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On a Unit-Boundary of a Function Algebra II

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§1. Introduction

In this paper we shall mention some properties of a unit-boundary of a function algebra, continued from the paper " On a unit-boundary of a function algebra ". [2]*

Our main result is the necessary & sufficient condition, called condition (*) or condition (**), for a unit-boundary to be identical with the Silov boundary.

Let X be a compact Hausdorff space and $C(X)$ and Banach algebra of all complex valued continuous functions on X with sup-norm.

We shall consider a function algebra A on X, that is, A is a closed subalgebra, of $C(X)$, which separates points of X and contains the constant functions.

Throughout this paper, let $M(A)$ be the maximal ideal space, $\Gamma(A)$ the Silov boundary and Cho(A) the Choquet boundary, of A, respectively. Further, let, for any set S, $f(S)$ be the set $\{f(x) : x \in S\}$.

§2. On the condition (*) & the condition (**)

DEFINITION. *A closed subset F of $M(A)$ is called a unit-boundary of A iff F satisfies the following condition;*

For any function f in A which does not vanish on F, there is a function g in A with $f \cdot g = 1$.

The above definition is written equivalently as follows;

A unit-boundary F is a closed subset of $M(A)$ with $f(F) = f(M(A))$.

We know that every unit-boundary contains the Silov boundary. (See the theorem in [2]).

Now we shall search for conditions on which a unit-boundary is identical with the Silov boundary $\Gamma(A)$.

Let us first remark that under any one of the following equivalent

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* The number in brackets refers to the paper in the reference.

conditions (a), (b) and (c) a unit-boundary F is a minimal one.

(a) For any closed proper subset K in F there exists a function f in A and a point x in $F-K$ such that $f(x)=0$, $f(y)\neq 0$ for all y in K.

(b) For any point x in F and any open nbhd U there exists a function f in A such that $f(U)\ni 0$, $f(F-U)\not\ni 0$.

(c) For any closed proper subset K in F there exists a function f in A such that $f(K)\neq f(M(A))$.

In general a minimal unit-boundary is not identical with the Silov boundary. To construct an example for this we make use of the example 1 in [2]. Let X be the set $\{(z, w) : |z|\leqq 1, |w|\leqq 1 ; |z|-|w|\leqq \frac{1}{3}\}$ and A the set $\{f\in C(X) : f$ is holomorphic in the interior of X and continuous on $X\}$. It is well known that every function in A can be extended holomorphically to the set $X=\{(z, w) : |z|\leqq 1, |w|\leqq 1\}$, and the Silov boundary is the set $\{(z, w) : |z|=1$ and $|w|=1\}$, one of the minimal unit-boundary is the set $\{(z, w) : |z|\leqq 1, |w|\leqq 1, |z|=|w|\}$. And this is not identical with the Silov boundary.

Naturally we would like to strengthen the above condition so as to obtain the one under which a minimal unit-boundary is identical with the Silov boundary.

Now we shall consider a unit-boundary F with the condition (*) in the following definition.

DEFINITION. *A function algebra A satisfies the condition* (*) *on a closed subset S in M(A) iff for any proper closed subset K in S there is a function f in A and a point x in S−K such that* $f(x)=0$ *and* (Re f) $(x)>0$ (*or* <0) *for all y in K.*
(a function Ref is a real part of f)

The condition (*) may be also stated as follows;

A function algebra A satisfies the condition (**) *on a closed subset S in M(A) iff for any closed proper subset K in S and any positive number* ε *there is a function h in A and a point x in S−K such that* $h(x)=1$ *and* $|h(y)|<\varepsilon$ *on K.*

It is easy to see that the condition (*) is stronger than the condition (a). Hence if A satisfies the condition (*) on a unit-boundary F, then F is a minimal unit-boundary.

LEMMA. *For a closed subset S in M(A), the condition* (*) *on S is equivalent to the condition* (**) *on S.*

PROOF. At first we shall show that (*)$\Rightarrow$(**).

For a closed subset K in S and a positive number ε, by (*) there is a function f in A and a point x in $S-K$ such that $f(x)=0$, (Re f) $(y)>0$

on K. By the continuity of the function Re f, we can assume without loss of generality that, for a suitable positive number δ, the following estimate holds: $\delta<(\mathrm{Re}\, f)\,(y)<1$ on K and $f(x)=0$.

Let g be $e^{-f}\equiv\sum_{n=1}^{\infty}\frac{1}{n\,!}(-f)^n$.

Then g is in A by the closedness of A under uniform norm. The function g has the following property: $g(x)=1$ and $|g(y)|<e^{-\delta}<1$.

Hence for any ε, there exists a natural number n such that $|g^n(y)|<\varepsilon$ on K. Let h be the function g^n. Then h is also a function in A which satisfies the condition (**).

Now we shall show that (**)$\Rightarrow$(*).

For the function h in (**), let us put $f=1-h$. Then f is in A and satisfies the condition (*).

PROPOSITION. *Let F be a unit-boundary of A. The necessary and sufficient condition for F to be identical with the Silov boundary of A is that A satisfies the condition (*) on F.*

PROOF. We show first that $F=\Gamma(A)$ under the assumption that A satisfies the condition (*).

By the theorem in [2], every unit-boundary contains the Silov boundary. If F contained $\Gamma(A)$ properly, then there would be a point x in $F-\Gamma(A)$. By the normality of $M(A)$, there is an open nbhd U of x such that $U\cap\Gamma(A)=\emptyset$. By the above lemma "(*)$\Leftrightarrow$(**)", for the closed set $F-U$ and any positive number ε, there is a function h in A and a point x' in U such that $h(x')=1$ and $|h(y)|<\varepsilon$ on $F-U$. The set $F-U$ contains $\Gamma(A)$ by the choice of U. Then the existence of the function h contradicts the definition of the Silov boundary.

Hence a unit-boundary F is identical with $\Gamma(A)$.

Now we shall show that A satisfies the condition (*) (or (**)) on $\Gamma(A)$. Let K be a closed subset in $\Gamma(A)$, x a point of $\Gamma(A)-K$ and U an open nbhd of x in $\Gamma(A)$ such that $U\cap K=0$. Since the point x is in $\Gamma(A)$, there are a point x' in $U\cap\mathrm{Cho}(A)$ and a nbhd V of x' in $\Gamma(A)$ with $V\subset U$. By E. Bishop & K. de Leeuw [1], for a point x' in Cho(A) and any positive number ε, there is a function f in Asuch that $\|f\|\leqq 1$, $|f(x'')|=1$ at some x'' in V and $|f(y)|<\varepsilon$ on $\Gamma(A)-V$. Now since the function f satisfies the condition (**), also the condition (*) on $\Gamma(A)$.

COROLLARY. *Let R be any representing space of a function algebra A on X. Then the necessary and sufficient condition for R to contain properly the Silov boundary $\Gamma(A)$ is that for some closed proper subset K of R and for every functian in A it holds that*

$$f(R)\subset \text{the convex closure of } f(K).$$

PROOF. If $R=\Gamma(A)$, then R satisfies the condition (*) on $\Gamma(A)$. Hence for any closed set K of R, there is a function f in A such that $f(x)=0$ for some point x in $R-K$ and (Re f) $(y)>0$ on K.

Then the convex closure of $f(K)$ does not contain the origin by the continuity of Re f.

Therefore $f(R)$ is not contained by the convex closure of $f(K)$. Now we shall show the converse.

We assume that R contains $\Gamma(A)$ properly and further that the condition did not hold i.e. for any closed set K, there would be a function f in A such that $f(R)$ is not contained by the convex closure of $f(K)$.

Let K be the Silov boundary $\Gamma(A)$. Then the set $f(R)-$ the convex closure of $f(\Gamma(A))$ is not empty, which shows that there is a complex number λ in the set.

For λ, there is a point x in $R-\Gamma(A)$ such that $f(x)=\lambda$. In the complex plane there is a straight line which separates λ and the convex closure of $f(\Gamma(A))$. Therefore for some real number θ and the function $g=e^{i\theta}(f-\lambda)$ in A the following estimate holds: $g(x)=0$ and (Re g) $(y)>0$ on $\Gamma(A)$.

Then as in the lemma, there is a function h in A such that $h(x)=1$ and $|h(y)|<\varepsilon$ on $\Gamma(A)$, which contradicts the definition of the Silov boundary.

References

[1] E. Bishop & K. De Leeuw: The representation of linear functionals by measures on sets of extreme points, Ann. Inst. Fourier (Grenoble) 9 (1959), 305-331.

[2] K. Nishizawa: On a unit-boundary of a function algebra, Natural Science Report, Ochanomizu Univ. 22 (1971).