such that $f_\alpha(x_\alpha)=1$ and $f_\alpha(y)=0$ for y in $M(A_0)-\{x_\alpha\}$, whence we have $\{x_\alpha : \alpha \in \mathfrak{A}\} \subset K_1 \cup K_2$. Therefore the set $\{x_\alpha\}$ is a finite set.

Let $M' = M(A_0) - \{x_\alpha : \alpha \in \mathfrak{A}\}$, $K_1' = K_1 - \{x_\alpha : \alpha \in \mathfrak{A}\}$ and $K_2' = K_2 - \{x_\alpha : \alpha \in \mathfrak{A}\}$.

Then M' has no isolated points. If $M' - K_1' \cup K_2'$ $(= M(A_0) - K_1 \cup K_2)$ were not empty, then by the condition (***) there would be a point x in $M' - K_1' \cup K_2'$ and a function f in A for any positive number $\varepsilon(<1/3)$ such that $f(x)=1$ and $|f(y)|<\varepsilon$ on $K_1' \cup K_2'$. On the other hand, there is a function g in A_0 such that $g(x_\alpha)=f(x_\alpha)$ for $\alpha \in \mathfrak{A}$ and $g(y)=0$ on M' by the Silov's theorem.

There is a function h in A such that $\|g-h\|_{K_1 \cup K_2} < \varepsilon$, as A_0 is the closure of $A | K_1 \cup K_2$.*

The function $f-h$ is in A and $|f(x_\alpha)-h(x_\alpha)|<\varepsilon$ for $\alpha \in \mathfrak{A}$ and $|f(y)-h(y)|<2\varepsilon$ on $K_1' \cup K_2'$.

Therefore $\|f-h\|_{K_1 \cup K_2} < 2\varepsilon$.

But $|f(x)-h(x)| = |1-h(x)| > 1-\varepsilon$.

The existence of the function $f-h$ contradicts the construction of $M(A_0)$. Hence $M(A_0) = K_1 \cup K_2$.

Again applying the Silov's theorem [4], there is a function f' in A_0 such that $f'(x)=0$ on K_1 and $f'(y)=1$ on K_2.

There is a function g' in A such that $|g'(x)|=\varepsilon$ on K_1 and $|g'(y)-1| < \varepsilon$ on K_2 by the closedness of A_0.

Thus we know that A is approximately normal in a weak sense on $M(A)$.

PROPOSITION. *A function algebra A satisfies the condition (***) on a closed subset F in M(A) if and only if for any closed subset K of F having no isolated points with respect to the relative topology of K and for the closure A_K in C(K) of A|K, the Silov boundary of A_K is identical with K.*

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* We shall denote $A | S$ the restriction of A to the set S.

PROOF. If, for some K, the Silov boundary of A_K were contained properly in K, then for $\Gamma(A_K)$ and any positive number ε, there would be a function f in A and some point x in $K-\Gamma(A_K)$ such that $f(x)=1$ and $|f(y)|<\varepsilon$ on $\Gamma(A_K)$ since A satisfies the condition (***) on F.

Then existence of the function f contradicts that $\Gamma(A_K)$ is the Silov boundary of A_K.

Then, for any closed subset K, $\Gamma(A_K)=K$.

Conversely we assume that $\Gamma(A_K)$ is equivalent to any closed subset K. For any point x in K and open nbhd U of x in K with respect to the relative topology, there is a point x' in Choquet boundary and a nbhd V of x' with $U\supset V$. For x', V and any positive number $\varepsilon(<1)$, there is a function g in A_K and a point x_0 in V such that $\|g\|\leqq 1$, $|g(x_0)|=1$ and $|g(y)|<\varepsilon/3$ on $K-V$. Since A_K is the closure of $A|K$, there is a function f in A such that $\|g-f\|_K<\varepsilon/3$. Let h be $f/f(x_0)$. Then h is in A and $h(x_0)=1$ and $|h(y)|<\varepsilon$ on $K-V$.

Therefore A satisfies the condition (***) on F.

COROLLARY 1. *If a function algebra A is maximal on the maximal ideal space $M(A)$, then A is approximately normal in a weak sense on $M(A)$.*

PROOF. If A is maximal, then the Silov boundary is equivalent to $M(A)$. Hence A satisfies the condition (**) on $M(A)$.

If A did not satisfy the condition (**) on some closed subset K having no isolated point, then there would be a proper subset F such that for any f in A and any point x in K, $|f(x)|\leqq\|f\|_F$.

Let A be a uniform closure of $A|F$. Then $M(A_F)$ is the set $\{x\in M(A):|f(x)|\leqq|f|_F \text{ for all } f \text{ in } A\}$ and A_F is also maximal on $M(A_F)$ and on F by the maximalty of A.

Hence $\Gamma(A_K)=F=M(A_F)$.

On the other hand $F\subsetneqq K\subset M(A_F)$ by the choice of F.

This contradiction shows us that A must satisfy the condition (***) on $M(A)$. Applying the theorem in § 2, A is approximately normal in a weak sense on $M(A)$.

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* DEFINITION. *A function algebra A is said to be "approximately regular in a weak sense on X" iff, for each point p in X and each closed subset K in X having at most finite isolated points and for any positive number* ε, *there is a function f in A such that* $|f(p)|=1$ *and* $|f(y)|<\varepsilon$ *on K. A is said to be "approximately normal in a weak sense on X" iff, for any two disjoint closed subsets* K_1, K_2 *in X having at most finite isolated points and for any positive number* ε, *there is a function f in A such that* $|f(x)-1|<\varepsilon$ *on* K_1 *and* $|f(y)|<\varepsilon$ *on* K_2.

— Postscript —

We can take away the assumption "in a weak sense" in the theorem. i.e. we get the sharp result as follows;

THEOREM. *Let A be a function algebra on M(A) satisfying the following condition; for any connected closed subset K of M(A) the Silov-boundary* $\Gamma(A_K)=K$. *Then A is approximately normal on M(A). And if A is approximately normal on M(A), then A satisfies the above condition.*

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